Section 5.3

Integration: "Integration by Substitution"

All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Introduction

- In this section we will study a technique called substitution. It can often be used to transform complicated integration problems into simpler ones.
- This method is directly related to the chain rule that we learned in chapter three for taking the derivative of a composition of functions f(g(x)).

Relationship to the Chain Rule

1. The chain rule gives us

   $\frac{d}{dx}[F(g(x))] = F'(g(x)) * g'(x)$ when F(x) is an antiderivative of f(x).

2. The corresponding integration formula is $\int F'(g(x)) * g'(x)dx = F(g(x)) + C$.

3. Since F is an antiderivative of f(x), that means that $\int f(x)dx = F(x)$ *and* $\frac{d}{dx}[F(x)] = F'(x) = f(x)$.

4. Therefore, we can do substitution of f(x) where F'(x) used to be and this gives us

   $\int f(g(x)) * g'(x)dx = F(g(x)) + C$.

5. This is where we may begin the u-substitution idea. If we let u = g(x), which is the innermost function for f(g(x)), and take its derivative, then we have $\frac{du}{dx} = g'(x)$.

6. Now we can do u-substitution and simplify:

$$\int f(g(x)) * g'(x)dx = \int f(u) * \frac{du}{dx}dx = \int f(u)du = F(u) + C$$

Guidelines for u-Substitution

Step 1. Look for some composition $f(g(x))$ within the integrand for which the substitution

$$u = g(x), \quad du = g'(x)\,dx$$

produces an integral that is expressed entirely in terms of u and its differential du. This may or may not be possible.

Step 2. If you are successful in Step 1, then try to evaluate the resulting integral in terms of u. Again, this may or may not be possible.

Step 3. If you are successful in Step 2, then replace u by $g(x)$ to express your final answer in terms of x.

Example

- I picked x^2+1 to be u because it is the "innermost function" and its derivative du/dx=2x which is also in the original problem and will make substitution easier.

Example 1 Evaluate $\int (x^2+1)^{50} \cdot 2x\,dx$.

Solution. If we let $u = x^2 + 1$, then $du/dx = 2x$, which implies that $du = 2x\,dx$. Thus, the given integral can be written as

$$\int (x^2+1)^{50} \cdot 2x\,dx = \int u^{50}\,du = \frac{u^{51}}{51} + C = \frac{(x^2+1)^{51}}{51} + C$$

Rules for Choosing u

- There is no rule for how to pick u that works 100% of the time.
- If you are starting with a composition of functions f(g(x)), setting u=g(x) will almost always work and that is a good thing to look for.
- Another strategy that often works well is to look for two parts of the original question where one part is the derivative of another. Make one du/dx and the other equal to u.

Example of f(g(x))

Find $\int \frac{dx}{\left(\frac{1}{3}x-8\right)^5}$	Since the function inside the parentheses is $\frac{1}{3}x - 8$, it is a good idea to make that u. Then $\frac{du}{dx} = \frac{1}{3}$. Cross multiply to get 3du = 1dx and solve for dx to get dx =3du.
$\int \frac{3du}{(u)^5} = 3\int u^{-5}du$	Substitute dx =3du and u = $\frac{1}{3}x - 8$. Factor out the constant, 3. Rewrite with a negative exponent to make the integration easier.
$3 * \frac{u^{-4}}{-4} + C$	Use the rule $\int x^r du = \frac{x^{r+1}}{r+1} + C$
$\frac{-3}{4\left(\frac{1}{3}x-8\right)^4} + C$ or $\frac{-3}{4}\left(\frac{1}{3}x - 8\right)^{-4} + C$	Simplify and resubstitute u = $\frac{1}{3}x - 8$.

Example when a function and its derivative are both in the question:

- The derivative of sin x is cos x and this original question has both. That often signals that we should make u = sin x and du/dx = cos x to make substitution work.

▶ **Example 7** Evaluate $\int \sin^2 x \cos x \, dx$.

Solution. If we let $u = \sin x$, then

$$\frac{du}{dx} = \cos x, \quad \text{so} \quad du = \cos x \, dx$$

Thus,

$$\int \sin^2 x \cos x \, dx = \int u^2 \, du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C$$ ◀

Another example

- Keep practicing and you will get better at these.
- Since 3-5t5 is inside the radical, it is a good choice for u.

Evaluate $\int t^4 \sqrt[3]{3-5t^5}dt$	Since the function inside the radical is $3-5t^5$, and its derivative has t^4 in it, those are both signs that it is a good idea to make that u. Then $\frac{du}{dx} = -25t^4$. Cross multiply to get du = $-25t^4$dx and divide both sides by -25 to get $\frac{-1}{25}du = t^4$dx
$\int \sqrt[3]{u} * \frac{-1}{25} du = \frac{-1}{25} \int u^{\frac{1}{3}} du$	Substitute $\frac{-1}{25}du = t^4$dx and u = $3-5t^5$. Factor out the constant, $\frac{-1}{25}$. Rewrite with a fractional exponent to make the integration easier.
$\frac{-1}{25} * \frac{u^{\frac{4}{3}}}{\frac{4}{3}}$+C	Use the rule $\int x^r du = \frac{x^{r+1}}{r+1} + C$
$\frac{-1}{25} * \frac{3}{4} * u^{\frac{4}{3}} + C$ ⇨ $\frac{-3}{100}(3-5t^5)^{\frac{4}{3}} + C$	Multiply by the reciprocal, simplify and resubstitute u = $3-5t^5$

The Duomo – Florence, Italy

I love the pink and green and the intricacy of the work.