Section 4.1

The Derivative in Graphing and Applications- "Analysis of Functions I: Increase, Decrease, and Concavity"

All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Overview

- In this chapter, we will study a variety of applications of the derivative.
  - Analyzing graphs
  - Optimization problems to find the smallest and largest value occurs
  - Motion of a particle along a line

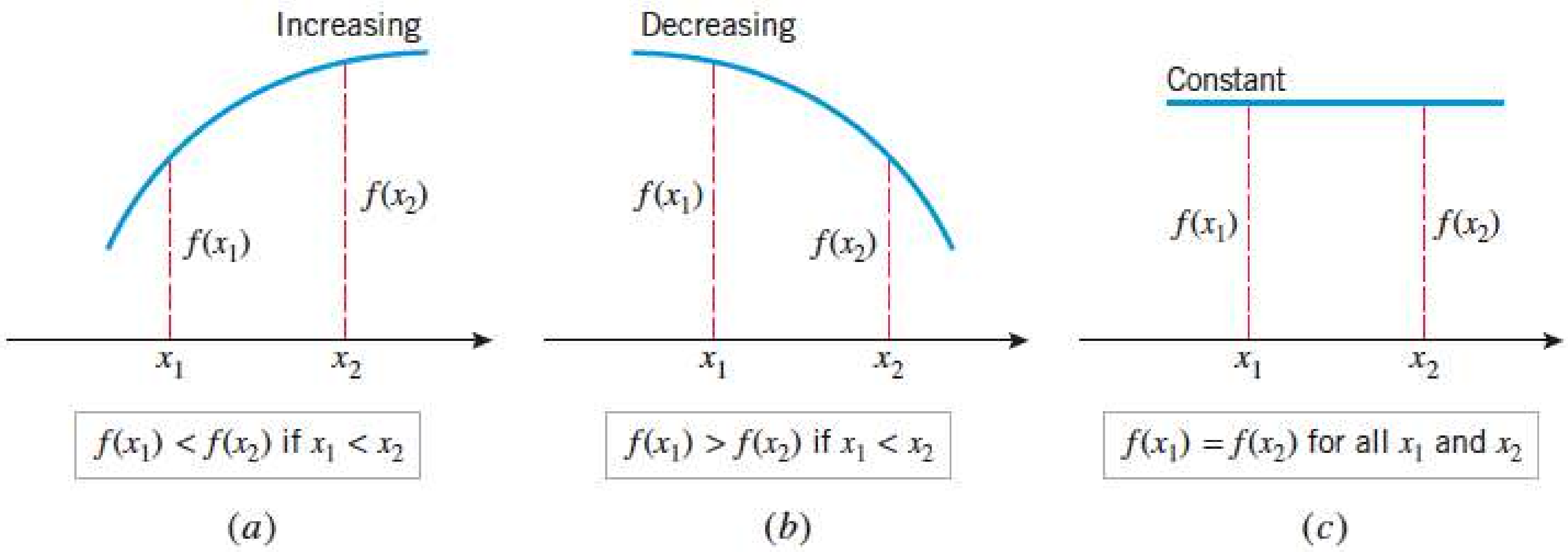
Analyzing Graphs

- Some problems require more precision than graphing calculators produce. Therefore, we need methods to determine the exact shape of a graph and locations of key features.

Increasing, Decreasing and Constant Functions/Intervals

- A differentiable function (one you are able to take the derivative of) is increasing on any interval where each tangent line to its graph has a positive slope.
- A differentiable function is decreasing on any interval where each tangent line to its graph has a negative slope.
- A differentiable function is constant on any interval where each tangent line to its graph has a zero slope.
- See graphs on next page to help visualize.

Graphical Interpretation



Related Theorem

4.1.2 THEOREM *Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).*

(a) *If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.*

(b) *If $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on $[a, b]$.*

(c) *If $f'(x) = 0$ for every value of x in (a, b), then f is constant on $[a, b]$.*

- Note: The derivative conditions are only required to hold inside the interval (a,b), even though the conclusions apply to the entire interval [a,b].

Example

- Find the intervals where the graph $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ is increasing, decreasing, and/or constant.

- Solution: By differentiating f(x), we obtain

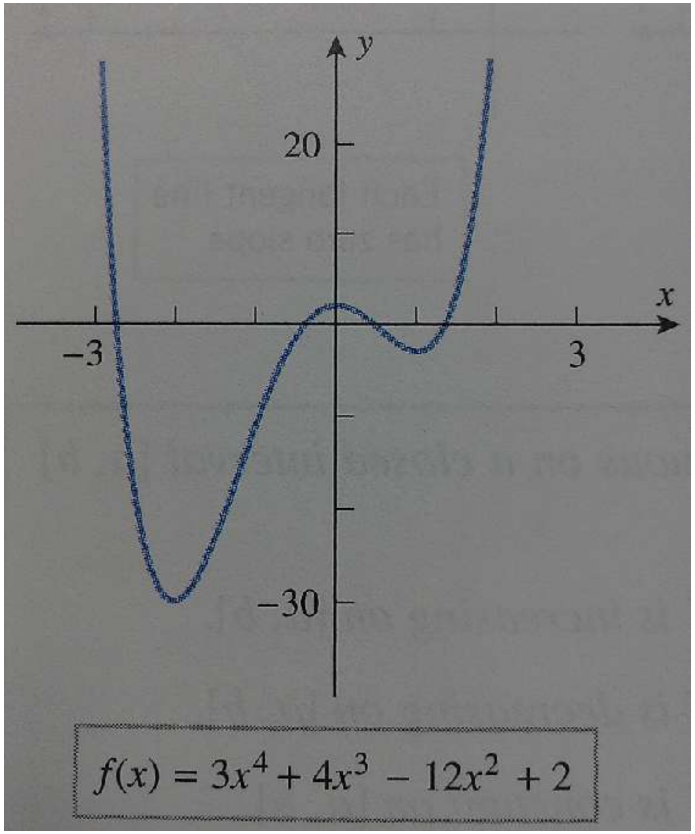
$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x + 2)(x - 1)$$

- Then do the zero product property which gives x=0, x=-2 and x=1. Since f'(x) =0 at these points, that is where the function is constant (these are called stationary points).
- Next, we need to make an interval table to find out where it is increasing and/or decreasing.

Interval Table and Graph

Interval	Test #	Test in f '(x)	Result	Effect
(-infinity, -2)	-3	$12(-3)^3+12(-3)^2-24(-3)$	-144	decreasing
(-2, 0)	-1	$12(-1)^3+12(-1)^2-24(-1)$	+24	increasing
(0, 1)	.5	$12(.5)^3+12(.5)^2-24(.5)$	-7.5	decreasing
(1, +infinity)	2	$12(2)^3+12(2)^2-24(2)$	+96	increasing

We can see the results of the zero product property and the interval table match the intervals on the graph where it is increasing, decreasing, and/or constant.



Concavity

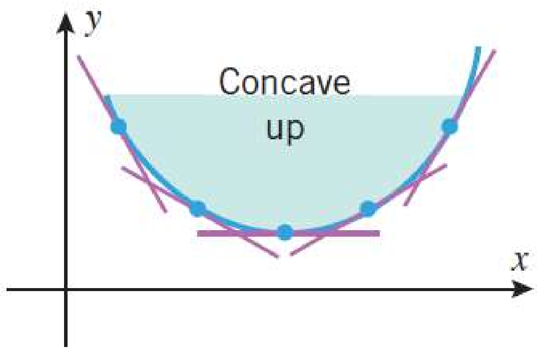
- The sign of f'(x) reveals where the graph of f(x) is increasing or decreasing or constant, it does not reveal direction of curvature a.k.a. concavity.
- For that we will need the second derivative because a function is "concave up" on an open interval (a,b) if its graph lies above its tangent lines and is "concave down" if its graph lies below its tangent lines.

4.1.3 DEFINITION If f is differentiable on an open interval, then f is said to be *concave up* on the open interval if f' is increasing on that interval, and f is said to be *concave down* on the open interval if f' is decreasing on that interval.

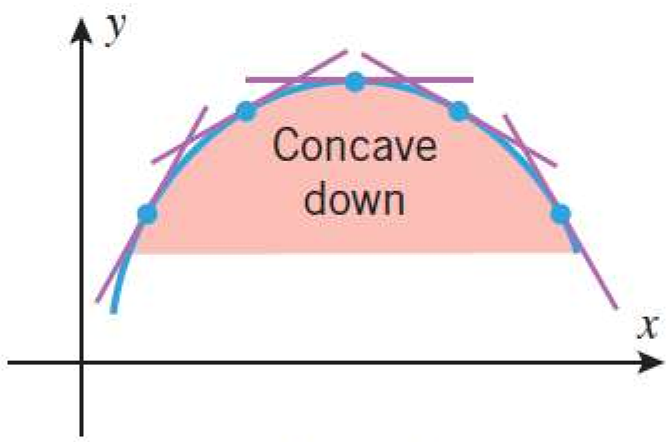
Visual Interpretation

Slopes are getting bigger as you move to the right.

Slopes are getting smaller as you move to the right



Increasing slopes



Decreasing slopes

How to find intervals of concave up and concave down

- A differentiable function is concave up on any interval where f''(x) is positive.
- A differentiable function is concave down on any interval where f''(x) is negative.

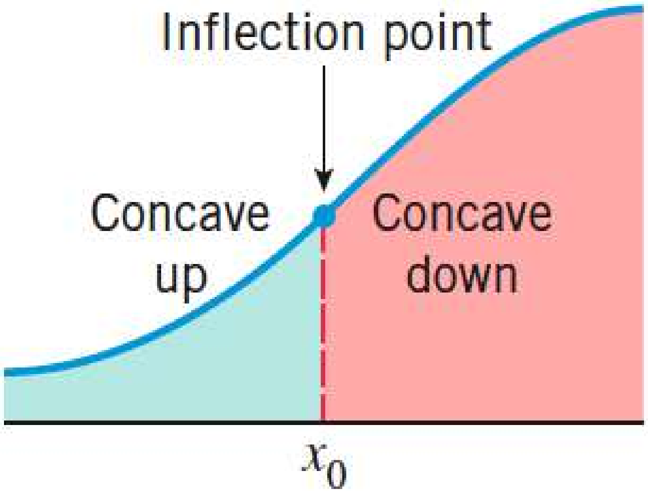
4.1.4 THEOREM *Let f be twice differentiable on an open interval.*

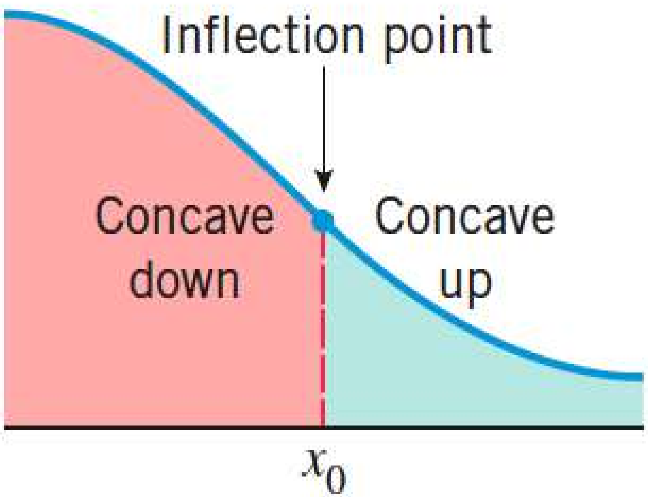
(a) *If $f''(x) > 0$ for every value of x in the open interval, then f is concave up on that interval.*

(b) *If $f''(x) < 0$ for every value of x in the open interval, then f is concave down on that interval.*

Inflection Points

- A differentiable function has an inflection point where the second derivative is zero.





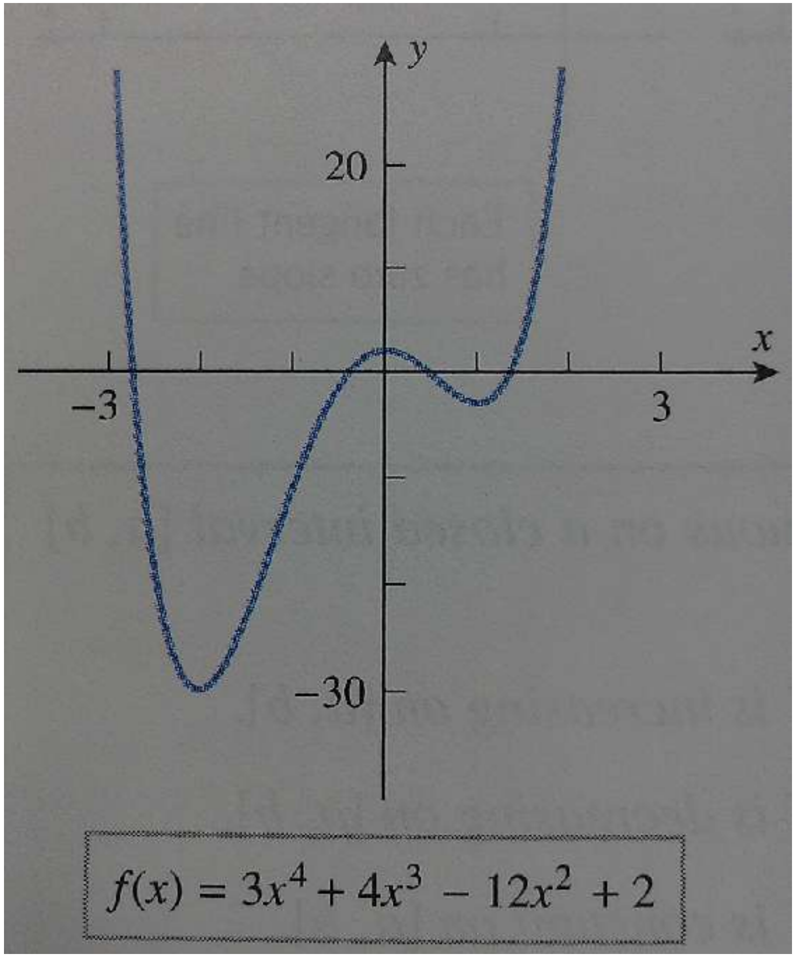
4.1.5 DEFINITION If f is continuous on an open interval containing a value x_0, and if f changes the direction of its concavity at the point $(x_0, f(x_0))$, then we say that f has an ***inflection point at*** x_0, and we call the point $(x_0, f(x_0))$ on the graph of f an ***inflection point*** of f (Figure 4.1.9).

Previous Example Continued

- Find the intervals where the graph of $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ is concave up, concave down, and/or constant.
- Remember, the first derivative was $f'(x) = 12x^3 + 12x^2 - 24x$ therefore, the second derivative is $f''(x) = 36x^2 + 24x - 24$.
- If you factor out the 12, set it equal to zero, and do the quadratic formula for $0 = 12(3x^2 + 2x - 2)$, you get x = .55 and x= -1.2.
- Those are called the inflection points and it is where the concavity changes. To determine whether concavity is up or down, you need an interval table.

Interval	Test #	Test in f ''(x)	Result	Effect
(-infinity, -1.2)	-2	$36(-2)^2+24*-2-24$	+72	Concave up
(-1.2,.55)	0	$36(0)^2+24*0-24$	-24	Concave down
(.55, +infinity)	1	$36(1)^2+24*1-24$	+36	Concave up

Graph



- We can see the results of the quadratic formula and the interval table match the intervals on the graph where it is concave up, concave down, or has an inflection point.

Inflection Points in Applications

- Inflection points mark the places on the curve f(x) where the rate of change of y with respect to x changes from increasing to decreasing or vice versa.