3

Exponential and Logarithmic Functions

3.1 Exponential Functions and Their Graphs

What You Should Learn

- Recognize and evaluate exponential functions with base *a*.
- Graph exponential functions with base *a*.
- Recognize, evaluate, and graph exponential functions with base *e*.
- Use exponential functions to model and solve real-life problems.

Exponential Functions

Exponential Functions

In this chapter you will study two types of nonalgebraic functions—*exponential functions and logarithmic functions.*

Definition of Exponential Function

The **exponential function f with base a** is denoted by

$$f(x) = a^x$$

where $a > 0$, $a \neq 1$, and x is any real number.

Exponential Functions

Note that in the definition of an exponential function, the base $a = 1$ is excluded because it yields

$f(x) = 1^x = 1.$ Constant function

This is a constant function, not an exponential function.

Example 1 – *Evaluating Exponential Functions*

Try this on your calculator, but do not write it down.

Use a calculator to evaluate each function at the indicated value of x.

Function	*Value*
a. $f(x) = 2^x$	$x = -3.1$
b. $f(x) = 2^{-x}$	$x = \pi$
c. $f(x) = 0.6^x$	$x = \frac{3}{2}$
d. $f(x) = 1.05^{2x}$	$x = 12$

Example 1 – *Solution*

Function Value	*Graphing Calculator Keystrokes*	*Display*
a. $f(-3.1) = 2^{-3.1}$	2 [^] [(−)] 3.1 [ENTER]	0.1166291
b. $f(\pi) = 2^{-\pi}$	2 [^] [(−)] π [ENTER]	0.1133147
c. $f\left(\frac{3}{2}\right) = (0.6)^{3/2}$	.6 [^] [(] 3 [÷] 2 [)] [ENTER]	0.4647580
d. $f(12) = (1.05)^{2(12)}$	1.05 [^] [(] 2 [×] 12 [)] [ENTER]	3.2250999

Graphs of Exponential Functions

Graphs of Exponential Functions

The graphs of all exponential functions have similar characteristics, as shown in Example 2 on the next slide.

Example 2 – *Graphs of $y = a^x$*

In the same coordinate plane, sketch the graph of each function by hand.

a. $f(x) = 2^x$ ***b.*** $g(x) = 4^x$

Solution:

The table below lists some values for each function. By plotting these points and connecting them with smooth curves, you obtain the graphs shown in Figure 3.1.

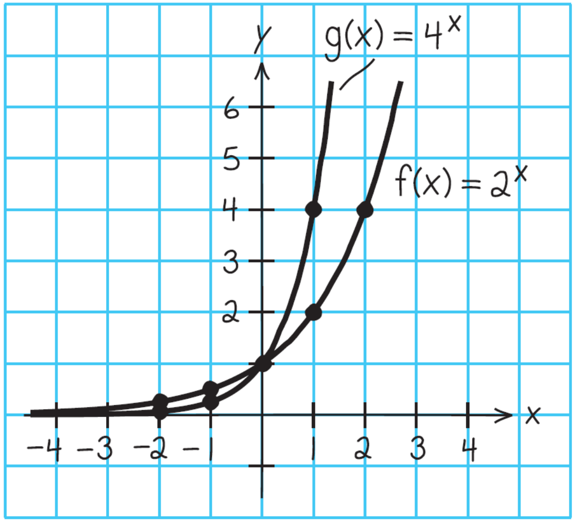


Figure 3.1

Example 2 – *Solution*

cont'd

Note that both graphs are increasing. Moreover, the graph of $g(x) = 4^x$ is increasing more rapidly than the graph of $f(x) = 2^x$. *You can tell if you compare the y values in the table below.*

x	-2	-1	0	1	2	3
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
4^x	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64

Graphs of Exponential Functions

Graph of $f(x) = a^x$, $a > 1$

Graph of $f(x) = a^{-x}$, $a > 1$

Domain:(,$-\infty$) ∞ Domain:(,) $-\infty$ ∞

Range :(0 , ∞) Range :(0 ,) ∞

Intercept :(0 ,1) Intercept :(0 ,1)

Increasing on :(,$-\infty$) ∞ Increasing on :(, $-$) ∞ ∞

The Natural Base *e*

The Natural Base *e*

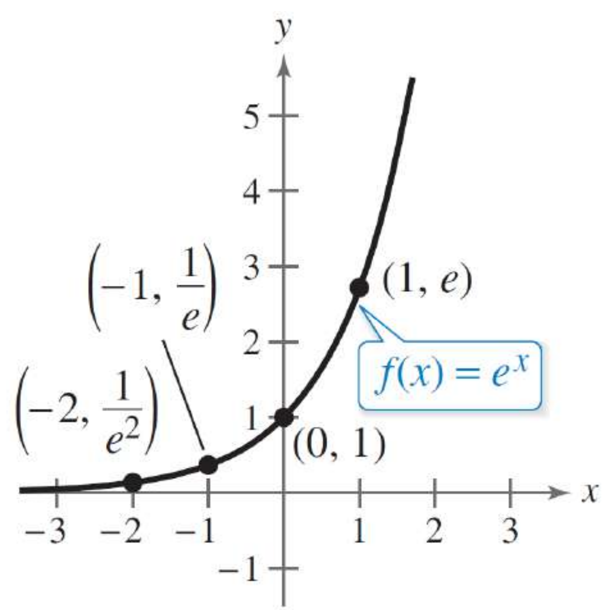
For many applications, the convenient choice for a base is the irrational number

$$e = 2.718281828 \ldots .$$

This number is called the **natural base.** The function

$$f(x) = e^x$$

is called the **natural exponential function** and its graph is shown in Figure 3.9.



The Natural Exponential Function

Figure 3.9

Example 6 – *Evaluating the Natural Exponential Functions*

Use a calculator to evaluate the function

$f(x) = e^x$

at each indicated value of *x*.

a. $x = -2$

b. $x = 0.25$

c. $x = -0.4$

d. $x = \frac{2}{3}$

Do this on your calculator, but do not write it down.

Example 6 – *Solution*

Function Value	*Graphing Calculator Keystrokes*	*Display*
a. $f(-2) = e^{-2}$	[e^x] [(−)] 2 [ENTER]	0.1353353
b. $f(0.25) = e^{0.25}$	[e^x] .25 [ENTER]	1.2840254
c. $f(-0.4) = e^{-0.4}$	[e^x] [(−)] .4 [ENTER]	0.6703200
d. $f\left(\frac{2}{3}\right) = e^{2/3}$	[e^x] [(] 2 [÷] 3 [)] [ENTER]	1.9477340

Applications

Applications

One of the most familiar examples of exponential growth is an investment earning *continuously compounded interest*.

To accommodate quarterly, monthly, or daily compounding of interest, let n be the number of compoundings per year and let t be the number of years.
(The product nt represents the total number of times the interest will be compounded.)

Please read the next two slides, but do not write them down.

Applications

Then the interest rate per compounding period is *r*/*n* and the account balance after *t* years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Amount (balance) with *n* compoundings per year

When the number of compoundings *n* increases without bound, the process approaches what is called **continuous compounding.** In the formula for *n* compoundings per year, let *m* = *n*/*r* . This produces

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = P\left(1 + \frac{1}{m}\right)^{mrt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}.$$

Applications

As *m* increases without bound, we have

$$\left(1 + \frac{1}{m}\right)^m$$

approaches *e*. So, for continuous compounding, it follows that

$$P\left[\left(1 + \frac{1}{m}\right)^m\right]^{rt} \Rightarrow P[e]^{rt}$$

and you can write $A = pe^{rt}$. This result is part of the reason that *e* is the "natural" choice for a base of an exponential function.

Applications

Formulas for Compound Interest

After t years, the balance A in an account with principal P and annual interest rate r (in decimal form) is given by the following formulas.

1. For n compoundings per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

2. For continuous compounding: $A = Pe^{rt}$

Example 8 – *Finding the Balance for Compound Interest*

A total of \$9000 is invested at an annual interest rate of 2.5%, compounded annually. Find the balance in the account after 5 years.

Solution:

In this case,

$P = 9000$, $r = 2.5\% = 0.025$, $n = 1$, $t = 5$.

Using the formula for compound interest with compoundings per year, you have

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$ Formula for compound interest

Example 8 – *Solution*

cont'd

$$= 9000\left(1 + \frac{0.025}{1}\right)^{1(5)}$$

Substitute for *P*, *r*, *n*, and *t.*

$= 9000(1.025)^5$

Simplify.

$\approx$ $10,182.67.

Use a calculator.

So, the balance in the account after 5 years will be about $10,182.67.