Topics in Differentiation

CHAPTER THREE

All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Topics in Differentiation

- We are now going to differentiate (take the derivative of) functions that are either difficult to differentiate in y= form (defined explicitly) or impossible to write in y= form and “differentiate directly”.

- Therefore, we need new methods such as implicit and logarithmic differentiation to find these derivatives in another and/or faster way.

Explicit vs. Implicit

- For more information on the difference between explicitly and implicitly defined, please read pages 185-186.

- In general, it is not necessary to solve an equation for y (in terms of x) in order to differentiate a function.
- On the next slide, I will take the derivative of a relatively easy function two different ways so that you can see the difference.

Example Two Ways

If we want to find the derivative of xy = 1, we could solve for y first if we wanted to: $y = \frac{1}{x} = x^{-1}$ and we could take its derivative by using the power rule which would give us $\frac{dy}{dx} = -1x^{-2} = \frac{-1}{x^2}$

Another way to obtain this derivative is to differentiate both sides of the original equation xy=1 before solving for y.

$xy = 1 \quad \rightarrow \quad y = \frac{1}{x}$		This is the original equation.
$\frac{d}{dx}[xy] = \frac{d}{dx}[1]$		$\frac{d}{dx}$ means to take the derivative.
$1(y) + y'(x) = 0$		On the left, I did the product rule f'g+g'f. On the right, the derivative of a constant is zero.
$y + x\frac{dy}{dx} = 0$		I changed y' to $\frac{dy}{dx}$. They mean the same thing.
$x\frac{dy}{dx} = -y$		Subtract y from both sides.
$\frac{dy}{dx} = \frac{-y}{x}$		Divide both sides by x.
$\frac{dy}{dx} = \frac{-\frac{1}{x}}{x}$	$\frac{dy}{dx} = \frac{-1}{x} * \frac{x}{1} = \frac{-1}{x^2}$	Substitute & Simplify

Example Results

- We got the same answer using both methods.
- Solving for y was obviously much faster, but it is not always possible to solve for y first (see example on next slide).
- Now we know another method.
  - Differentiate “implicitly”
  - Solve for dy/dx
  - Then substitution

Example Where We Cannot Solve for y First

Use implicit differentiation to find $\frac{dy}{dx}$ if $5y^2$ + sin y = x^2:

$\frac{d}{dx}[5y^2 + \sin y] = \frac{d}{dx}[x^2]$ — $\frac{d}{dx}$ means to take the derivative of both sides.

(10y) * y′ + (cos y) * y′= 2x — On the left, I did the power rule and the derivative of sin y. The new part is the y′. See explanation on next slide.

(10y) * $\frac{dy}{dx}$ + (cos y) * $\frac{dy}{dx}$ = 2x — I changed y′ to $\frac{dy}{dx}$. They mean the same thing.

$\frac{dy}{dx}$ (10y+ cos y) = 2x — Factor out $\frac{dy}{dx}$ on the left side of the equation.

$\frac{dy}{dx} = \frac{2x}{10y + \cos y}$ — Divide both sides by 10y+ cos y.

Normally, we would substitute & simplify. Since we cannot solve the original equation for y, we are forced to leave the answer in terms of x and y (with both x's and y's in the answer.

Second Derivative Example

Find $\frac{d^2y}{dx^2}$ if x cos y = y.

$\frac{d}{dx}[x\cos y] = \frac{d}{dx}[y]$	$\frac{d}{dx}$ means to take the derivative of both sides.	f is x and g is cos y so f' = 1 and g' = -sin y *y'
1*cos y –(-sin y)* y' * x= y'	On the left, I did the product rule.	
cos y +x sin y*$\frac{dy}{dx}$ = $\frac{dy}{dx}$	I changed y' to $\frac{dy}{dx}$ since they mean the same thing and I simplified.	
cos y + = $\frac{dy}{dx}$ – x sin y*$\frac{dy}{dx}$	Subtract x sin y*$\frac{dy}{dx}$ from both sides.	
cos y + = $\frac{dy}{dx}$ (1- x sin y)	Factor out $\frac{dy}{dx}$ on the left side of the equation.	
$\frac{dy}{dx} = \frac{\cos y}{1 - x\sin y}$	Divide both sides by 1- x sin y.	
	*That was just the first derivative☺. Second derivative is on the next slide.	

2nd Derivative Example (cont.)

Find $\frac{d^2y}{dx^2}$ if x cos y = y. (Continued)

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{\cos y}{1+ x\sin y}\right]$$

$\frac{d}{dx}$ of the derivative gives us the second derivative.

$$\frac{d^2y}{dx^2} = \frac{[(-\sin y * y') * (1+ x\sin y) - (\sin y + x\cos y * y')*\cos y]}{(1+ x\sin y)^2}$$

On the right, I did the quotient rule.

$$\frac{d^2y}{dx^2} = \frac{[(-\sin y * \frac{dy}{dx}) * (1+ x\sin y) - (\sin y + x\cos y * \frac{dy}{dx})*\cos y]}{(1+ x\sin y)^2}$$

I changed y′ to $\frac{dy}{dx}$ since they mean the same thing and I simplified.

$$\frac{d^2y}{dx^2} = \frac{[(-\sin y * \frac{\cos y}{1+x\sin y}) * (1+ x\sin y) - (\sin y - x\cos y * \frac{\cos y}{1+x\sin y})*\cos y]}{(1+ x\sin y)^2}$$

Substitute $\frac{\cos y}{1+x\sin y}$ for $\frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = \frac{\frac{-\sin y\cos y}{1+x\sin y} - \frac{x\sin^2 y\cos y}{1+x\sin y} - \frac{\sin y\cos y}{1+x\sin y} - \frac{x\sin^2 y\cos y}{1+x\sin y} - \frac{x\cos^3 y}{1+x\sin y}}{(1+ x\sin y)^2}$$

Distribute and get a common denominator.

$$\frac{d^2y}{dx^2} = \frac{-2\sin y\cos y - x\cos y\,(\sin^2 y+\sin^2 y+\cos^2 y)}{(1+ x\sin y)^2}$$

Combine like terms and factor out $x\cos y$ of several terms.

$$\frac{d^2y}{dx^2} = \frac{-\sin 2y - y\,(\sin^2 y+1)}{(1+ x\sin y)^2}$$

Apply trig identities $\sin^2 y + \cos^2 y = 1$ and sin double angle formula and substitute x cos y = y. DONE!!! ☺

Natural Log vs. Common Log

Natural Logarithm	Common Logarithm
The derivative when you are taking the natural log of just x. $$\frac{d}{dx}[\ln x] = \frac{1}{x}, \qquad x > 0$$	The derivative when you are taking the logarithm with a base other than e of just x. $$\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}, \qquad x > 0$$
The derivative when you are taking the natural log of something other than x (u). You must use the Chain Rule. $$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$$	The derivative when you are taking the log with a base other than e of something other than x (u). You must use the Chain Rule. $$\frac{d}{dx}[\log_b u] = \frac{1}{u \ln b} \cdot \frac{du}{dx}$$

Why is the Natural Logarithm preferred?

- Among all possible bases, base e produces the simplest formula for the derivative of $\log_b x$. This is the main reason (for now ☺) that the natural logarithm function is preferred over other logarithms in Calculus.

- Look at the previous slide and notice how much smaller the formulas are in the first column than those in the second column.

Examples

Find $\frac{d}{dx}[\ln(x^2+1)]$.

$$\frac{d}{dx}[\ln(x^2+1)]$$

Apply $\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$ and the Chain Rule

$$= \frac{1}{x^2+1} * the\ derivative\ of\ the\ inside$$

$$= \frac{1}{x^2+1} * 2x \quad = \frac{2x}{x^2+1}$$

Find $\frac{d}{dx}[\ln(\cos x)]$.

$$\frac{d}{dx}[\ln(\cos x)]$$

Apply $\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$ and the Chain Rule

$$= \frac{1}{\cos x} * the\ derivative\ of\ the\ inside$$

$$= \frac{1}{\cos x} * -\sin x \quad = \frac{-\sin x}{\cos x}$$ = - tan x

Logarithmic Differentiation

- Some derivatives are very long and messy to calculate directly, especially when they are composed of products, quotients, and powers all in one function like $y = \frac{x^2 \sqrt[5]{7x-14}}{(1+x^2)^4}$
- You would have to do the product and chain rules within the quotient rule and it would take quite awhile.
- Therefore, we are going to first take the natural logarithm of both sides and use log properties to simplify before we even start to find the derivative.

Logarithmic Differentiation Example

$$y = \frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4}$$

$$ln\, y = \ln\left(\frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4}\right)$$

Take the natural logarithm of both sides

$$ln\, y = \ln x^2 \sqrt[3]{7x-14} - \, ln(1+x^2)^4$$

Apply the quotient property of logarithms.

$$ln\, y = \ln x^2 + ln(7x-14)^{\frac{1}{3}} - \, ln(1+x^2)^4$$

Apply the product property of logarithms.

$$ln\, y = 2\ln x + \frac{1}{3}\ln(7x-14) - 4\ln(1+x^2)$$

Apply the power rule of logarithms.

$$\frac{1}{y}\frac{dy}{dx} = 2 * \frac{1}{x} * 1 + \frac{1}{3} * \frac{1}{7x-14} * 7 - 4 * \frac{1}{1+x^2} * 2x$$

Take the derivative of both sides and use

$$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$$

$$y * \frac{1}{y}\frac{dy}{dx} = \left(\frac{2}{x} + \frac{7}{3(7x-14)} - \frac{8x}{1+x^2}\right) * y$$

To get $\frac{dy}{dx}$ alone, multiply both sides by y. I also simplified the right side.

$$\frac{dy}{dx} = \left(\frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2}\right) * \frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4}$$

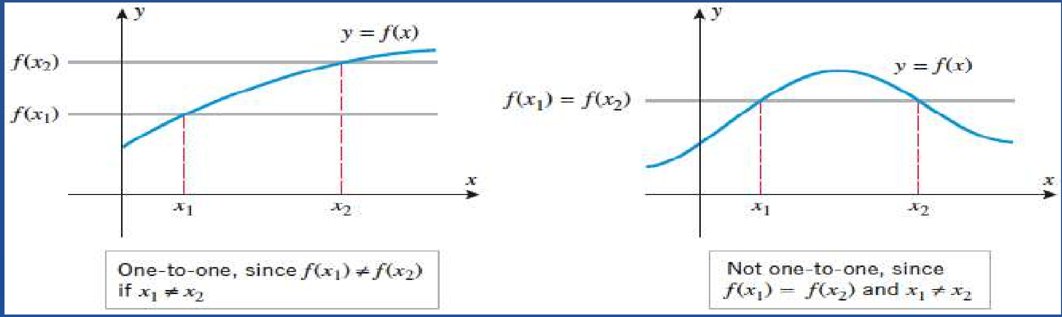
Substitute the original function in for y.

Logarithmic Differentiation Steps

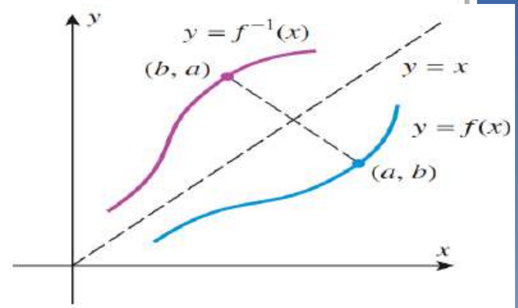
1. Take the natural log of both sides (whatever you do to one side, you must do to the other) since base e is easier than any other base (see slide #5).
2. Apply ALL of the possible properties of logarithms.
3. Take the derivative of both sides.
4. Multiply both sides by y.
5. Substitute the original in for y and simplify.

One-to-One and Inverse Functions (reminders from Section 0.4)

0.4.4 THEOREM *(The Horizontal Line Test)* *A function has an inverse function if and only if its graph is cut at most once by any horizontal line.*



0.4.5 THEOREM *If f has an inverse, then the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about the line $y = x$; that is, each graph is the mirror image of the other with respect to that line.*



One-to-One and Inverse Functions (continued)

- Note: Sometimes, it is necessary to restrict the domain of an inverse $f^{-1}(x)=x$ or of an original f(x) in order to obtain a function (see examples on page 44).

- A function f(x) has an inverse iff it is one-to-one (invertible), each x has one y and each y has one x (must pass vertical and horizontal line tests).

Increasing or Decreasing Functions are One-to-One

- If the graph of a function is always increasing ($f'(x)>0$) or always decreasing ($f'(x)<0$), then it will pass the horizontal and vertical line tests which means that the function is one-to-one.

Derivative of Exponential Functions

- After the last few slides, you probably do not want to see a proof here. ☺ If you would like to read the proof(s) of these derivatives, please see pages 198-199.

Base b (not e)	Base e (only)
The derivative when you have any base other than e to just the x power. $$\frac{d}{dx}[b^x] = b^x \ln b$$	The derivative when you have only base e to just the x power. $$\frac{d}{dx}[e^x] = e^x$$
The derivative when you have any base other than e to something other than just the x power (u). $$\frac{d}{dx}[b^u] = b^u \ln b \cdot \frac{du}{dx}$$	The derivative when you have only base e to to something other than just the x power (u). $$\frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx}$$

Logarithmic Differentiation Example

$y = (x^2 + 1)^{\sin x}$

$ln\, y = \ln(x^2 + 1)^{\sin x}$ — Take the natural logarithm of both sides

$ln\, y = (\sin x) * \ln\,(x^2 + 1)$ — Apply the power property of logarithms.

$$\frac{1}{y}\frac{dy}{dx} = (\cos x) * \ln(x^2 + 1) + \frac{1}{x^2 + 1}(2X) * \ sinx$$

Take the derivative of both sides using the product rule and $\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$.

$$y * \frac{1}{y}\frac{dy}{dx} = ((\cos x) * \ln(x^2 + 1) + \frac{2x * sinx}{x^2 + 1}) * y$$

To get $\frac{dy}{dx}$ alone, multiply both sides by y. I also simplified the right side.

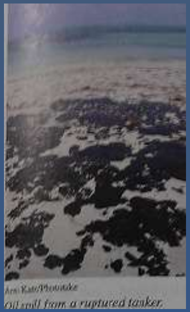
$$\frac{dy}{dx} = ((\cos x) * \ln(x^2 + 1) + \frac{2xsinx}{x^2 + 1}) * (x^2 + 1)^{\sin x}$$

Substitute the original function in for y.

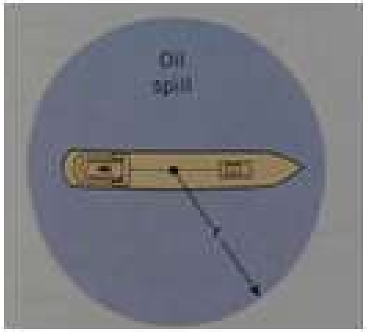
Related Rates Problems

- In related rates problems, we are trying to find out the rate at which some quantity is changing related to other quantities whose rates of change are known.

Oil Spill Example:



Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at a constant rate of 2 ft/s. How fast is the area of the spill increasing when the radius of the spill is 60 ft?



1. Let t=number of seconds after time of the spill
   r=radius of the spill after t seconds
   A=area of the spill after t seconds (square feet)

2. We know $\frac{dr}{dt} = 2\frac{ft}{second}$ *since it was given in the question*

   We are trying to find $\frac{dA}{dt}$ *when* $r = 60\ feet$

3. $A = \pi r^2$ — Area of a circle
4. $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$ — Take the derivative of both sides with respect to t implicitly ($which\ is\ \frac{d}{dt}$).

5. $\frac{dA}{dt}$ $when\ r\ is\ 60\ feet = 2\pi(60)(2)$ — Substitute and solve

$$= 240\pi\frac{ft^2}{sec} \approx 754\ ft^2/sec$$

Strategy for Solving Related Rates Problems

- If you noticed the numbers #1-5 on the last example, those are the steps that I suggest you follow when solving these problems.

1. Draw and label
2. Have and need
3. Relating equation
4. d/dt both sides
5. Substitute/solve

A Strategy for Solving Related Rates Problems

Step 1. Assign letters to all quantities that vary with time and any others that seem relevant to the problem. Give a definition for each letter.

Step 2. Identify the rates of change that are known and the rate of change that is to be found. Interpret each rate as a derivative.

Step 3. Find an equation that relates the variables whose rates of change were identified in Step 2. To do this, it will often be helpful to draw an appropriately labeled figure that illustrates the relationship.

Step 4. Differentiate both sides of the equation obtained in Step 3 with respect to time to produce a relationship between the known rates of change and the unknown rate of change.

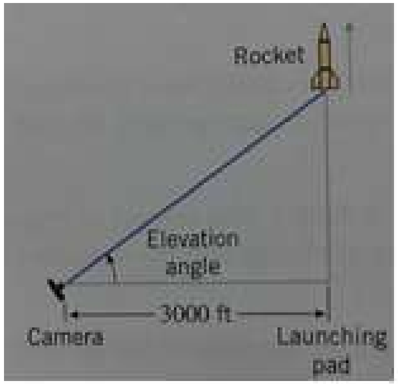
Step 5. *After* completing Step 4, substitute all known values for the rates of change and the variables, and then solve for the unknown rate of change.

Rocket Example

The camera is mounted at a point 3000 feet from the base of a rocket launching pad. If the rocket is rising vertically at 880 feet/second when it is 4000 feet above the launching pad, how fast must the camera elevation angle change at that instant to keep the camera aimed at the rocket?

1. Draw and label:

   Let t=number of seconds after time of the launch
   φ=camera elevation angle in radians after t seconds
   h=height of the rocket after t seconds (feet)



2. We know $\frac{dh}{dt}$ $when\ the\ height\ is\ 4000\ ft = 880\ \frac{ft}{second}$

   We are trying to find $\frac{d\varphi}{dt}$ $when\ height = 4000\ feet$

Given in question

3. Equation relating the variables h and φ:

   $\tan\varphi = \frac{h}{3000}$ may want to rewrite as: $\tan\varphi = \frac{1}{3000}h$

4. Take the derivative of both sides with respect to t implicitly.

   $sec^2\ \varphi\frac{d\varphi}{dt} = \frac{1}{3000}\frac{dh}{dt}$

Since an angle is involved, we must use trig. We have the adjacent side and are talking about height which is the opposite side→ $tangent$

5. $sec^2\ \varphi\frac{d\varphi}{dt} = \frac{1}{3000}(880)$

   ASIDE: $3000^2 + 4000^2 = x^2$

   $9000000+16000000=x^2$

   x=5000

   $\left(\frac{5}{3}\right)^2\frac{d\varphi}{dt} = \frac{1}{3000}(880)$

   $\sec\varphi = \frac{5000}{3000} = \frac{5}{3}$

   $\frac{d\varphi}{dt}$ $when\ h\ is\ 4000\ feet = \frac{880}{3000}\left(\frac{9}{25}\right) \approx .11\ rad/sec * \frac{180}{\pi} \approx 6.05\ deg/sec$

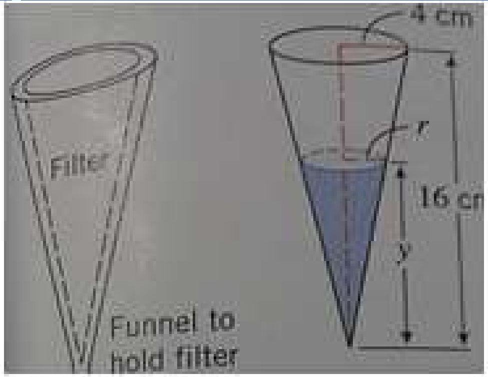
Substitution

We still have too many variables, so we need to solve for $\sec\varphi = \frac{hyp}{adj}$ here.

More substitution and solve

Similar Triangles Example

Suppose that liquid is to be cleared of sediment by allowing it to drain through a conical filter that is 16 cm high and has a radius of 4 cm at the top (see picture at right). Suppose also that the liquid is forced out of the cone at a constant rate of $2\frac{cm^3}{sec}$. At what rate is the depth of the liquid changing at the instant when the liquid in the cone is 8 cm deep?



1. Draw and label:

Let t=number of seconds after initial observation

V=Volume of liquid in the cone after t seconds (cm^3)

y=depth of the liquid in the cone after t seconds (cm)

r=radius of the liquid surface at time t (cm)

2. We know $\frac{dV}{dt} = -2\frac{cm^3}{sec}$

We are trying to find $\frac{dy}{dt}$ $when\ depth = 8\ cm$

Given in question (negative because Volume is decreasing in the cone).

3. Equation relating the variables V and y:

$V = \frac{1}{3}\pi r^2 y$ Which has too many variables, so we need to replace r.

$V = \frac{1}{3}\pi\left(\frac{y}{4}\right)^2 y$ Simplify before going on to 4 to make derivative easier.

$V = \frac{\pi}{48}y^3$

Volume of a cone $= \frac{1}{3}\pi r^2 h$

Similar triangles: $\frac{r}{4} = \frac{y}{16}$

Solve above for r→ $r = \frac{y}{4}$

4. Take the derivative of both sides with respect to t implicitly.

$\frac{dV}{dt} = \frac{\pi}{48} * 3y^2\frac{dy}{dt}$

5. $-2 = \frac{\pi}{48} * 3(8)^2\frac{dy}{dt}$ → $-2 = 4\pi\frac{dy}{dt}$

$\frac{dy}{dt}$ $when\ depth\ is\ 8\ cm = \frac{-1}{2\pi} \approx -.16\ cm/min$

Substitution, then divide to solve for dy/dt.

Indeterminate Forms

- In this section, we will discuss a general method for using derivatives to find limits.
- This method will allow us to find limits that we were previously only able to find by graphing (like the squeezing theorem).
- Many computers use this method (internally) when calculating limits.

0/0 Type Indeterminate Form: L'Hopital's Rule

3.6.1 THEOREM (*L'Hôpital's Rule for Form 0/0*) *Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that*

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

If $\lim_{x \to a} [f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$, or as $x \to +\infty$.

Warning and Steps

- ** Please notice that the numerator and denominator are differentiated individually. (you take their derivatives separately) It is NOT the same as (f/g)’, so you are NOT using the quotient rule.**

Applying L’Hôpital’s Rule

Step 1. Check that the limit of $f(x)/g(x)$ is an indeterminate form of type $0/0$.

Step 2. Differentiate f and g separately.

Step 3. Find the limit of $f'(x)/g'(x)$. If this limit is finite, $+\infty$, or $-\infty$, then it is equal to the limit of $f(x)/g(x)$.

Example Two Ways

Find the limit: $\lim_{x\to 2}\frac{x^2-4}{x-2}$

Using L'Hopital's Rule: Step 1: $\lim_{x\to 2}\frac{x^2-4}{x-2} = \frac{2^2-4}{2-2} = \frac{0}{0}\surd$ Step 2: $\lim_{x\to 2}\frac{\frac{d}{dx}(x^2-4)}{\frac{d}{dx}(x-2)} = \lim_{x\to 2}\frac{2x}{1} = \frac{2(2)}{1} = 4$ Step 3: 4 is finite, therefore, $\lim_{x\to 2}\frac{x^2-4}{x-2} = 4$	Check that it is indeterminate form 0/0 Take the derivative of the top and bottom SEPARATELY!!!!!!!!! ☺ If this limit is finite, $+\infty, or -\infty$, then it is equal to the limit of f(x)/g(x).
Using methods from Chapter 2 $\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2} x+2$ $= 2+2 = 4$ We get the same answer both ways.	Simplify first Then evaluate the limit by substitution

Another example

- There are five more similar examples on pages 221-222 if you are interested.
- For the example below, I cannot use algebraic methods from Chapter 2 and I do not know how to graph it without a calculator.
- This is when L'Hopital's Rule is especially useful.

Solution (a). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{\sin 2x}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}[\sin 2x]}{\frac{d}{dx}[x]}=\lim_{x\to 0}\frac{2\cos 2x}{1}=2$$

Make sure you do step 1!

WARNING

Applying L'Hôpital's rule to limits that are not indeterminate forms can produce incorrect results. For example, the computation

$$\lim_{x \to 0} \frac{x+6}{x+2} = \lim_{x \to 0} \frac{\frac{d}{dx}[x+6]}{\frac{d}{dx}[x+2]} = \lim_{x \to 0} \frac{1}{1} = 1$$

is ***not valid***, since the limit is not an indeterminate form. The correct result is

$$\lim_{x \to 0} \frac{x+6}{x+2} = \frac{0+6}{0+2} = 3$$

Another version of L'Hopital's Rule

- This version is used to find the limit of ratios in which the numerator and denominator both have infinite limits.

3.6.2 THEOREM ***(L'Hôpital's Rule for Form ∞/∞)*** *Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that*

$$\lim_{x \to a} f(x) = \infty \quad \textit{and} \quad \lim_{x \to a} g(x) = \infty$$

If $\lim_{x \to a} [f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$, or as $x \to +\infty$.

Example this version of L’Hopital’s Rule

- Similar steps are required as the original version of L’Hopital’s Rule.
  1. Verify that numerator and denominator have infinite limits.
  2. Derivative of numerator and denominator SEPARATELY.
  3. Find the new limit.

Solution (a). The numerator and denominator both have a limit of $+\infty$, so we have an indeterminate form of type ∞/∞. Applying L’Hôpital’s rule yields

$$\lim_{x \to +\infty} \frac{x}{e^x} = \lim_{x \to +\infty} \frac{1}{e^x} = 0$$

Repeated applications of L'Hopital's Rule Example

- Sometimes you will get an indeterminate form again after applying L'Hopital's Rule the first time and it will be necessary to do it again.

Solution (b). The numerator has a limit of $-\infty$ and the denominator has a limit of $+\infty$, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x \to 0^+} \frac{\ln x}{\csc x} = \lim_{x \to 0^+} \frac{1/x}{-\csc x \cot x} \qquad (4)$$

This last limit is again an indeterminate form of type ∞/∞. Moreover, any additional applications of L'Hôpital's rule will yield powers of $1/x$ in the numerator and expressions involving $\csc x$ and $\cot x$ in the denominator; thus, repeated application of L'Hôpital's rule simply produces new indeterminate forms. We must try something else. The last limit in (4) can be rewritten as

$$\lim_{x \to 0^+} \left(-\frac{\sin x}{x} \tan x\right) = -\lim_{x \to 0^+} \frac{\sin x}{x} \cdot \lim_{x \to 0^+} \tan x = -(1)(0) = 0$$

Thus,

$$\lim_{x \to 0^+} \frac{\ln x}{\csc x} = 0$$

Other indeterminate forms

- There are several other types of indeterminate forms that you will see on pages 224 & 225.
- Many of you keep forgetting about the other types, so I suggest you look at these two pages and see if you have questions.

One Idea to Consider

WARNING

It is tempting to argue that an indeterminate form of type $0 \cdot \infty$ has value 0 since "zero times anything is zero." However, this is fallacious since $0 \cdot \infty$ is not a product of numbers, but rather a statement about limits. For example, here are two indeterminate forms of type $0 \cdot \infty$ whose limits are *not* zero:

$$\lim_{x \to 0} \left(x \cdot \frac{1}{x} \right) = \lim_{x \to 0} 1 = 1$$

$$\lim_{x \to 0^+} \left(\sqrt{x} \cdot \frac{1}{x} \right) = \lim_{x \to 0^+} \left(\frac{1}{\sqrt{x}} \right) = +\infty$$

Rock Buggy