Chapter Six Overview

Applications of the Definite Integral in Geometry, Science, and Engineering

All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Introduction

- In the last chapter, the definite integral was introduced as the limit of Riemann sums and we used them to find area:

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

- However, Riemann sums and definite integrals have applications that extend far beyond the area problem.
- In this chapter, we will use Riemann sums and definite integrals to find volume and surface area of a solid, length of a plane curve, work done by a force, etc.
- While these problems sound different, the calculations we will use will all follow a nearly identical procedure.

A Review of Riemann Sums

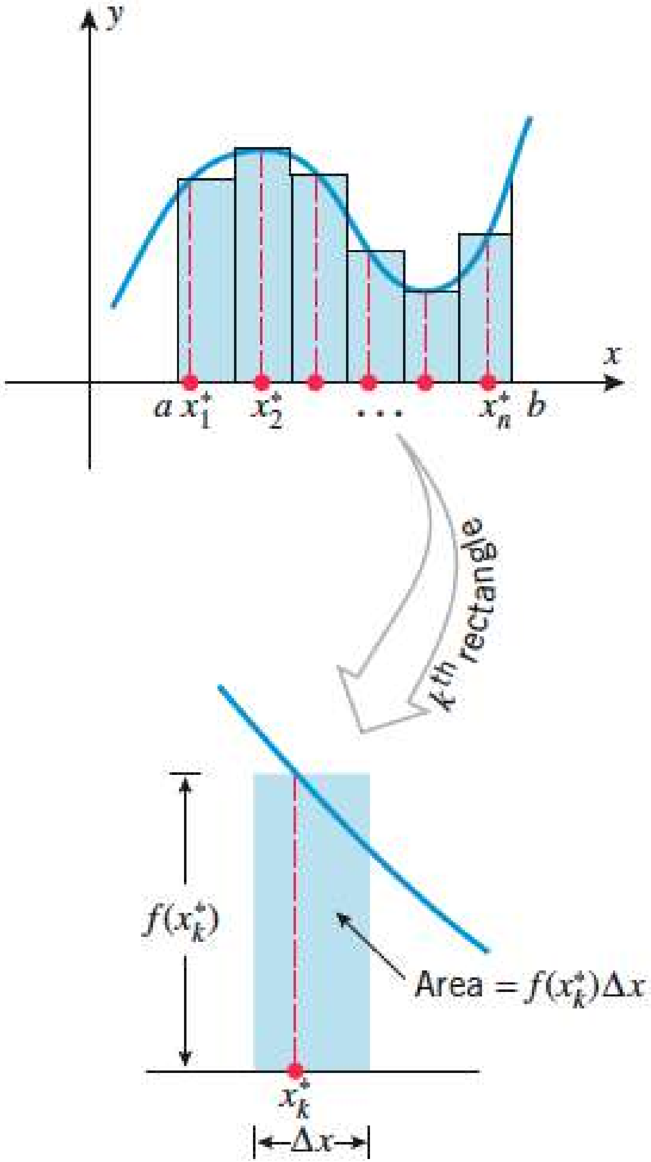
- The distance between a and b is b-a.
- Since we divided that distance into n subintervals, each is :

$$\Delta x = \frac{b-a}{n}$$

- In each subinterval, draw a rectangle whose height is the value of the function f(x) at an arbitrarily selected point in the subinterval (a.k.a. x_k*) which gives f(x_k*).
- Since the area of each rectangle is base * height, we get the formula you see on the right for each rectangle:
- Area=

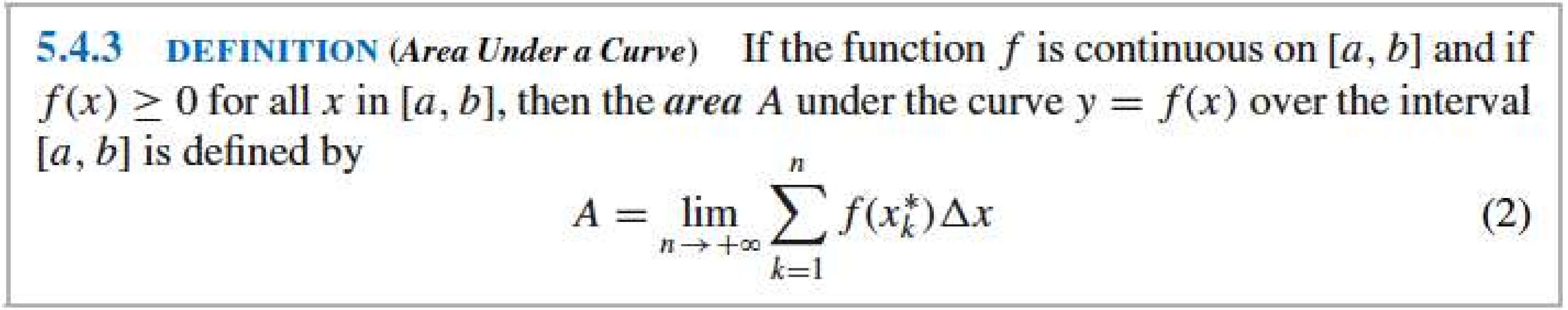
b*h = x * f(x_k*) = f(x_k*) x .

▲ ▲



Review of Riemann Sums - continued

- Remember, that was the area for each rectangle. We need to find the sum of the areas of all of the rectangles between a and b which is why we use sigma notation.
- As we discussed in a previous section, the area estimate is more accurate with the more number of rectangles used. Therefore, we will let n approach infinity.

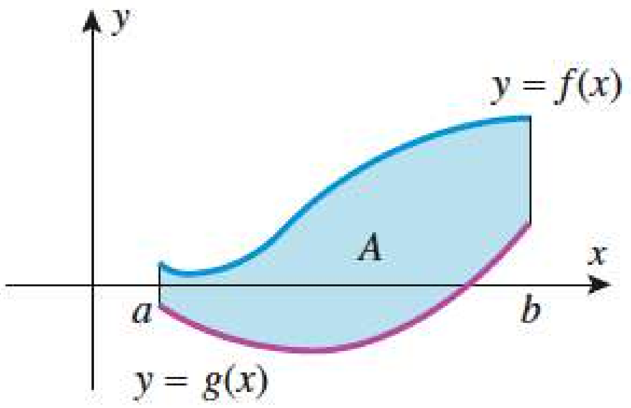


5.4.3 DEFINITION *(Area Under a Curve)* If the function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$, then the ***area*** A under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

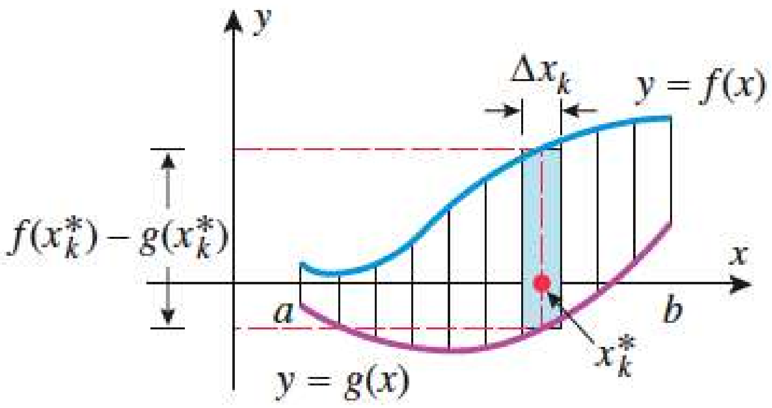
$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \tag{2}$$

Area Between y = f(x) and y = g(x)

- To find the area between two curves, we will divide the interval [a,b] into n subintervals (like we did in section 5.4) which subdivides the area region into n strips (see diagram below).



(a)



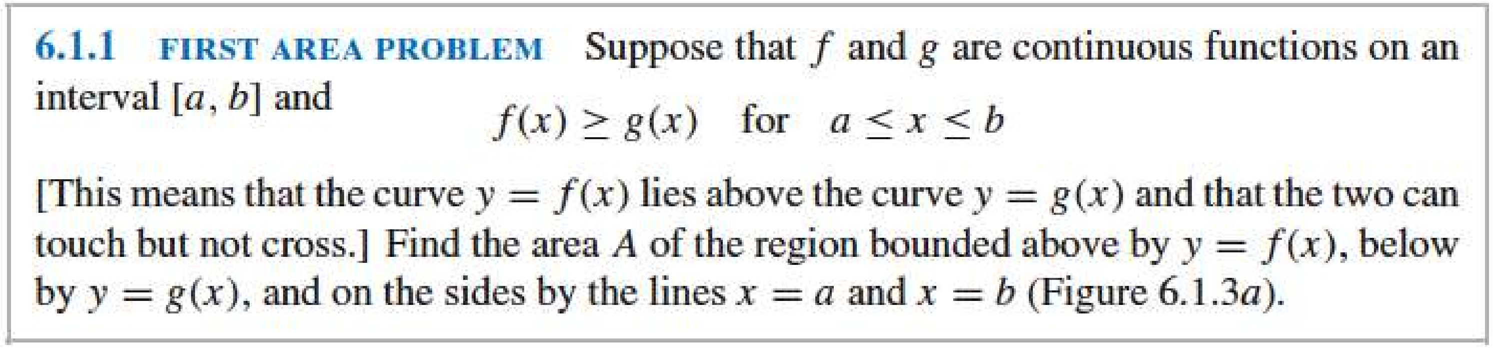
(b)

Area Between y = f(x) and y = g(x) continued

- To find the height of each rectangle, subtract the function output values $f(x_k*) - g(x_k*)$. The base is Δx_k .
- Therefore, the area of each strip is

  base * height = Δx_k * $[f(x_k*) - g(x_k*)]$.
- We do not want the area of one strip, we want the sum of the areas of all of the strips. That is why we need the sigma.
- Also, we want the limit as the number of rectangles "n" increases to approach infinity, in order to get an accurate area.

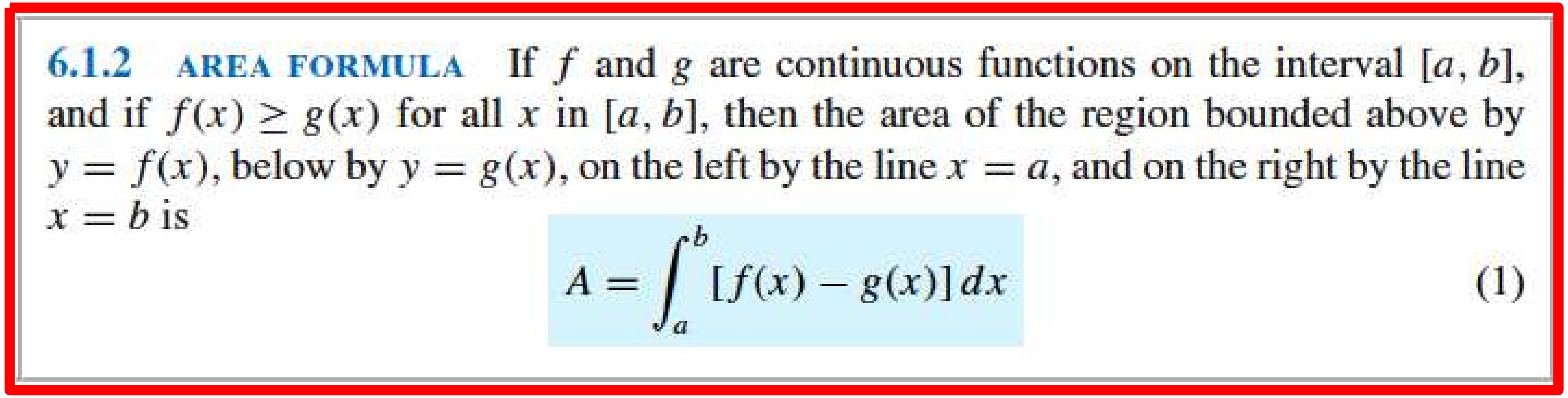
Assuming One Curve is Always Above the Other



6.1.1 FIRST AREA PROBLEM Suppose that f and g are continuous functions on an interval $[a, b]$ and

$$f(x) \geq g(x) \quad \text{for} \quad a \leq x \leq b$$

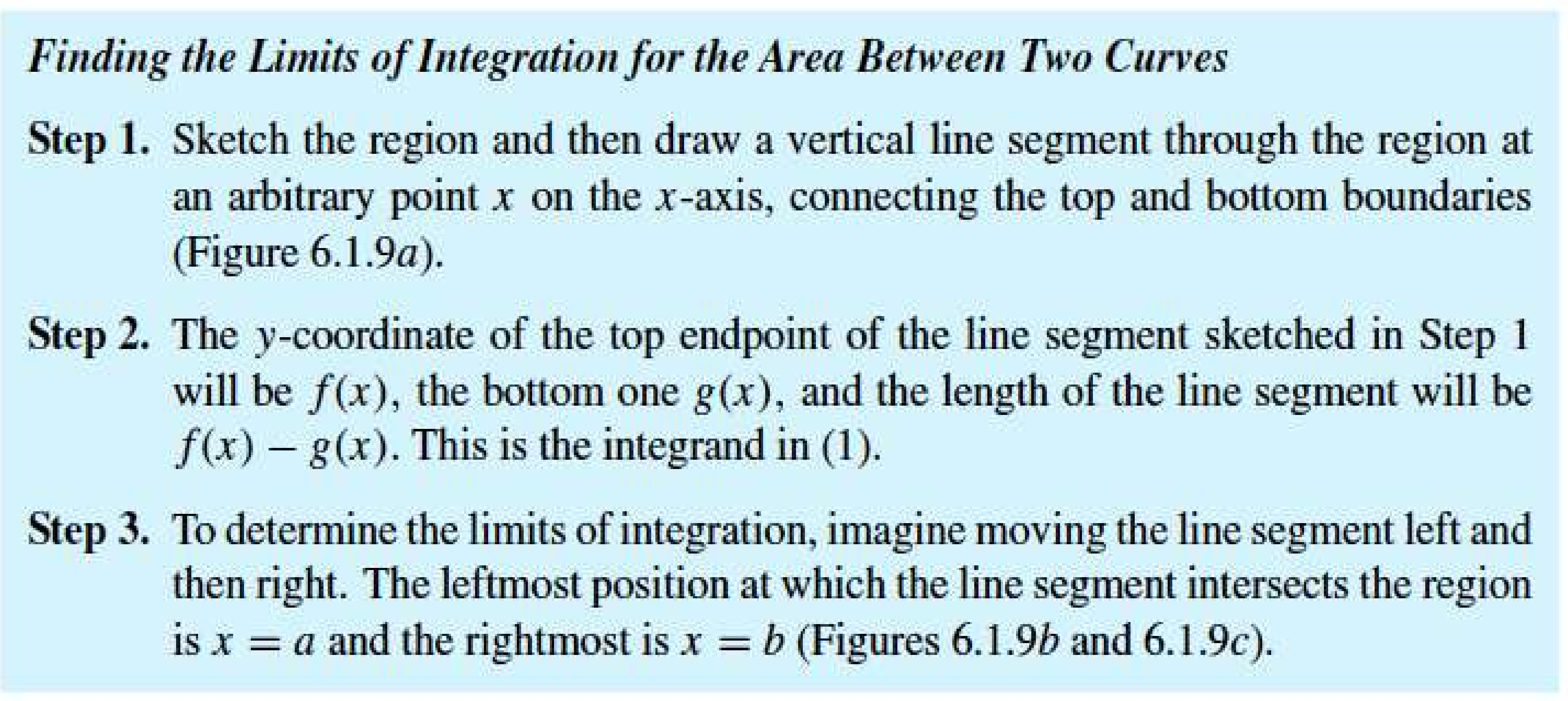
[This means that the curve $y = f(x)$ lies above the curve $y = g(x)$ and that the two can touch but not cross.] Find the area A of the region bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.1.3a).



6.1.2 AREA FORMULA If f and g are continuous functions on the interval $[a, b]$, and if $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,dx \tag{1}$$

Summary of Steps Involved



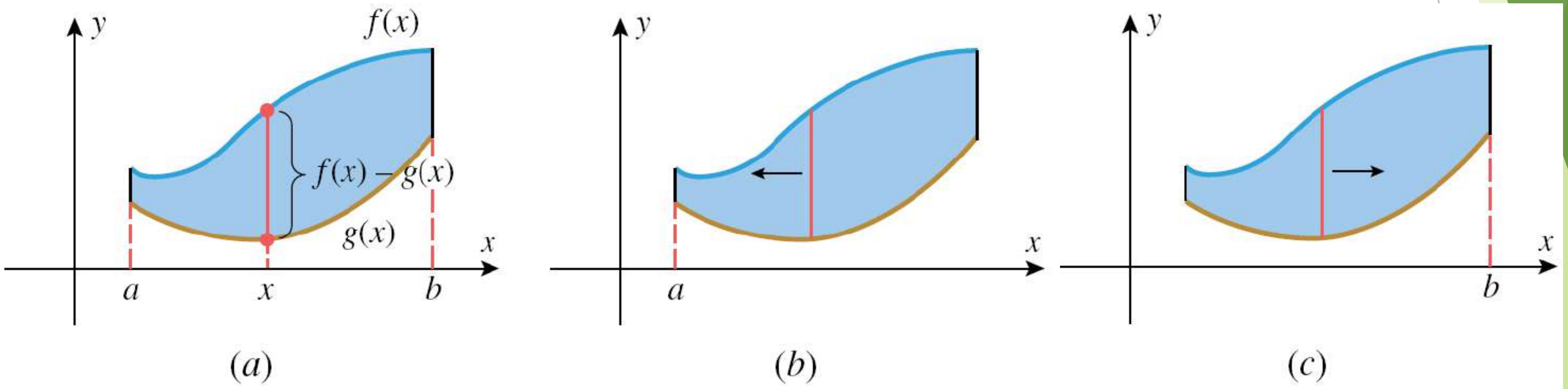
Finding the Limits of Integration for the Area Between Two Curves

Step 1. Sketch the region and then draw a vertical line segment through the region at an arbitrary point x on the x-axis, connecting the top and bottom boundaries (Figure 6.1.9*a*).

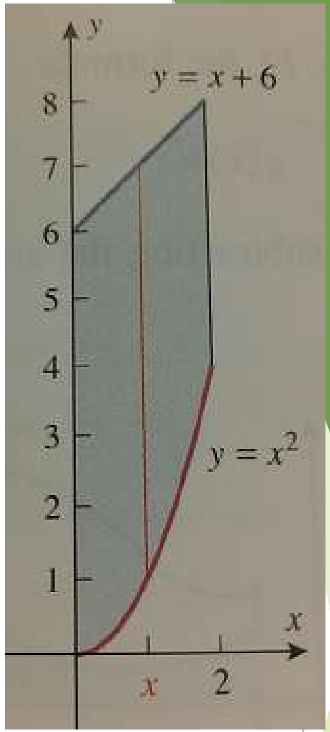
Step 2. The y-coordinate of the top endpoint of the line segment sketched in Step 1 will be $f(x)$, the bottom one $g(x)$, and the length of the line segment will be $f(x) - g(x)$. This is the integrand in (1).

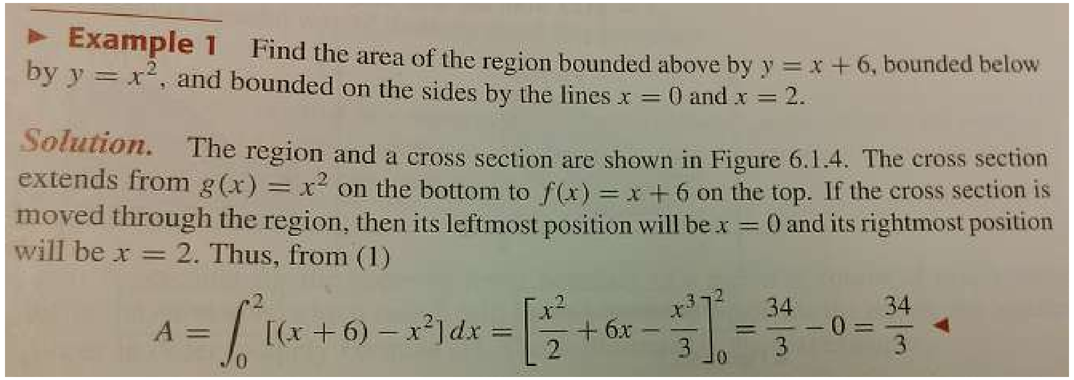
Step 3. To determine the limits of integration, imagine moving the line segment left and then right. The leftmost position at which the line segment intersects the region is $x = a$ and the rightmost is $x = b$ (Figures 6.1.9*b* and 6.1.9*c*).

Picture of Steps Two and Three From Previous Slide:



Straightforward Example



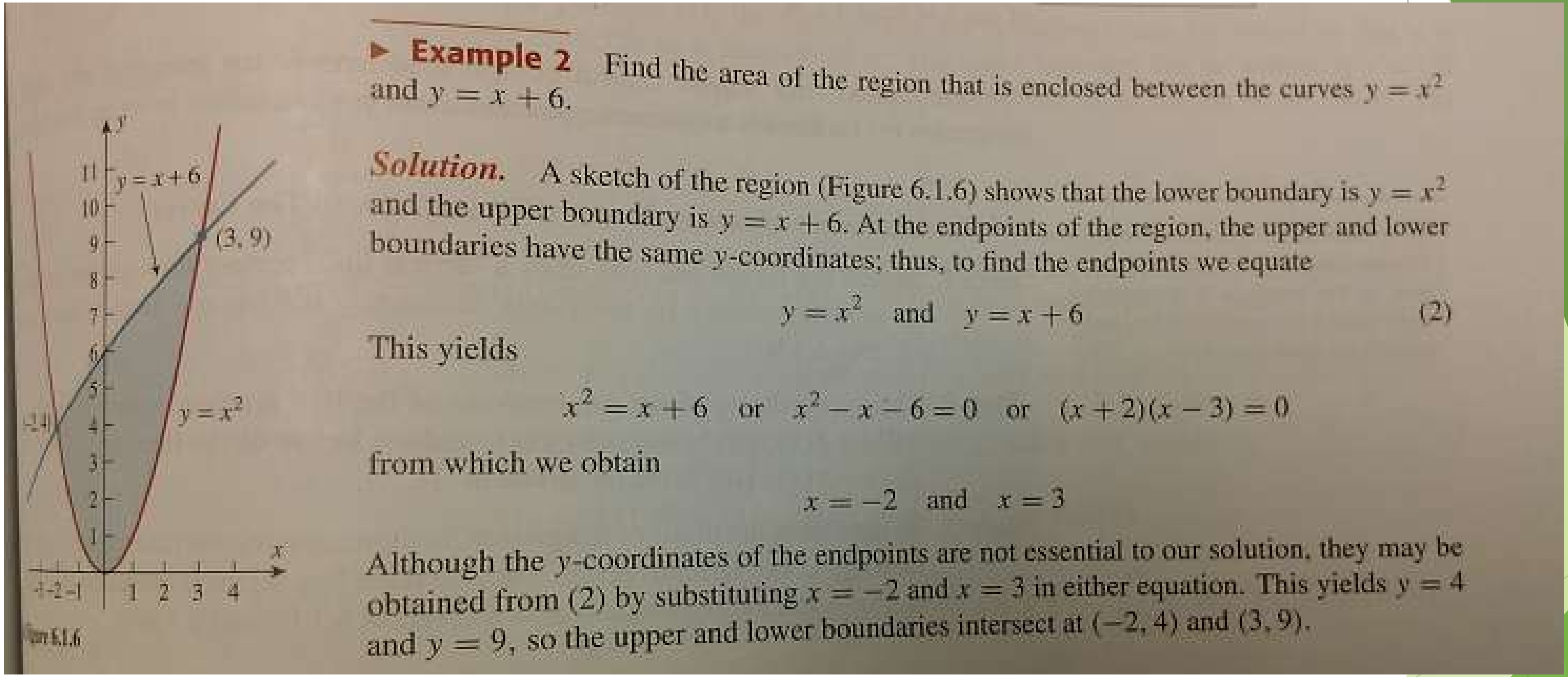


▸ **Example 1** Find the area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$, and bounded on the sides by the lines $x = 0$ and $x = 2$.

Solution. The region and a cross section are shown in Figure 6.1.4. The cross section extends from $g(x) = x^2$ on the bottom to $f(x) = x + 6$ on the top. If the cross section is moved through the region, then its leftmost position will be $x = 0$ and its rightmost position will be $x = 2$. Thus, from (1)

$$A = \int_0^2 [(x+6) - x^2]\,dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^2 = \frac{34}{3} - 0 = \frac{34}{3}$$ ◂

Sometimes, you will have to find the limits of integration by solving for the points of intersection first:



Example 2 Find the area of the region that is enclosed between the curves $y = x^2$ and $y = x + 6$.

Solution. A sketch of the region (Figure 6.1.6) shows that the lower boundary is $y = x^2$ and the upper boundary is $y = x + 6$. At the endpoints of the region, the upper and lower boundaries have the same y-coordinates; thus, to find the endpoints we equate

$$y = x^2 \quad \text{and} \quad y = x + 6 \tag{2}$$

This yields

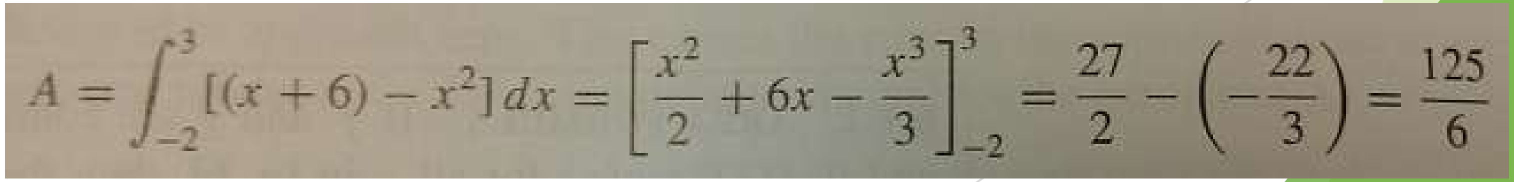
$$x^2 = x + 6 \quad \text{or} \quad x^2 - x - 6 = 0 \quad \text{or} \quad (x + 2)(x - 3) = 0$$

from which we obtain

$$x = -2 \quad \text{and} \quad x = 3$$

Although the y-coordinates of the endpoints are not essential to our solution, they may be obtained from (2) by substituting $x = -2$ and $x = 3$ in either equation. This yields $y = 4$ and $y = 9$, so the upper and lower boundaries intersect at $(-2, 4)$ and $(3, 9)$.

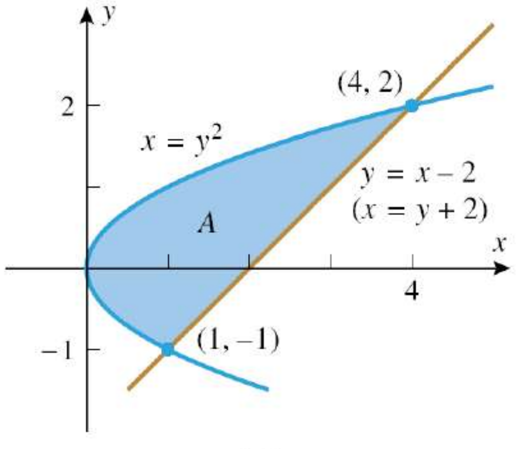
Figure 6.1.6



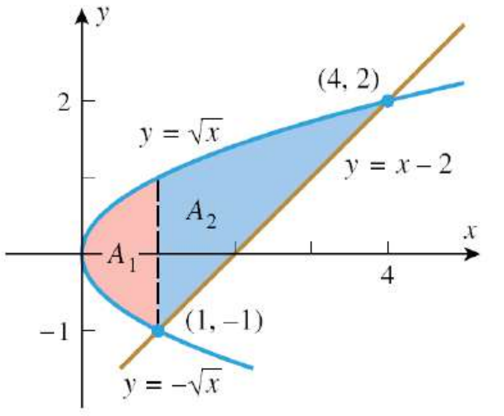
$$A = \int_{-2}^{3} [(x + 6) - x^2]\, dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3} = \frac{27}{2} - \left(-\frac{22}{3}\right) = \frac{125}{6}$$

Inconsistent Boundaries

- If you look at the area in figure (a), the upper and lower boundaries are not the same for the left portion of the graph as they are for the right portion.
- On the left, the $x = y^2$ curve is the upper and lower boundary.
- On the right, the $x = y^2$ curve is the upper boundary, but the line $y = x – 2$ is the lower boundary.
- Therefore, in order to calculate the area using x as our variable, we must divide the region into two pieces, find the area of each, then add those areas to find the total area (see figure (b)).
- See work on page 417 if interested.



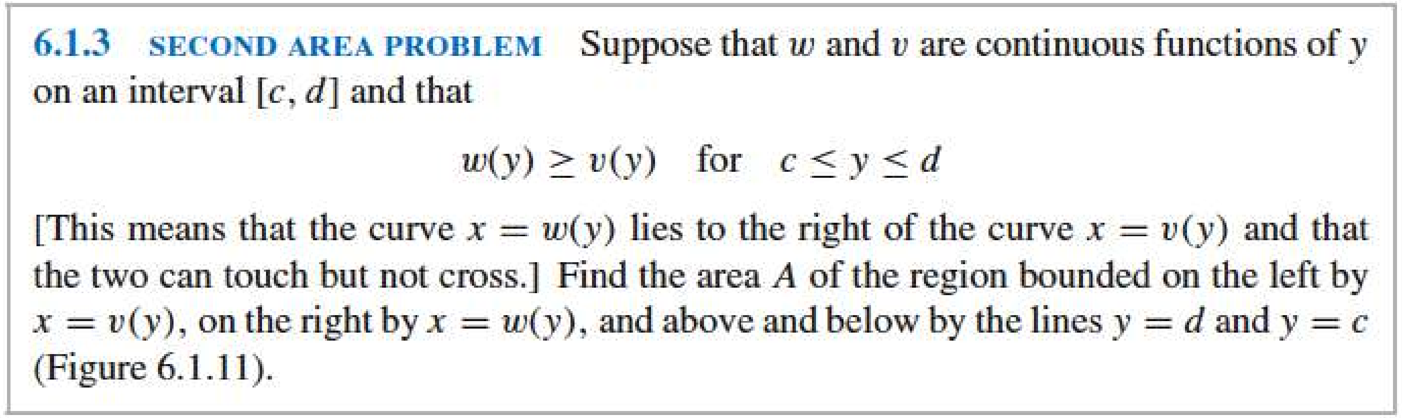
(a)



(b)

Reversing the Roles of x and y

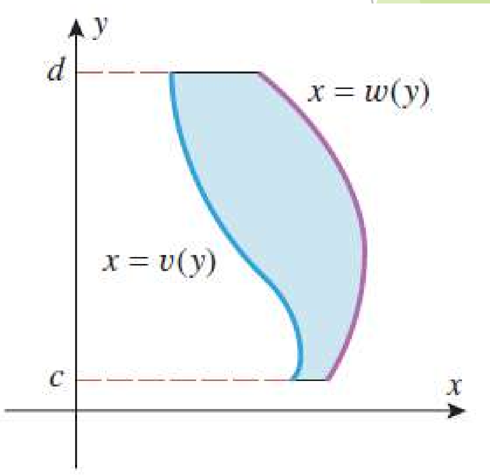
- Instead, we could reverse the roles of x and y to make it easier to find the area.
- Solve for x in terms of y, find the lower and upper limits of integration in terms of y, and integrate with respect to y.



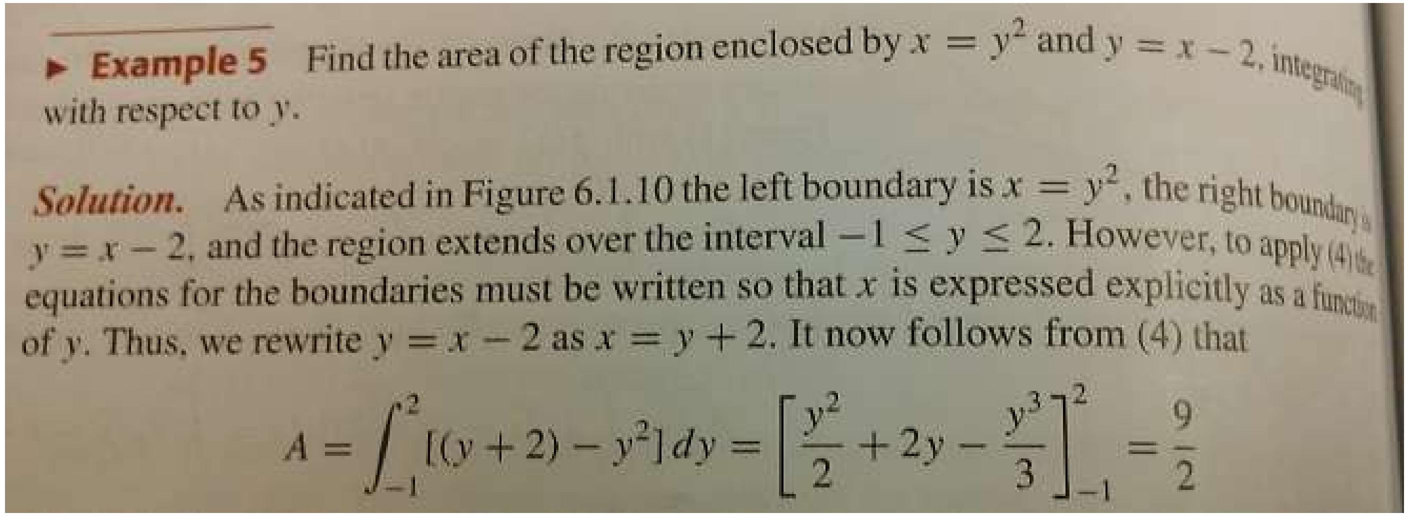
6.1.3 SECOND AREA PROBLEM Suppose that w and v are continuous functions of y on an interval $[c, d]$ and that

$$w(y) \geq v(y) \quad \text{for} \quad c \leq y \leq d$$

[This means that the curve $x = w(y)$ lies to the right of the curve $x = v(y)$ and that the two can touch but not cross.] Find the area A of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, and above and below by the lines $y = d$ and $y = c$ (Figure 6.1.11).



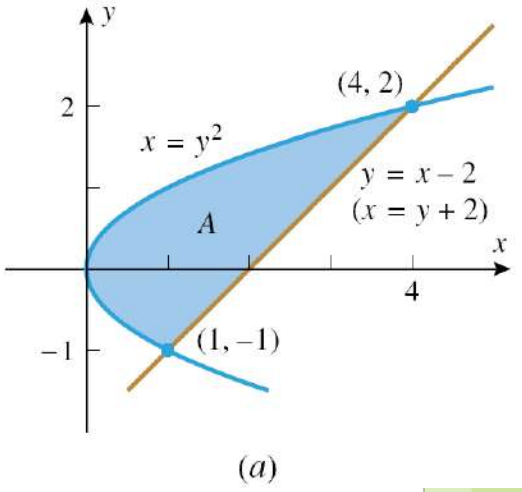
Reverse x and y to find the area on slide #13 instead of breaking into two sections.



Example 5 Find the area of the region enclosed by $x = y^2$ and $y = x - 2$, integrating with respect to y.

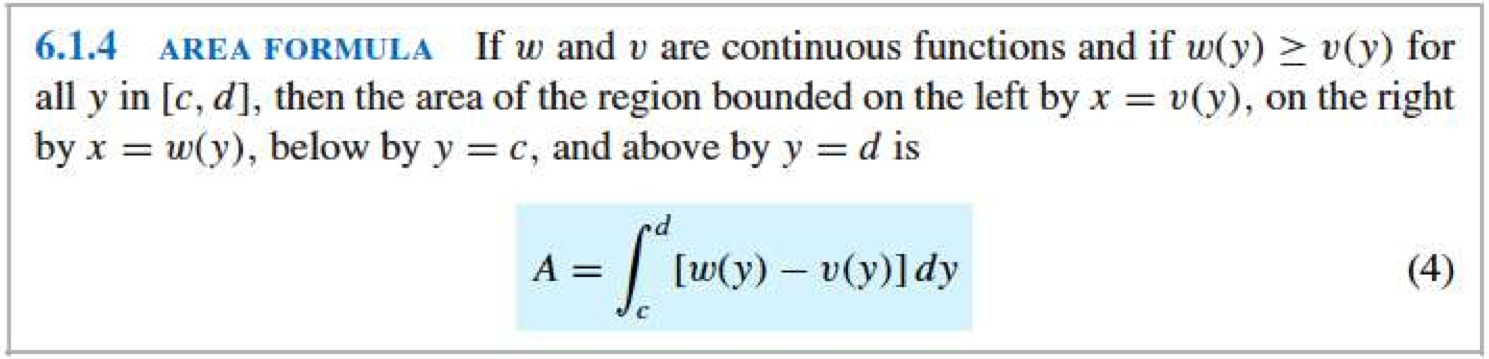
Solution. As indicated in Figure 6.1.10 the left boundary is $x = y^2$, the right boundary is $y = x - 2$, and the region extends over the interval $-1 \le y \le 2$. However, to apply (4) the equations for the boundaries must be written so that x is expressed explicitly as a function of y. Thus, we rewrite $y = x - 2$ as $x = y + 2$. It now follows from (4) that

$$A = \int_{-1}^{2} [(y+2) - y^2]\, dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$



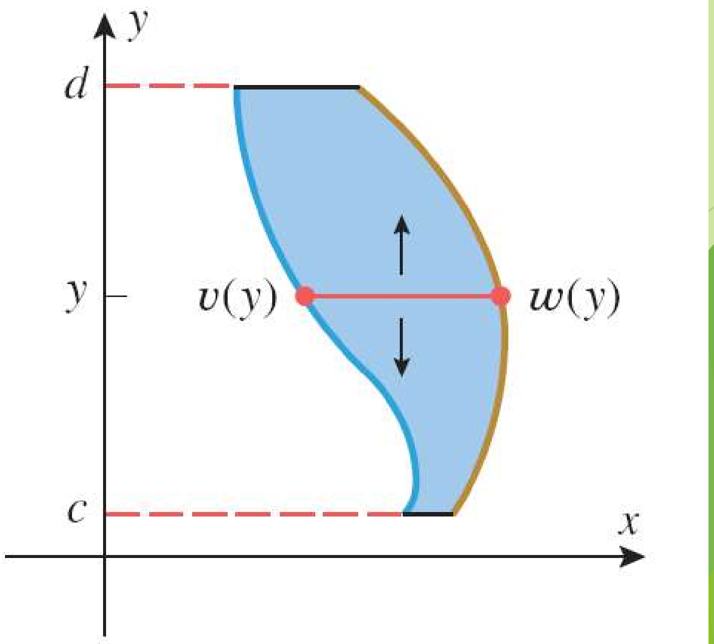
- You get exactly the same answer whether you break the area into two sections or if you reverse x and y.
- This is a much easier and quicker calculation that we had to perform when we reversed x and y.
- We avoided having to do two separate integrals and add our results.

Formula and Picture for Reversing the Roles of x and y

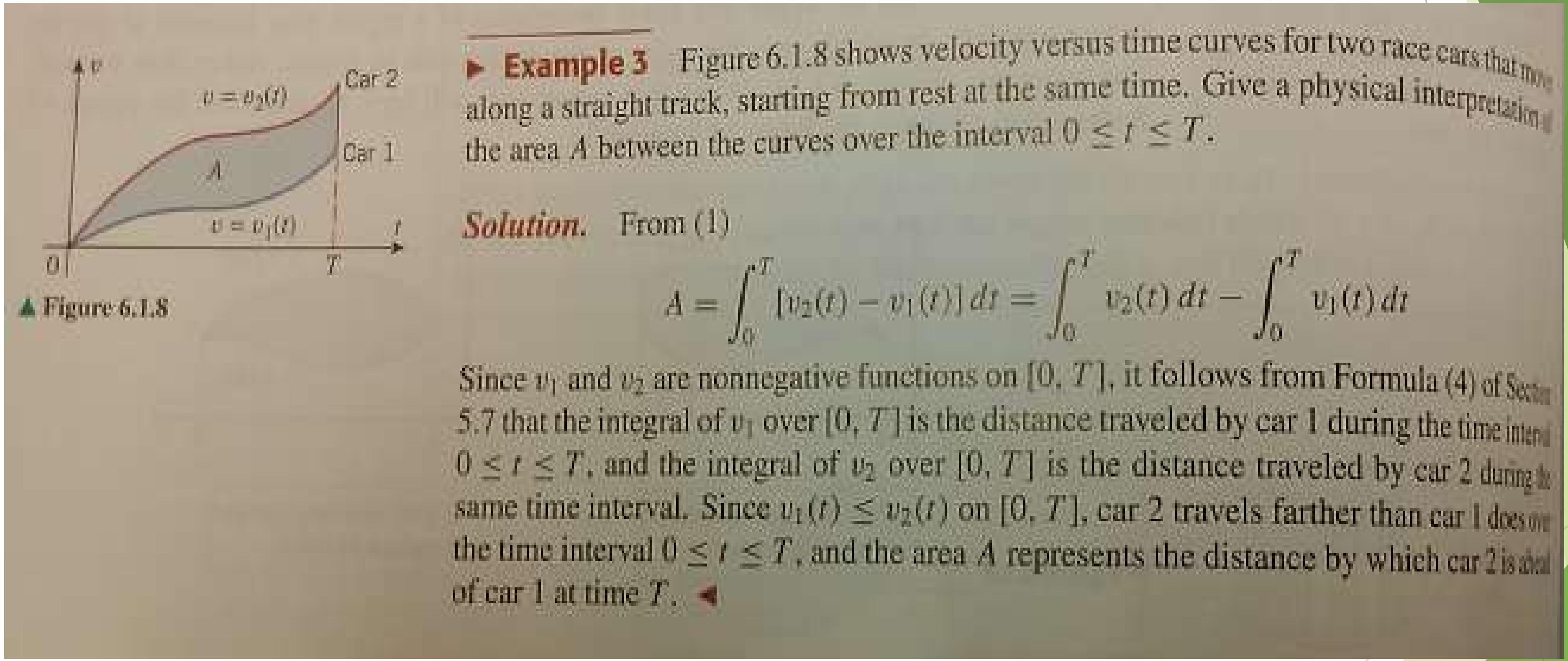


6.1.4 AREA FORMULA If w and v are continuous functions and if $w(y) \geq v(y)$ for all y in $[c, d]$, then the area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below by $y = c$, and above by $y = d$ is

$$A = \int_c^d [w(y) - v(y)]\, dy \tag{4}$$



Application of Area Between Two Curves



▲ **Figure 6.1.8**

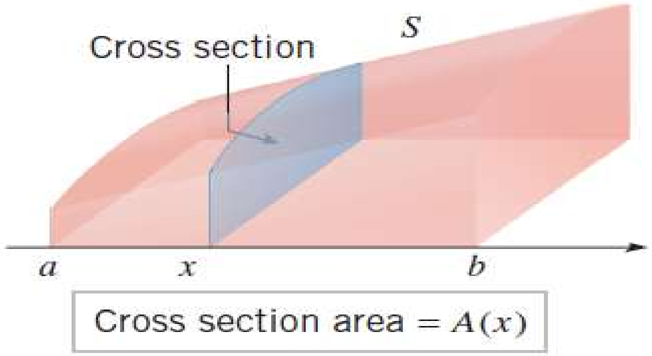
▶ **Example 3** Figure 6.1.8 shows velocity versus time curves for two race cars that mo along a straight track, starting from rest at the same time. Give a physical interpretation of the area A between the curves over the interval $0 \le t \le T$.

Solution. From (1)

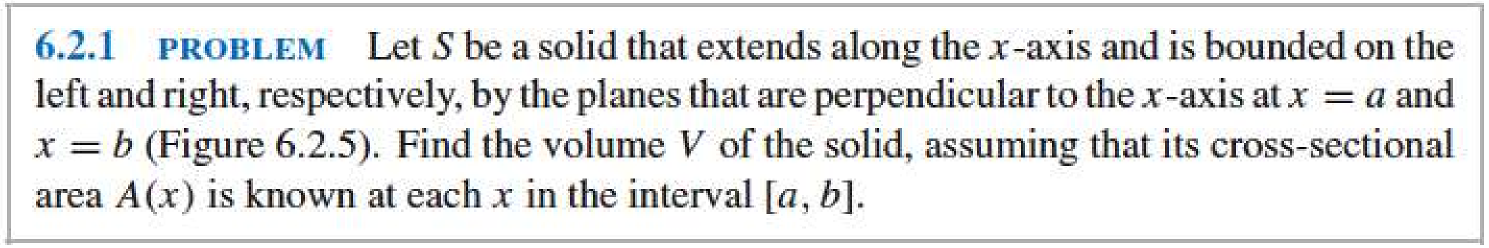
$$A = \int_0^T [v_2(t) - v_1(t)]\,dt = \int_0^T v_2(t)\,dt - \int_0^T v_1(t)\,dt$$

Since v_1 and v_2 are nonnegative functions on $[0, T]$, it follows from Formula (4) of Sectio 5.7 that the integral of v_1 over $[0, T]$ is the distance traveled by car 1 during the time interv $0 \le t \le T$, and the integral of v_2 over $[0, T]$ is the distance traveled by car 2 during th same time interval. Since $v_1(t) \le v_2(t)$ on $[0, T]$, car 2 travels farther than car 1 does ov the time interval $0 \le t \le T$, and the area A represents the distance by which car 2 is ahea of car 1 at time T. ◀

General Idea/Definition of Volume



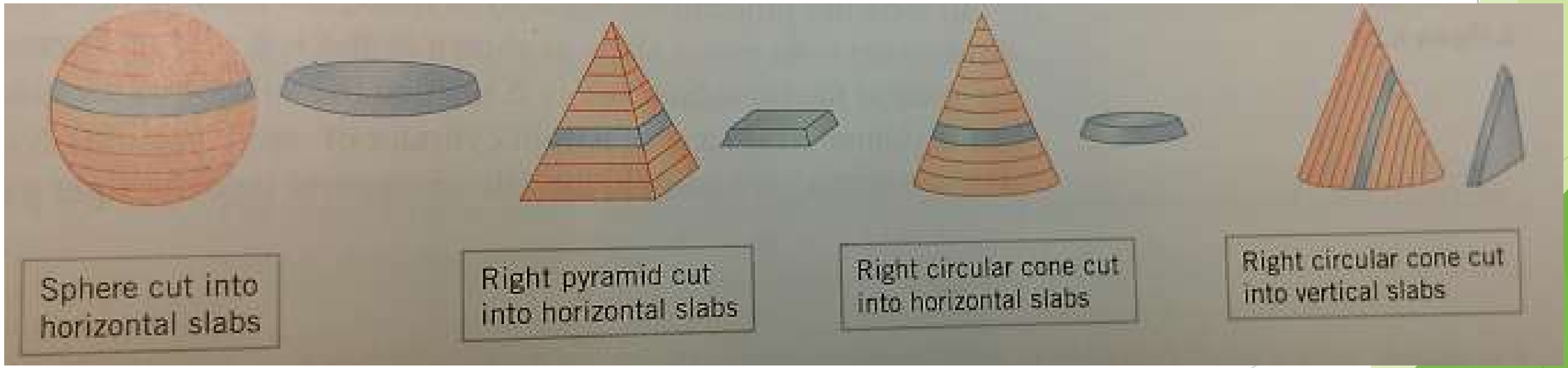
- In order to find volume we are first going to slice a three dimensional space (such as the one seen at the right) into infinitesimally narrow slices of area.
- Then, find the area of each slice separately.
- Next, find the sum of all of the areas (which will bring in the Sigma notation).
- Calculate the limit as the number of slices approaches infinity to get an accurate measure of volume.



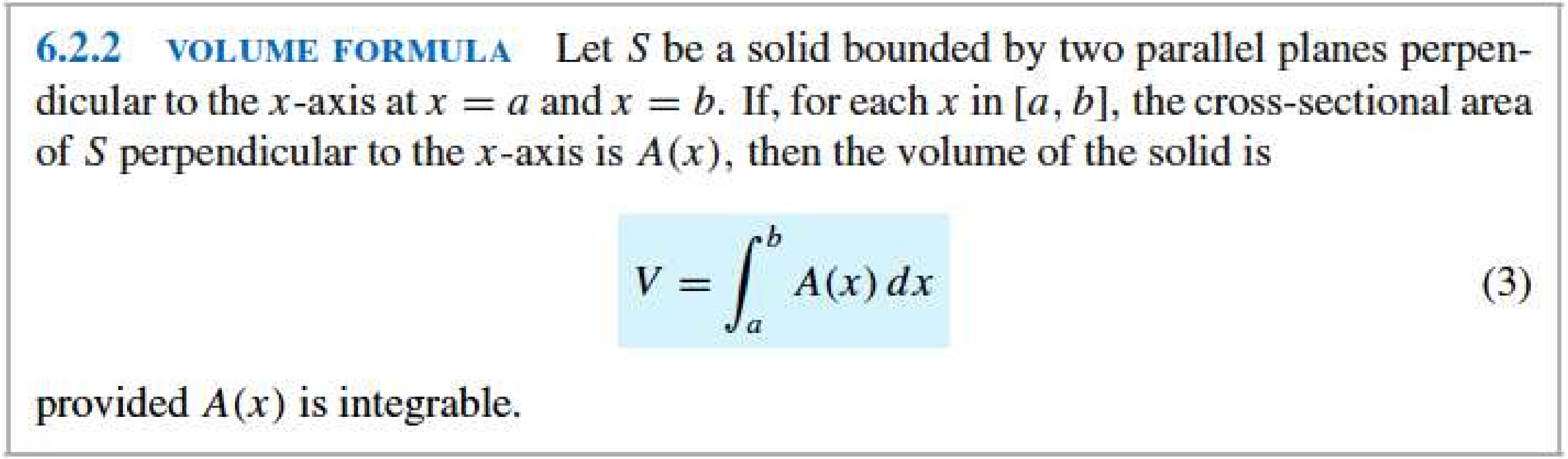
6.2.1 PROBLEM Let S be a solid that extends along the x-axis and is bounded on the left and right, respectively, by the planes that are perpendicular to the x-axis at $x = a$ and $x = b$ (Figure 6.2.5). Find the volume V of the solid, assuming that its cross-sectional area $A(x)$ is known at each x in the interval $[a, b]$.

Which Area Formula?

- The formula depends upon the shape of the cross-section. It could use a circle, square, triangle, etc.



The volume of a solid can be obtained by integrating the cross-sectional area from one end of the solid to the other.



6.2.2 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. If, for each x in $[a, b]$, the cross-sectional area of S perpendicular to the x-axis is $A(x)$, then the volume of the solid is

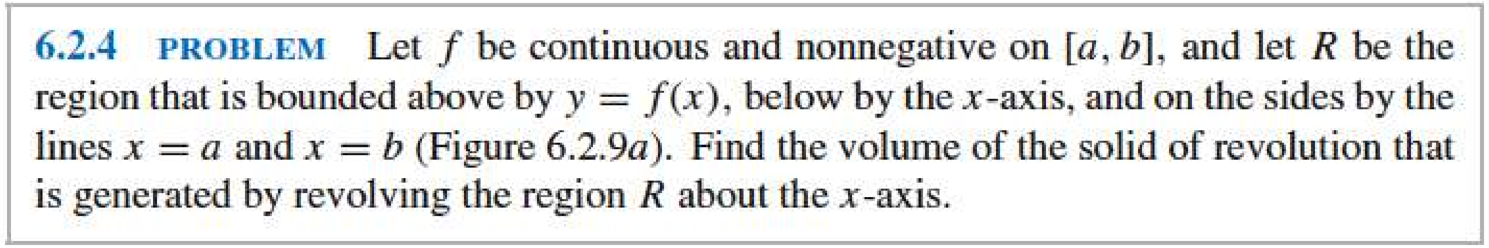
$$V = \int_a^b A(x)\,dx \tag{3}$$

provided $A(x)$ is integrable.

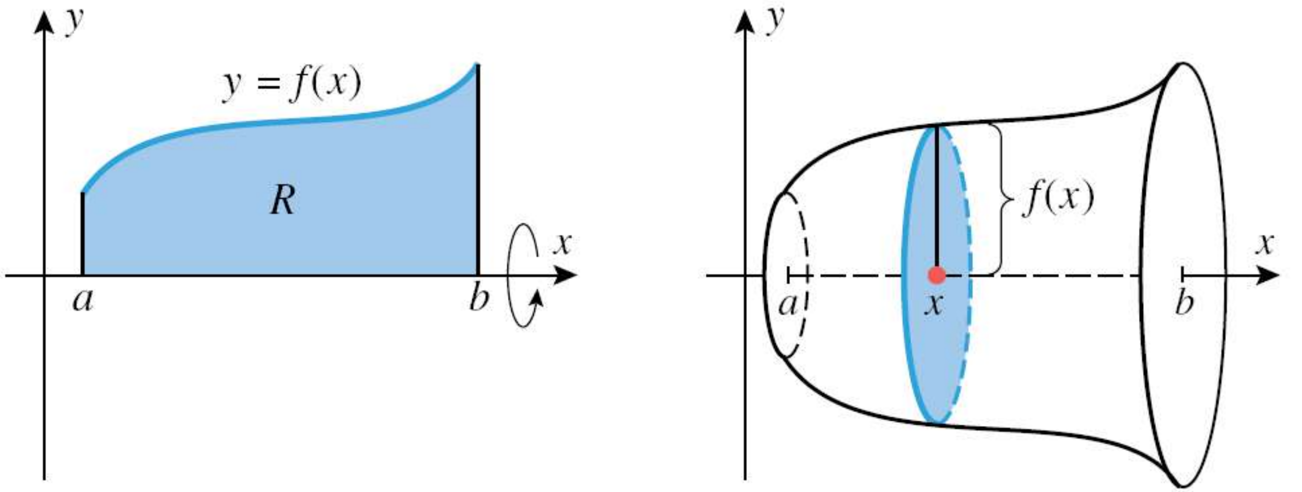
Rotating Area using Discs to Find Volume

- One of the simplest examples of a solid with congruent cross sections is
- We are going to use all of the same ideas from Section 6.1 and more, so I am going to remind you of the process.

- The difference is that after we find the area of each strip, we are going to rotate it around the x-axis, y-axis, or another line in order to find the resulting volume.



6.2.4 PROBLEM Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.9a). Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis.



(a) (b)

$$V = \int_a^b \pi[f(x)]^2\,dx$$

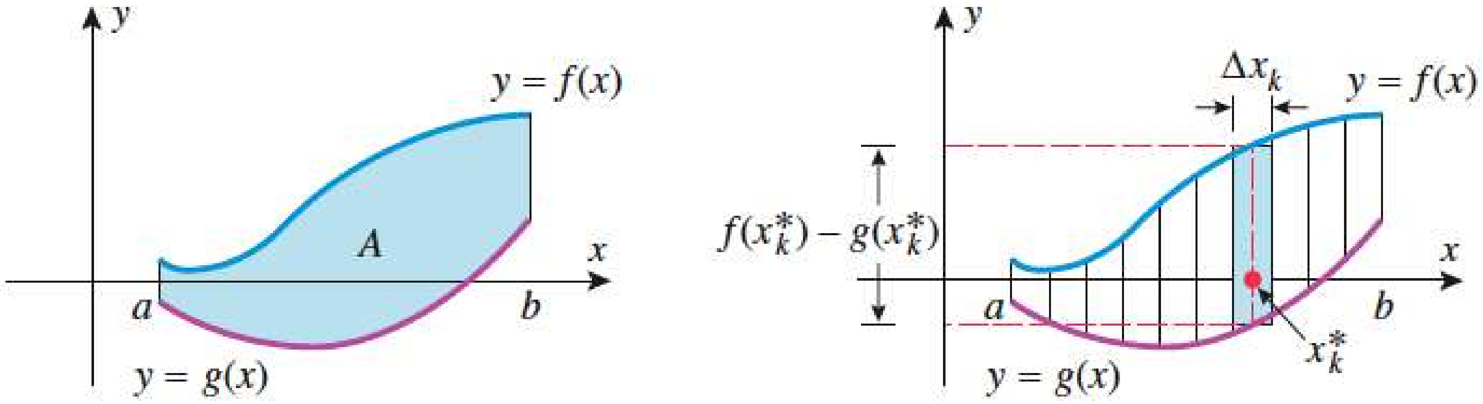
Rotating Area using Washers to Find Volume

- We are going to use all of the same ideas from Section 6.1 and more, so I am going to remind you of the process.

- The difference is that after we find the area of each strip, we are going to rotate it around the x-axis, y-axis, or another line in order to find the resulting volume.

Remember: Area Between y = f(x) and y = g(x)

- To find the area between two curves, we will divide the interval [a,b] into n subintervals (like we did in section 5.4) which subdivides the area region into n strips (see diagram below).

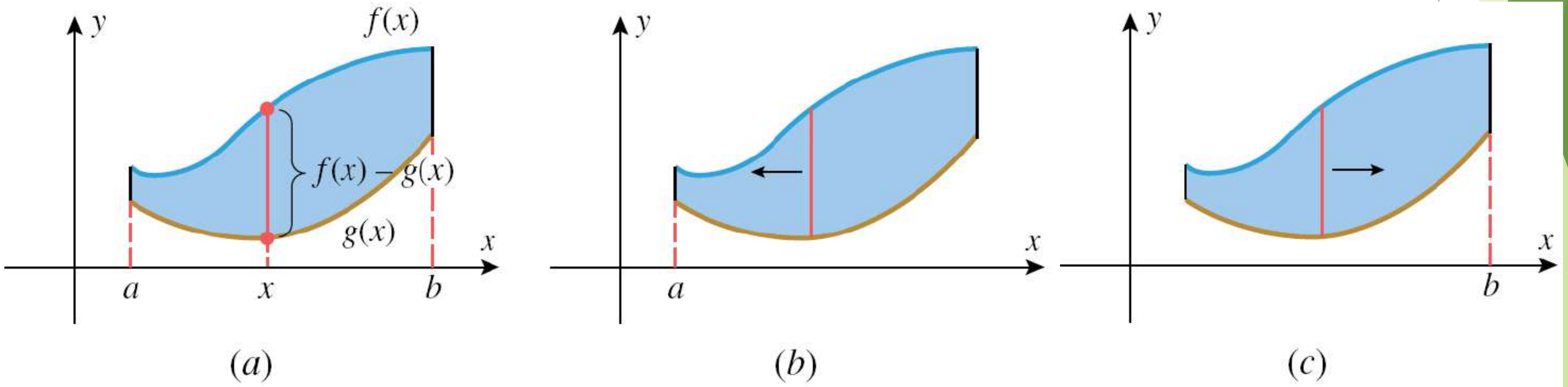


(a) (b)

Area Between y = f(x) and y = g(x) continued

- To find the height of each rectangle, subtract the function output values $f(x_k{}^*) - g(x_k{}^*)$. The base is Δx_k .
- Therefore, the area of each strip is
  base * height = Δx_k * $[f(x_k{}^*) - g(x_k{}^*)]$.
- We do not want the area of one strip, we want the sum of the areas of all of the strips. That is why we need the sigma.
- Also, we want the limit as the number of rectangles “n” increases to approach infinity, in order to get an accurate area.

Picture of Steps Two and Three From Previous Slide:

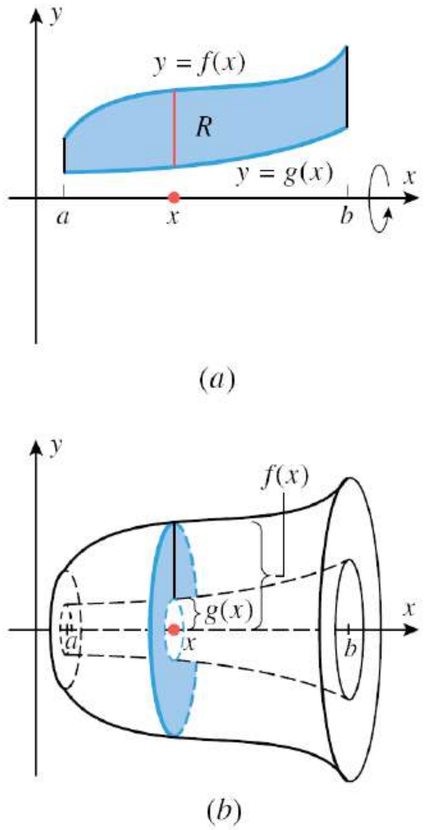


Volume

- To find the height of each rectangular strip, subtract the function output values $f(x_k^*) - g(x_k^*)$. The base is Δx_k
- Therefore, the area of each strip is
- base * height = Δx_k * $[f(x_k^*) - g(x_k^*)]$.
- We do not want the area of one strip, we want the sum of the areas of all of the strips. That is why we need the sigma.
- Also, we want the limit as the number of rectangles “n” increases to approach infinity, in order to get an accurate area.

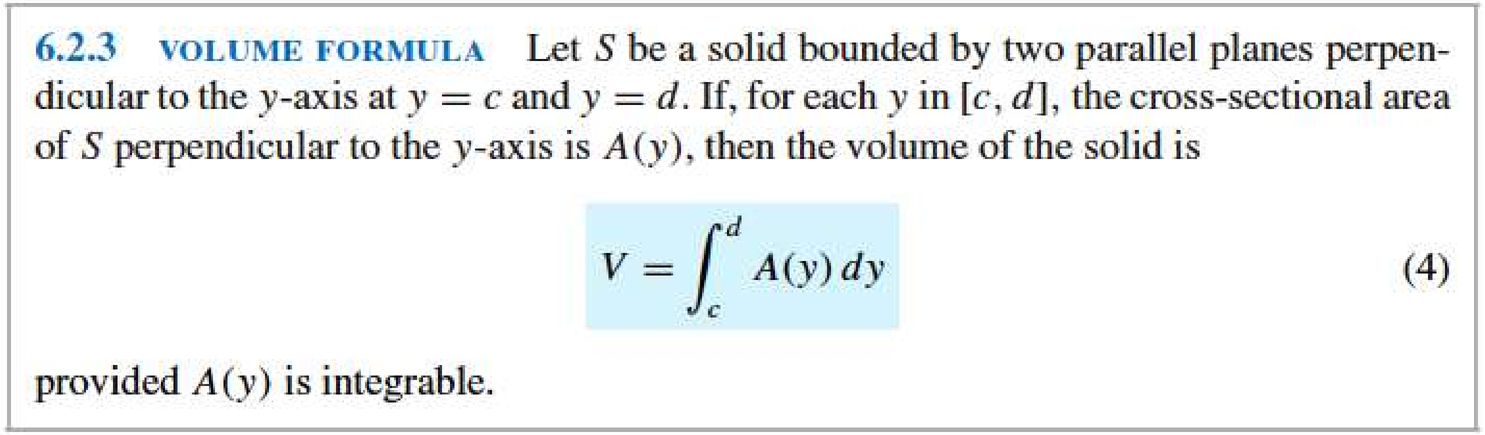
6.2.5 PROBLEM Let f and g be continuous and nonnegative on $[a, b]$, and suppose that $f(x) \geq g(x)$ for all x in the interval $[a, b]$. Let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.12a). Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis (Figure 6.2.12b).

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx$$



(a)

(b)



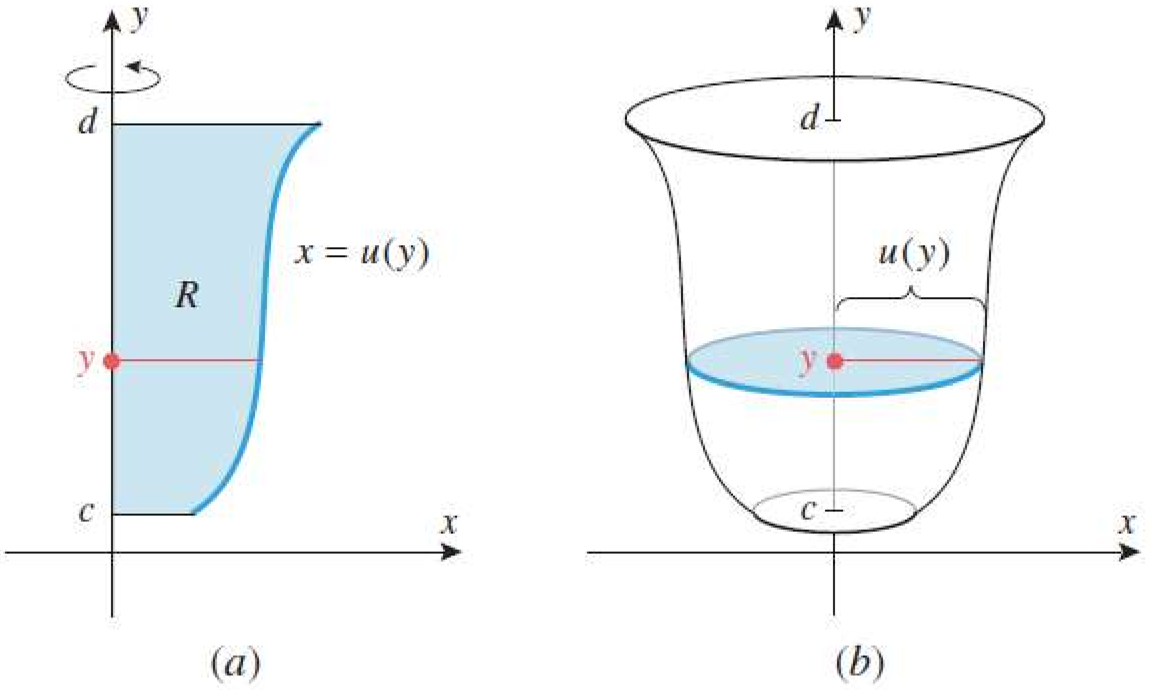
6.2.3 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the y-axis at $y = c$ and $y = d$. If, for each y in $[c, d]$, the cross-sectional area of S perpendicular to the y-axis is $A(y)$, then the volume of the solid is

$$V = \int_c^d A(y)\,dy \tag{4}$$

provided $A(y)$ is integrable.

$$V = \int_c^d \pi [u(y)]^2 \, dy$$

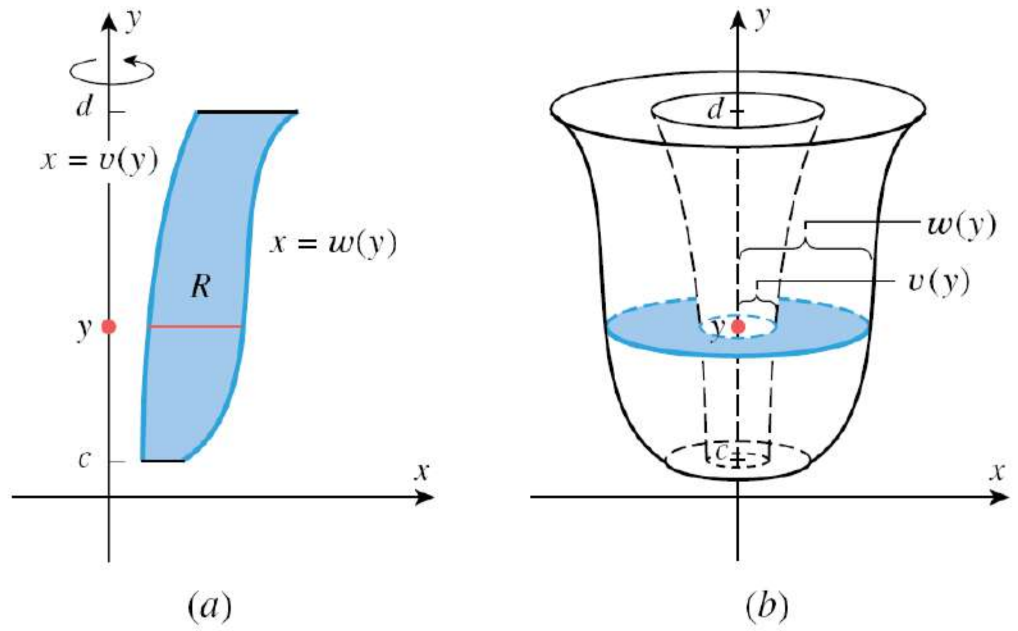
Disks



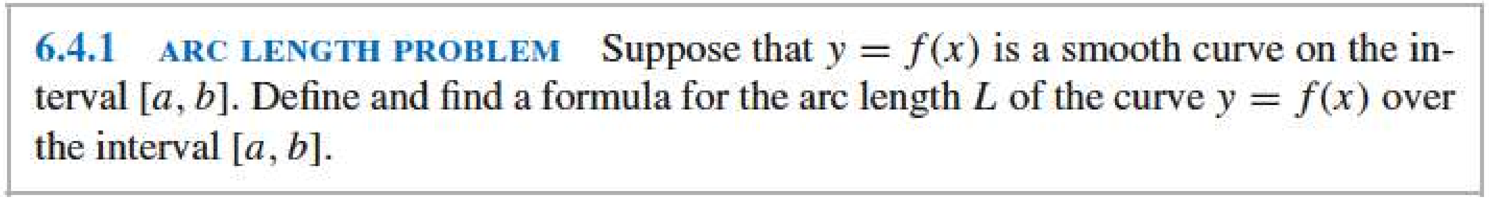
Disks

$$V = \int_{c}^{d} \pi([w(y)]^2 - [v(y)]^2)\, dy$$

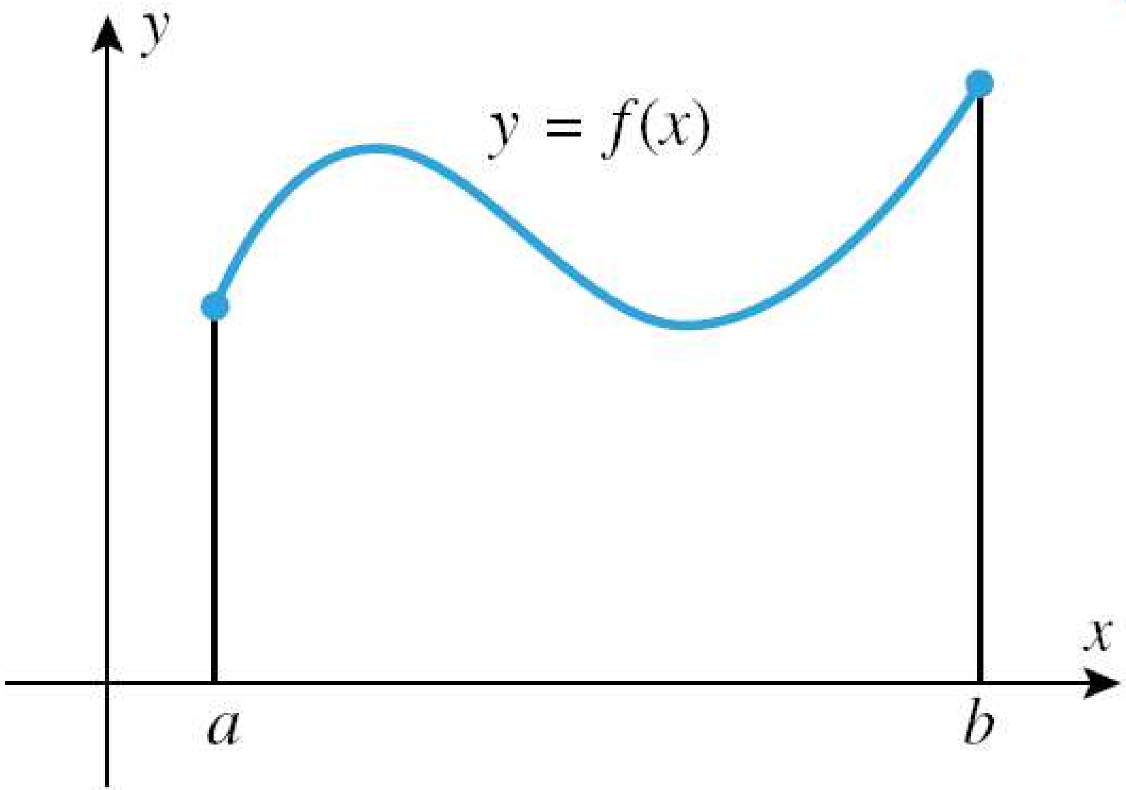
Washers

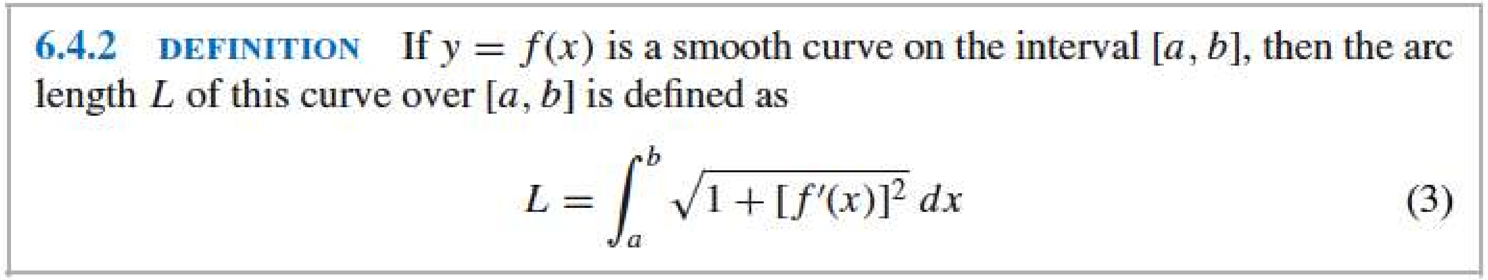


Washers



6.4.1 ARC LENGTH PROBLEM Suppose that $y = f(x)$ is a smooth curve on the interval $[a, b]$. Define and find a formula for the arc length L of the curve $y = f(x)$ over the interval $[a, b]$.



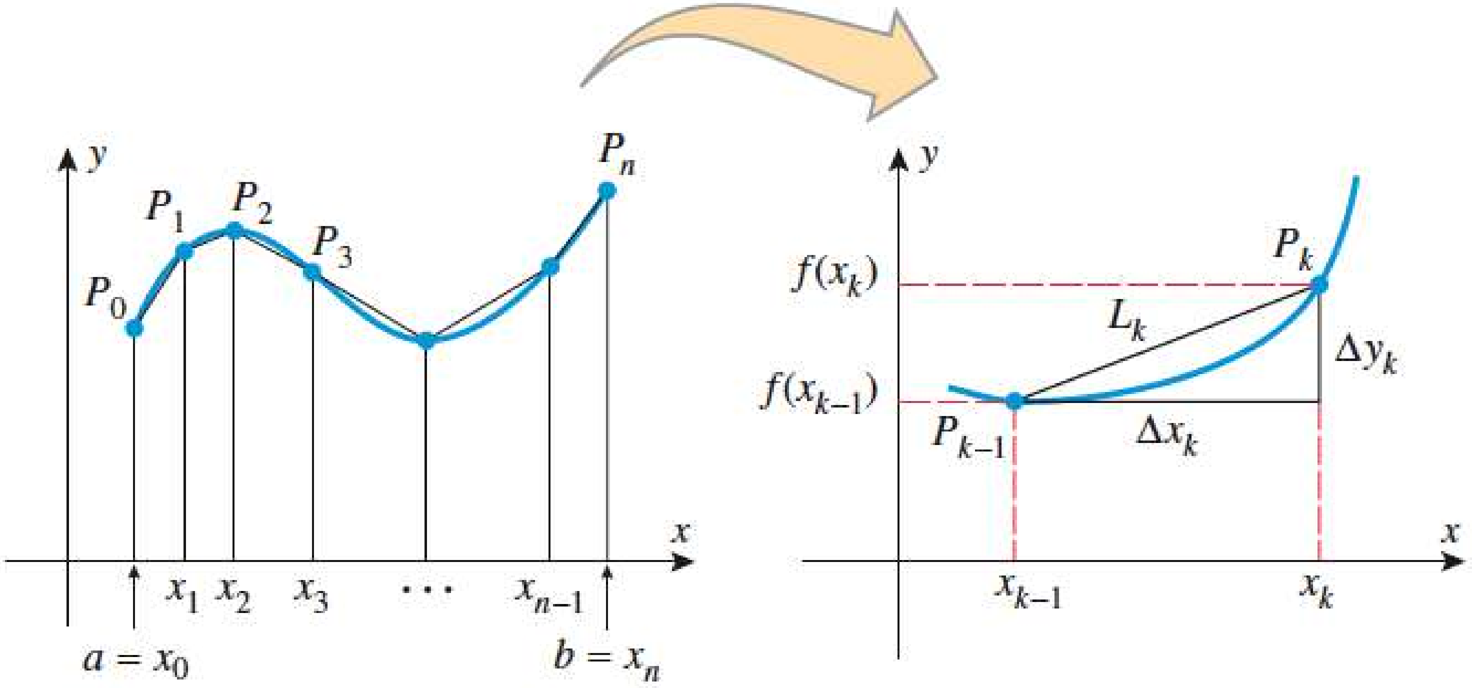


6.4.2 DEFINITION If $y = f(x)$ is a smooth curve on the interval $[a, b]$, then the arc length L of this curve over $[a, b]$ is defined as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \tag{3}$$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

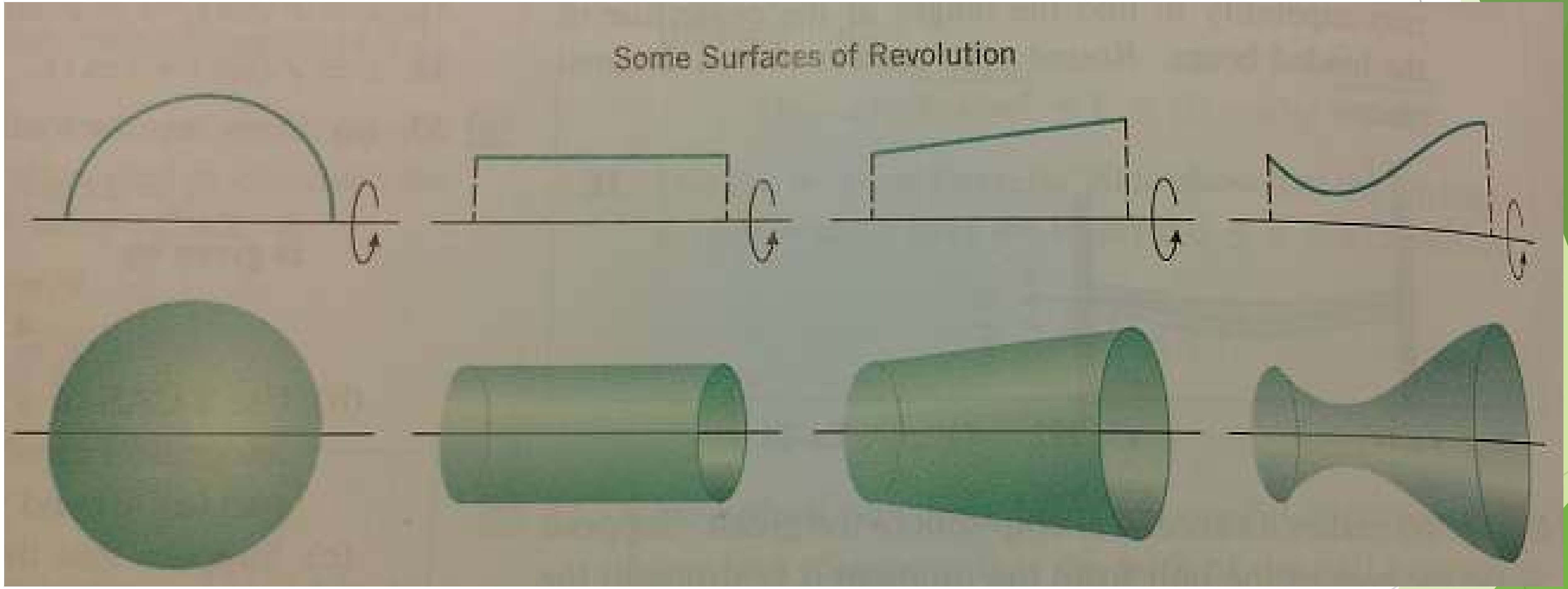

y
P_n
P_1
P_2
P_3
P_0
x
x_1
x_2
x_3
$\cdots$
x_{n-1}
$a = x_0$
$b = x_n$
y
P_k
$f(x_k)$
L_k
Δy_k
$f(x_{k-1})$
P_{k-1}
Δx_k
x
x_{k-1}
x_k

(a) (b)

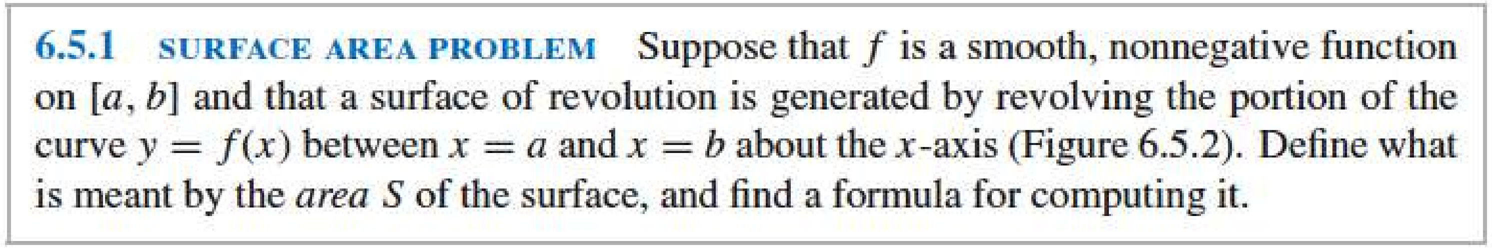
Introduction

- In this section we will find the area of a surface that is generated by revolving a plane curve about a line.
- It is very similar to section 6.4, therefore, the equation is very similar to that of arc length.
- The difference is that we need to revolve it around a line like we did in sections 6.2 and 6.3.
- Since we are only rotating the outside curve (not the area between it and the axis or line), each small section will be approximated by the circumference of the circle infinitesimally narrow width.

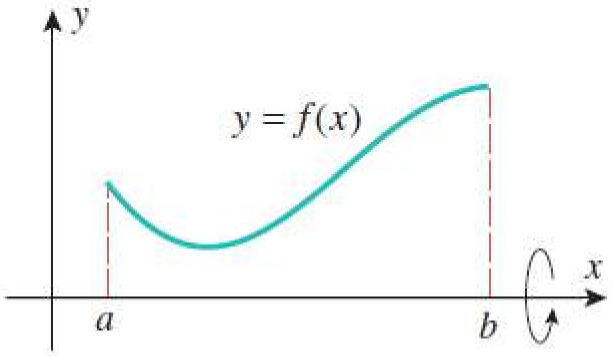
Examples Rotated about the x-axis

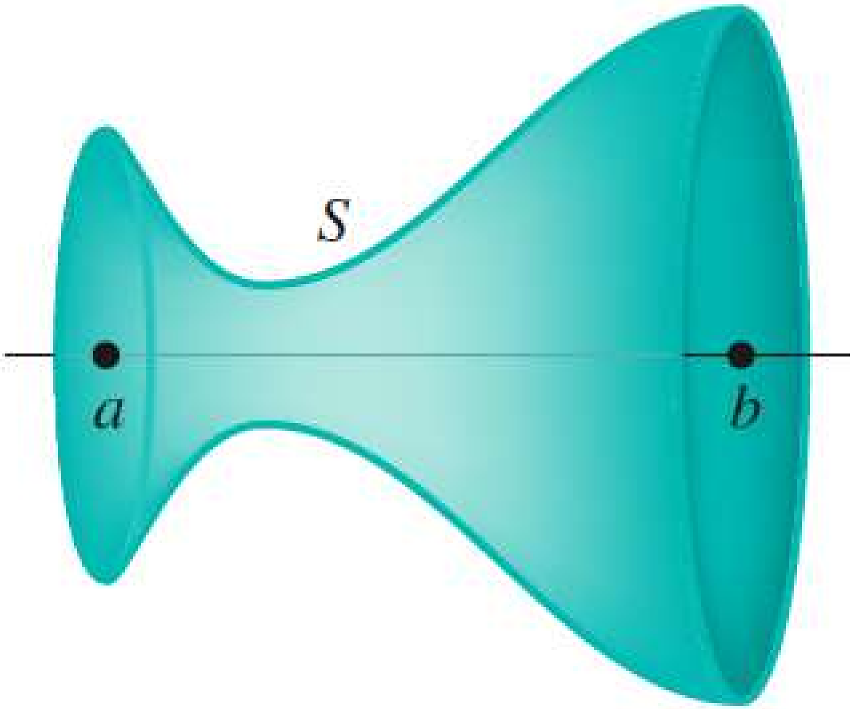


Around the x-axis

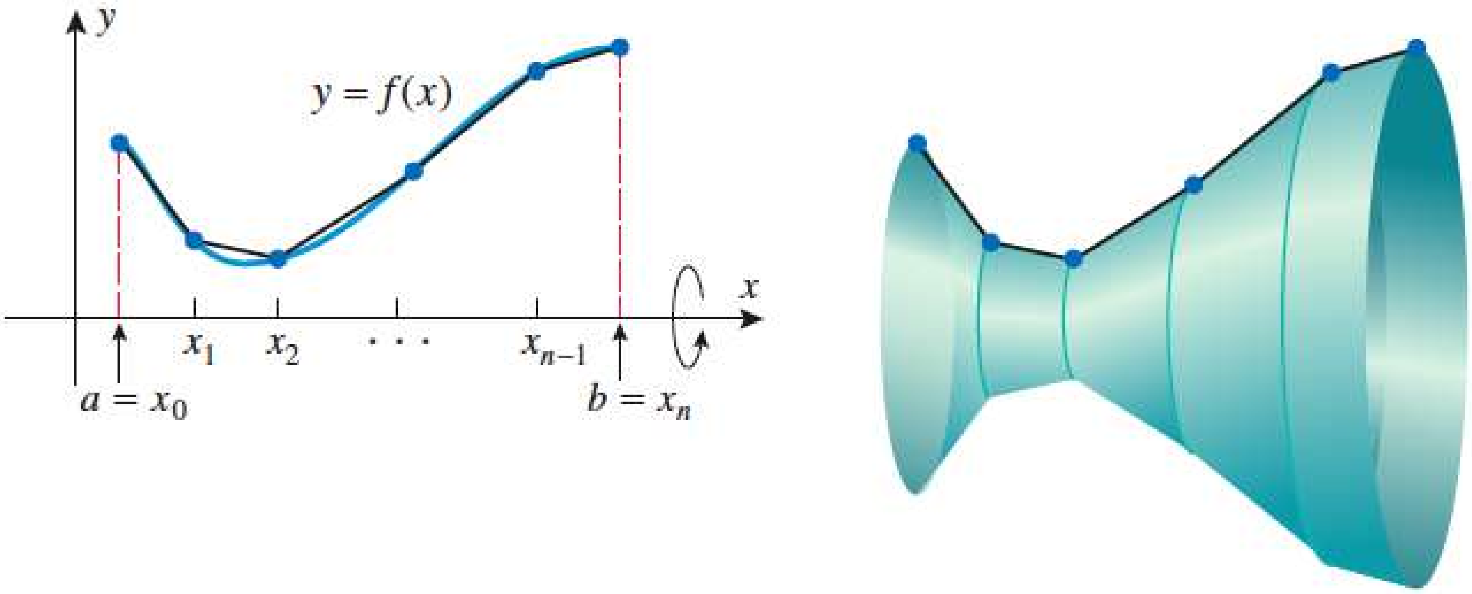


6.5.1 SURFACE AREA PROBLEM Suppose that f is a smooth, nonnegative function on $[a, b]$ and that a surface of revolution is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis (Figure 6.5.2). Define what is meant by the *area* S of the surface, and find a formula for computing it.





Break the Surface into Small Sections



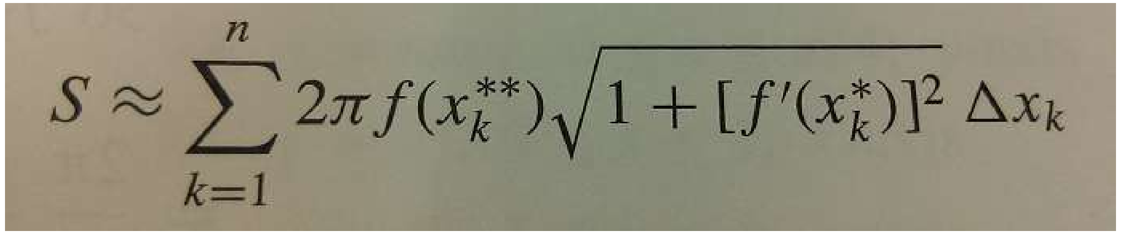
(a) (b)

Calculate Surface Area Using Riemann Sums

- Each of the subintervals into which we broke the surface area on the previous slide (figure b) is called a frustum which is a portion of a right circular cone.
- As we allow the number of subintervals to approach infinity, the width of each approaches zero.
- Each subinterval gets closer and closer to resembling a circle who's circumference is 2πr.
- We calculated the length of each subinterval in figure a on the previous slide last class using the distance formula.

Combine the Distance Formula and the Circumference

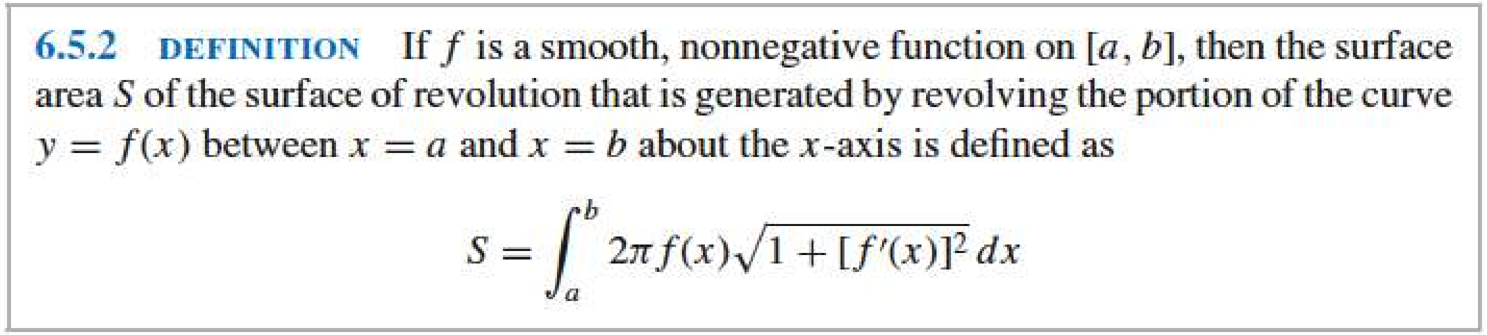
- The arc length (distance) formula from last class and the circumference which represents the rotation around a line combine together to generate the following Riemann sum-like expression:



$$S \approx \sum_{k=1}^{n} 2\pi f(x_k^{**})\sqrt{1 + [f'(x_k^*)]^2}\,\Delta x_k$$

- When we take the limit as the number of subintervals approaches infinity, this Riemann sum will give us the exact surface area.

Integral – when about the x-axis

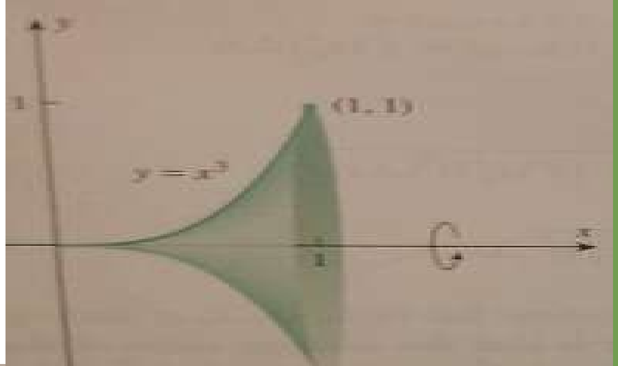


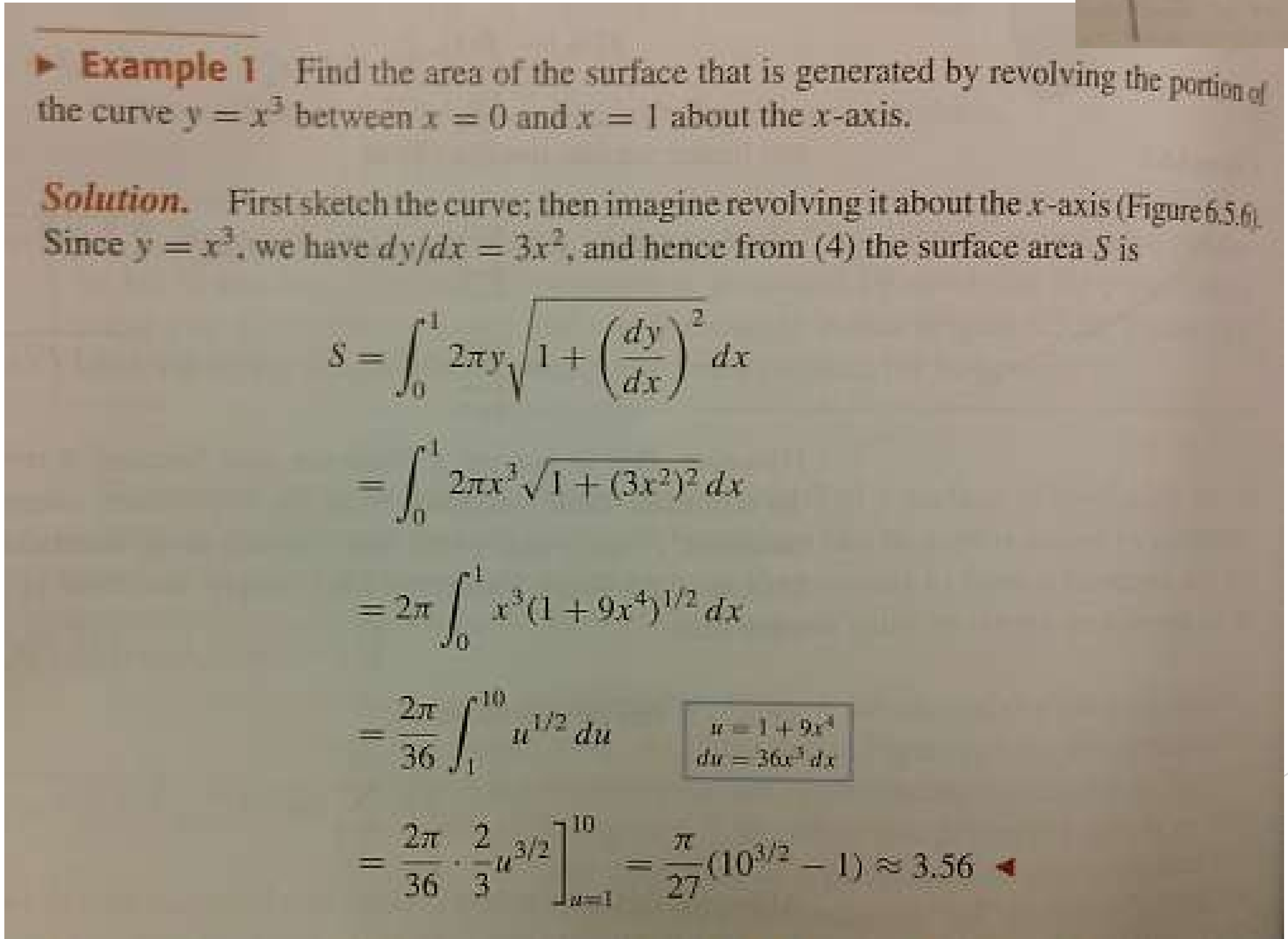
6.5.2 DEFINITION If f is a smooth, nonnegative function on $[a, b]$, then the surface area S of the surface of revolution that is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is defined as

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example about the x-axis





► **Example 1** Find the area of the surface that is generated by revolving the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis.

Solution. First sketch the curve; then imagine revolving it about the x-axis (Figure 6.5.6). Since $y = x^3$, we have $dy/dx = 3x^2$, and hence from (4) the surface area S is

$$S = \int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^1 2\pi x^3\sqrt{1+(3x^2)^2}\,dx$$

$$= 2\pi\int_0^1 x^3(1+9x^4)^{1/2}\,dx$$

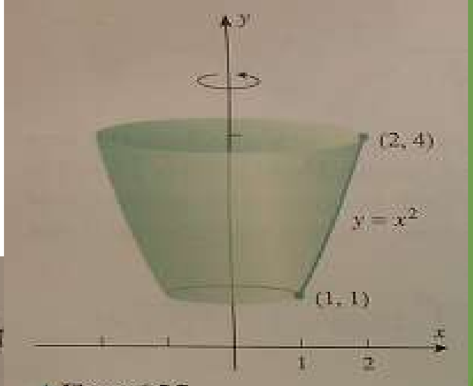
$$= \frac{2\pi}{36}\int_1^{10} u^{1/2}\,du \qquad \boxed{\begin{aligned} u &= 1+9x^4 \\ du &= 36x^3\,dx \end{aligned}}$$

$$= \frac{2\pi}{36}\cdot\frac{2}{3}u^{3/2}\Big]_{u=1}^{10} = \frac{\pi}{27}(10^{3/2}-1) \approx 3.56 \blacktriangleleft$$

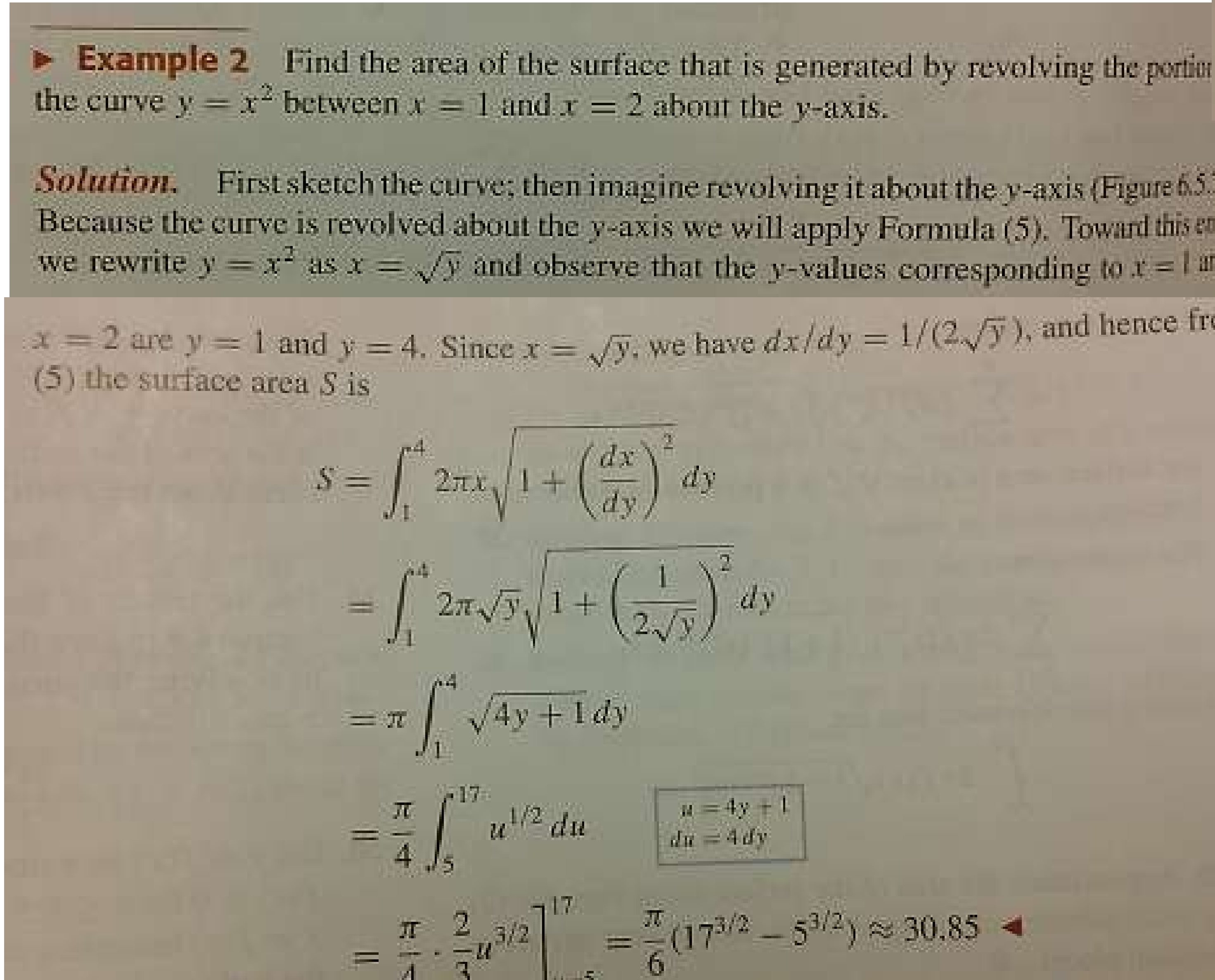
Integral – when about the y-axis

$$S = \int_{c}^{d} 2\pi g(y)\sqrt{1 + [g'(y)]^{2}}\, dy = \int_{c}^{d} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^{2}}\, dy$$

Example about the y-axis



▲ Figure 6.5.7



Example 2 Find the area of the surface that is generated by revolving the portion the curve $y = x^2$ between $x = 1$ and $x = 2$ about the y-axis.

Solution. First sketch the curve; then imagine revolving it about the y-axis (Figure 6.5.7). Because the curve is revolved about the y-axis we will apply Formula (5). Toward this end, we rewrite $y = x^2$ as $x = \sqrt{y}$ and observe that the y-values corresponding to $x = 1$ and $x = 2$ are $y = 1$ and $y = 4$. Since $x = \sqrt{y}$, we have $dx/dy = 1/(2\sqrt{y})$, and hence from (5) the surface area S is

$$S = \int_1^4 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_1^4 2\pi\sqrt{y}\sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\, dy$$

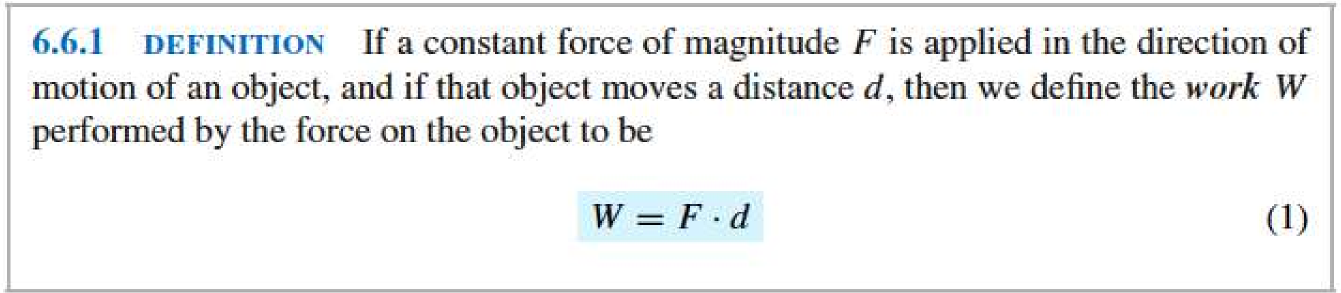
$$= \pi \int_1^4 \sqrt{4y+1}\, dy$$

$$= \frac{\pi}{4}\int_5^{17} u^{1/2}\, du \qquad \boxed{\begin{array}{l} u = 4y+1 \\ du = 4\,dy \end{array}}$$

$$= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big]_{5}^{17} = \frac{\pi}{6}(17^{3/2} - 5^{3/2}) \approx 30.85$$

Introduction

- In this section we will use our integration skills to study some basic principles of “work”, which is one of the fundamental concepts in physics and engineering.
- A simple example:
  - When you push a car that has run out of gas for a certain distance you are performing work and the effect of your work is to make the car move.
  - The energy of motion caused by the work is called the kinetic energy of the car.
  - There is a principle of physics called the work-energy relationship. We will barely touch on this principle.
  - Our main goal (on later slides) is to see how integration is related.



6.6.1 DEFINITION If a constant force of magnitude F is applied in the direction of motion of an object, and if that object moves a distance d, then we define the *work* W performed by the force on the object to be

$$W = F \cdot d \qquad (1)$$

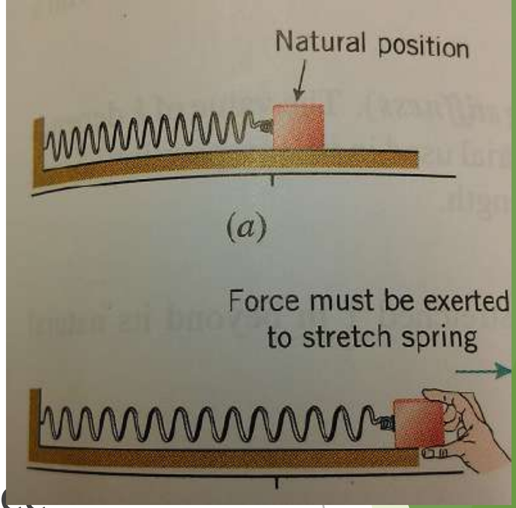
Examples of W = F * d

- An object moves five feet along a line while subjected to a constant force of 100 pounds in its direction of motion.
  - The work done is W = F * d = 100 * 5 = 500 ft-lbs

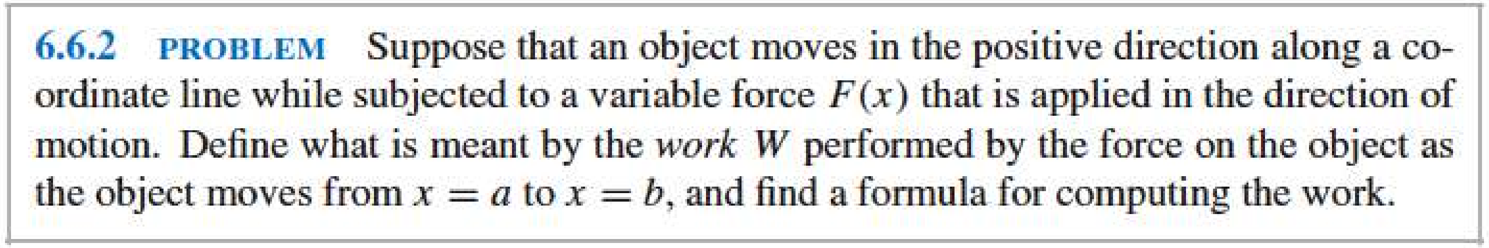
- An object moves 25 meters along a line while subjected to a constant force of four Newtons in its direction of motion.
  - The work done is W = F * d = 4*25 = 100 N-m

    =100 Joules

Work Done by a Variable Force Applied in the Direction of Motion



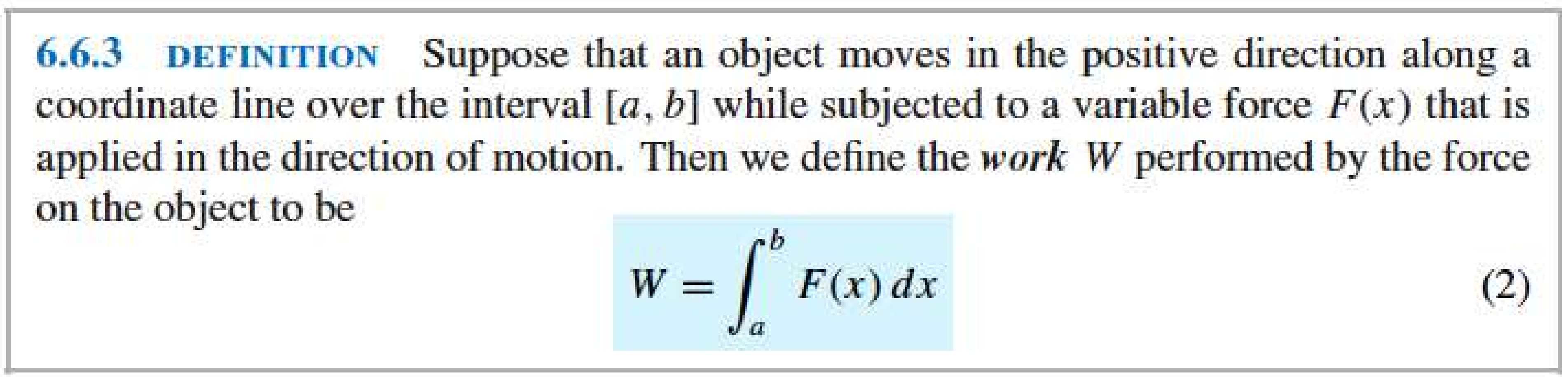
- If we wanted to pull the block attached to the spring horizontally, then we would have to apply more and more force to the block to overcome the increasing force of the stretching spring.



6.6.2 PROBLEM Suppose that an object moves in the positive direction along a coordinate line while subjected to a variable force $F(x)$ that is applied in the direction of motion. Define what is meant by the *work* W performed by the force on the object as the object moves from $x = a$ to $x = b$, and find a formula for computing the work.

- We will break the interval of the stretch [a,b] into subintervals and approximate the work on each subinterval.
- By adding the approximations to the work we will obtain a Riemann sum.
- The limit of the Riemann sum as n increases will give us an integral for work W.

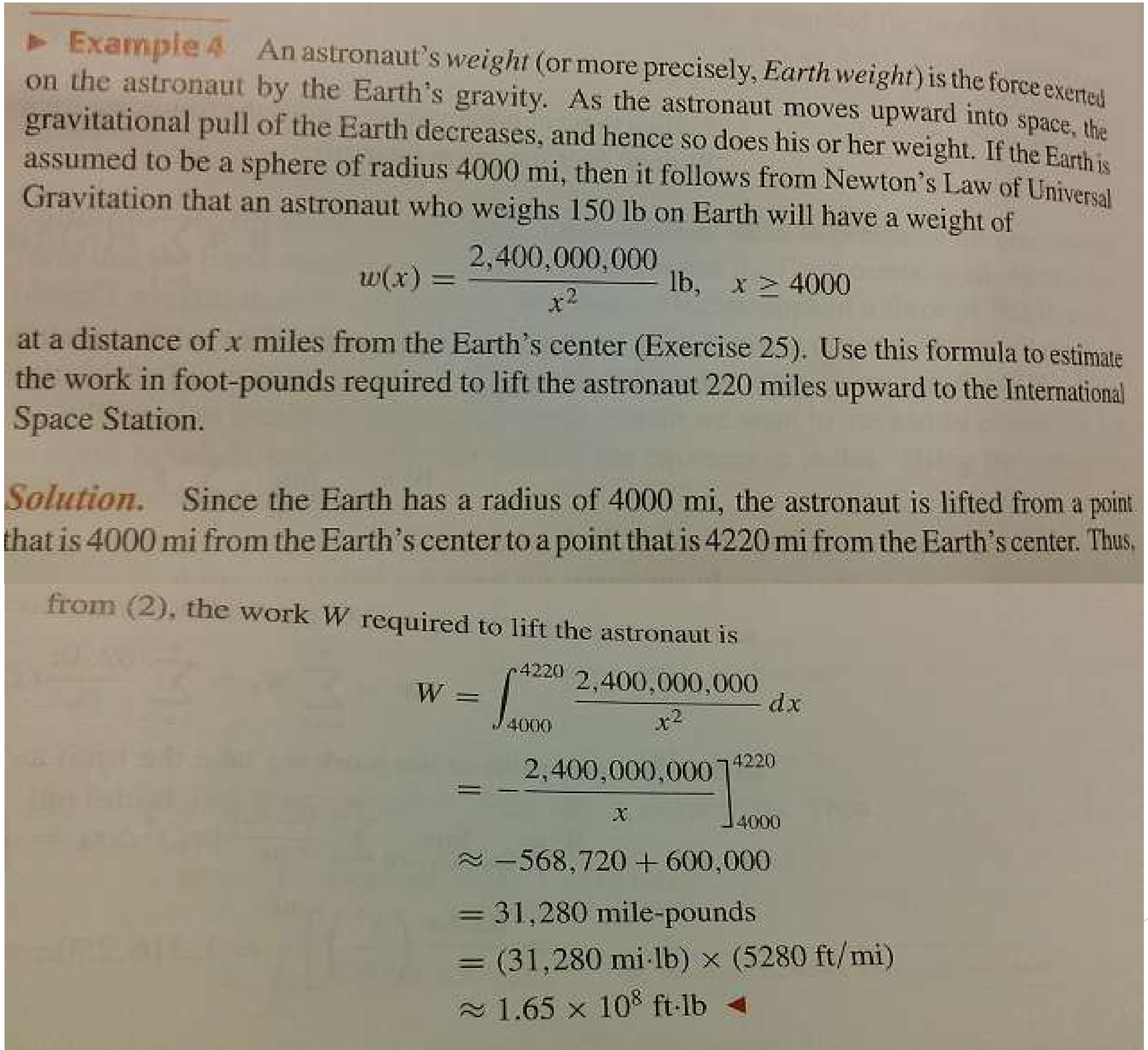
Integral for Work, W (with a Variable Force)



6.6.3 DEFINITION Suppose that an object moves in the positive direction along a coordinate line over the interval $[a, b]$ while subjected to a variable force $F(x)$ that is applied in the direction of motion. Then we define the *work* W performed by the force on the object to be

$$W = \int_a^b F(x)\,dx \tag{2}$$

Example



Example 4 An astronaut's *weight* (or more precisely, *Earth weight*) is the force exerted on the astronaut by the Earth's gravity. As the astronaut moves upward into space, the gravitational pull of the Earth decreases, and hence so does his or her weight. If the Earth is assumed to be a sphere of radius 4000 mi, then it follows from Newton's Law of Universal Gravitation that an astronaut who weighs 150 lb on Earth will have a weight of

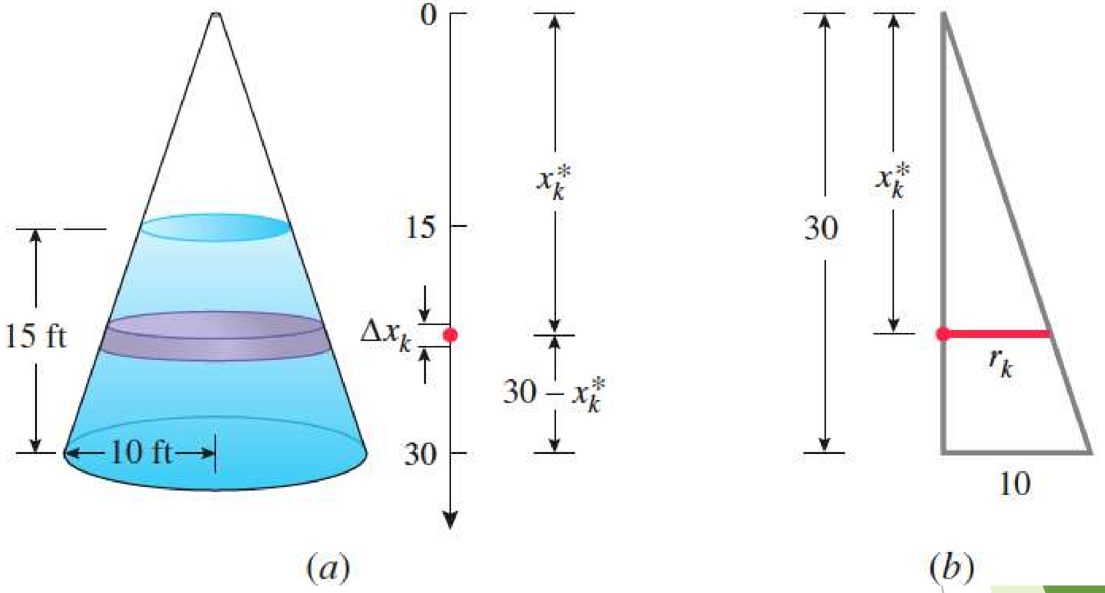
$$w(x) = \frac{2{,}400{,}000{,}000}{x^2} \text{ lb}, \quad x \geq 4000$$

at a distance of x miles from the Earth's center (Exercise 25). Use this formula to estimate the work in foot-pounds required to lift the astronaut 220 miles upward to the International Space Station.

Solution. Since the Earth has a radius of 4000 mi, the astronaut is lifted from a point that is 4000 mi from the Earth's center to a point that is 4220 mi from the Earth's center. Thus, from (2), the work W required to lift the astronaut is

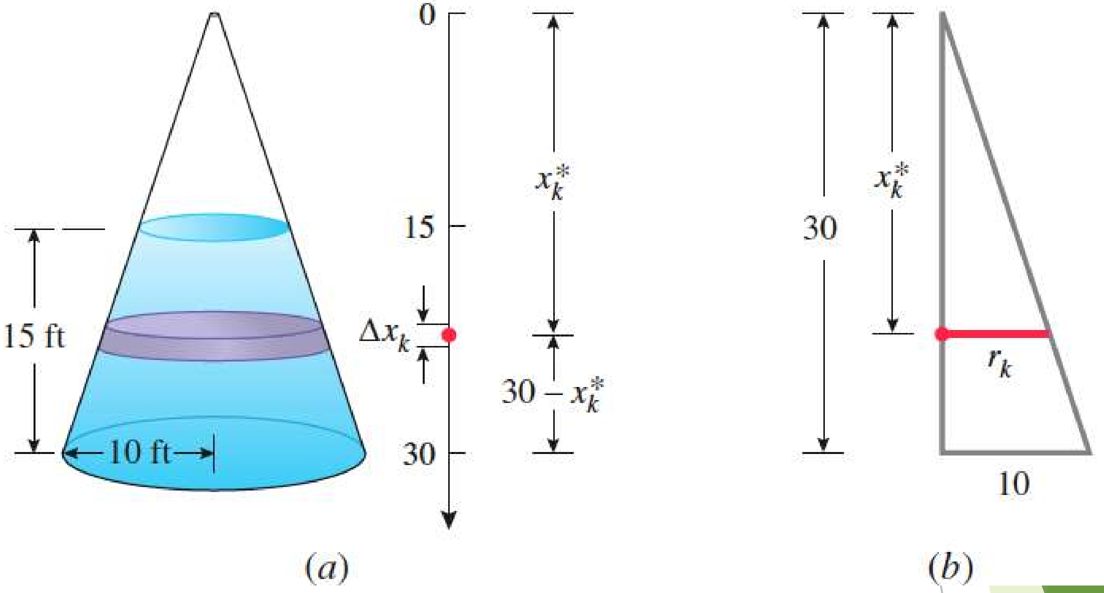
$$\begin{aligned} W &= \int_{4000}^{4220} \frac{2{,}400{,}000{,}000}{x^2}\,dx \\ &= -\frac{2{,}400{,}000{,}000}{x}\bigg]_{4000}^{4220} \\ &\approx -568{,}720 + 600{,}000 \\ &= 31{,}280 \text{ mile-pounds} \\ &= (31{,}280 \text{ mi·lb}) \times (5280 \text{ ft/mi}) \\ &\approx 1.65 \times 10^8 \text{ ft·lb} \end{aligned}$$

Calculating Work from Basic Principles Example

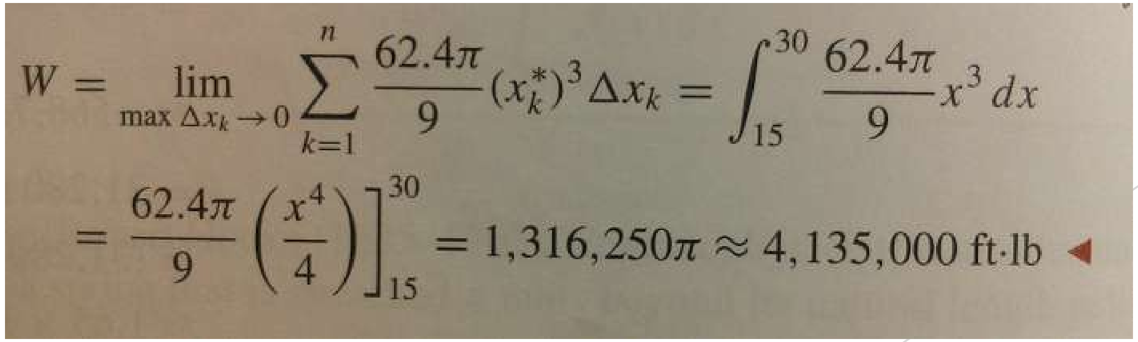


- As you can see in the figure above right, we have a cone shaped (conical) container of radius 10 ft and height 30 ft. Suppose that this container is filled with water to a depth of 15 ft. How much work is required to pump all of the water out through the hole in the top of the container?
  - We will divide the water into thin layers, approximate the work required to move each layer to the top of the container and add them to obtain a Riemann sum.
  - The limit of the Riemann sum produces an integral for the total work.

Calculating Work from Basic Principles Example con’t

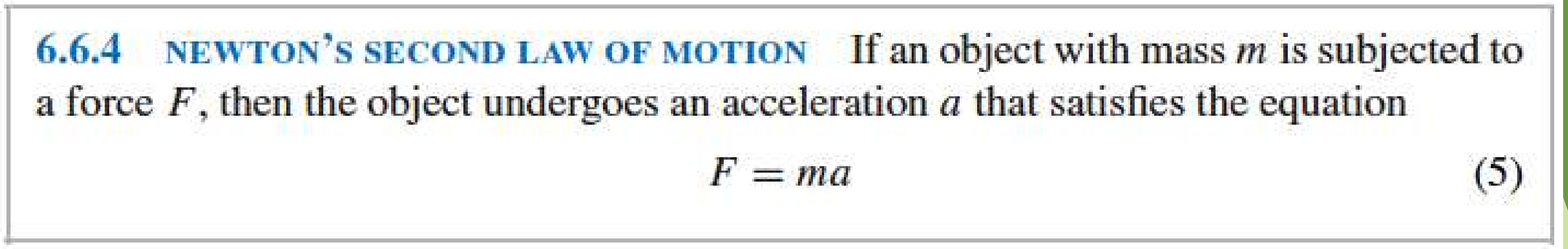


- The volume of a cone is $\frac{1}{3}\quad{}^{2}\mathrm{h}$
- By similar triangles the ratio of radius to height $\frac{}{h} = \frac{10}{30} = \frac{1}{3}$
- When these are combined with the weight density of water which is 62.4 pounds per cubic foot, we get the following Riemann sum and integral:



$$W = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \frac{62.4\pi}{9}(x_k^*)^3 \Delta x_k = \int_{15}^{30} \frac{62.4\pi}{9} x^3\, dx$$

$$= \frac{62.4\pi}{9}\left(\frac{x^4}{4}\right)\Bigg]_{15}^{30} = 1{,}316{,}250\pi \approx 4{,}135{,}000 \text{ ft}\cdot\text{lb}$$



6.6.4 NEWTON'S SECOND LAW OF MOTION If an object with mass m is subjected to a force F, then the object undergoes an acceleration a that satisfies the equation

$$F = ma \tag{5}$$