CHAPTER ONE

"Limits and Continuity":

ALL GRAPHICS ARE ATTRIBUTED TO:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

SUMMARY

- The concept of a "limit" is the fundamental building block on which all calculus concepts are based.
- We study limits informally, with the goal of developing an intuitive feel for the basic ideas.
- We will also focus on computational methods and precise definitions.

GEOMETRIC PROBLEMS LEADING TO LIMITS

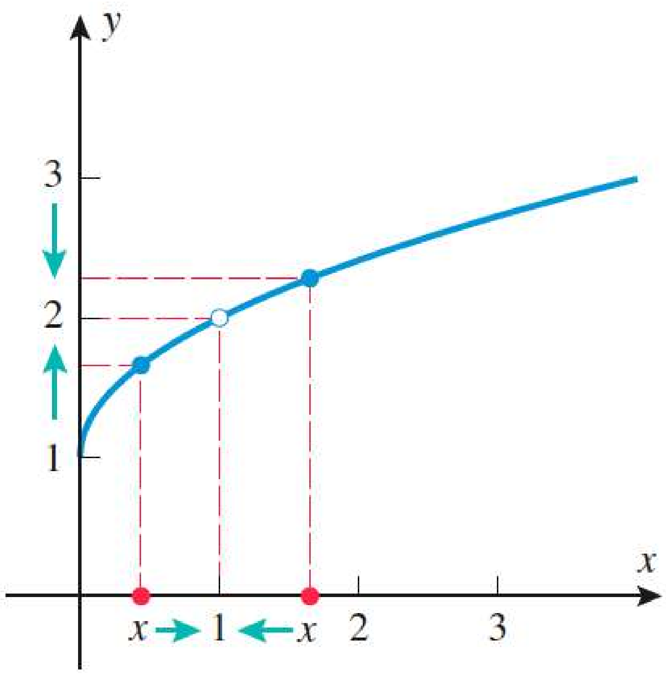
THE TANGENT LINE PROBLEM Given a function f and a point $P(x_0, y_0)$ on its graph, find an equation of the line that is tangent to the graph at P (Figure 1.1.1).

THE AREA PROBLEM Given a function f, find the area between the graph of f and an interval $[a, b]$ on the x-axis (Figure 1.1.2).

LIMITS

- The most basic use of limits is to describe how a function behaves as x (the independent variable) approaches a given value.

- In this figure, as x gets closer and closer to 1 from either the left or the right, y values get closer and closer to 2.



TERMINOLOGY

- We can find one sided or two sided limits. Below is the notation for one sided limits.

1.1.2 ONE-SIDED LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write

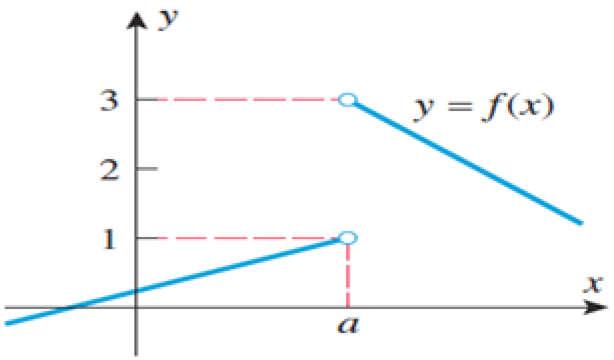
$$\lim_{x \to a^+} f(x) = L \tag{14}$$

and if the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write

$$\lim_{x \to a^-} f(x) = L \tag{15}$$

Expression (14) is read "the limit of $f(x)$ as x approaches a from the right is L" or "$f(x)$ approaches L as x approaches a from the right." Similarly, expression (15) is read "the limit of $f(x)$ as x approaches a from the left is L" or "$f(x)$ approaches L as x approaches a from the left."

ONE SIDED LIMITS EXAMPLES



Follow the graph from left to right as you get closer to a and you will find that

$\lim_{x \to a-} f(x) = 1$

since your y values are getting closer to 1 as x gets closer to a from the left.

Follow the graph from right to left as you get closer to a and you will find that

$\lim_{x \to a+} f(x) = 3$

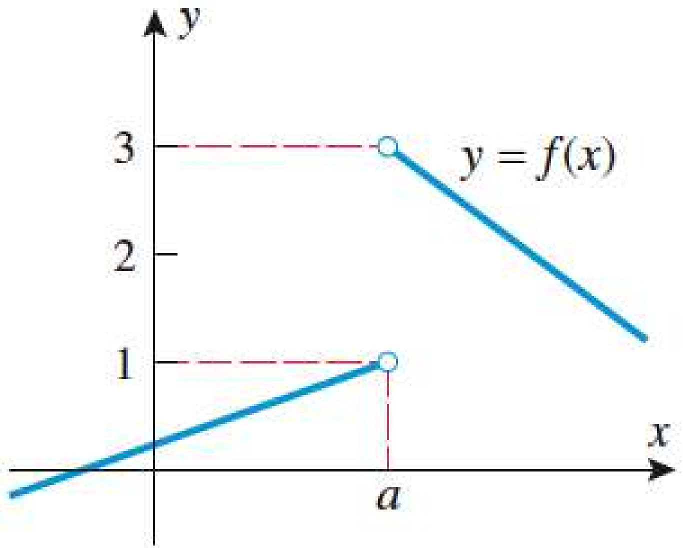
since your y values are getting closer to 3 as x gets closer to a from the right.

THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS

1.1.3 THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS The two-sided limit of a function $f(x)$ exists at a if and only if both of the one-sided limits exist at a and have the same value; that is,

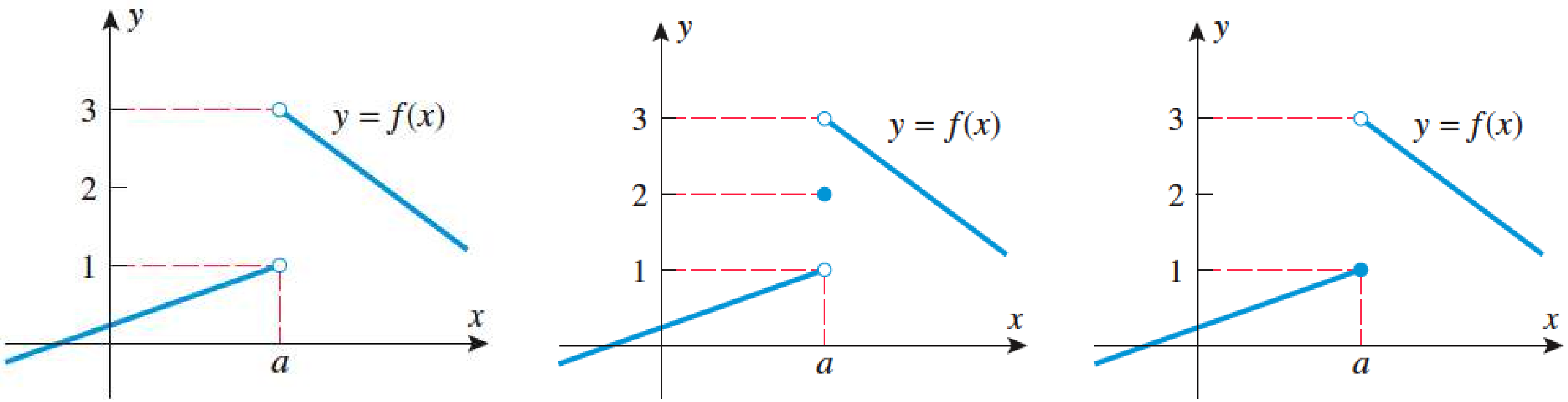
$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

- Therefore, the two sided limit at does not exist for the figure on the right because its one sided limits are not equal (1 does not equal 3).



"AS YOU APPROACH" NOT AT

- All of these graphs have the same one sided limits and none of the two sided limits exist. It does not matter what happens right at the a value when determining limits.



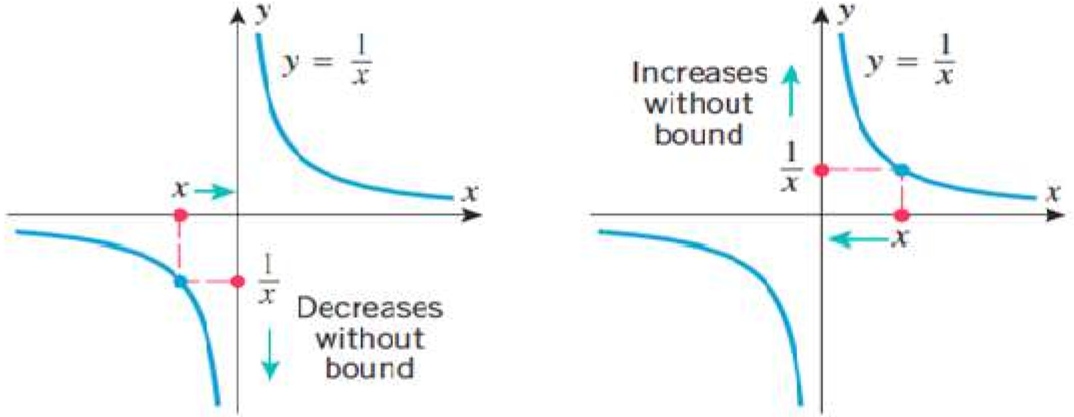
INFINITE LIMITS

- Sometimes one-sided or two-sided limits fail to exist because the values of the function increase or decrease without bound.

- Positive and negative infinity (on the next slide are not real numbers), they simply describe particular ways in which the limits fail to exist.

- You cannot manipulate infinity algebraically (you cannot add, subtract, etc).

INFINITE LIMITS EXAMPLE



Follow the graph from <u>left to right</u> as your x value gets closer to 0 and you will find that

$\lim_{x \to 0-} f(x) = -\infty$

since your y values are decreasing without bound as x gets closer to 0 <u>from the left</u>.

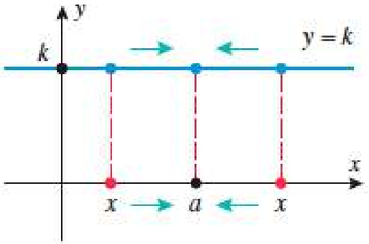
Follow the graph from <u>right to left</u> as x value gets closer to 0 and you will find that

$\lim_{x \to 0+} f(x) = +\infty$

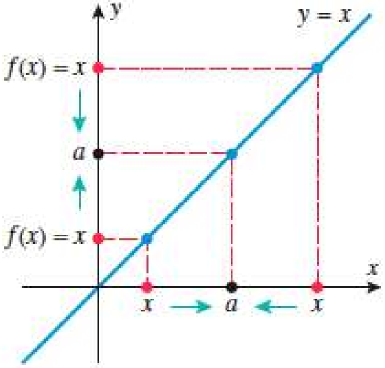
since your y values are increasing without bound as x gets closer to 0 <u>from the right</u>.

SOME BASIC LIMITS

- The limit of a constant = that constant because the y value never changes. (1.2.1 a)
- The limit of y=x as x approaches any value is just that value since x and y are equal.(1.2.1b)



$\lim_{x \to a} k = k$



$\lim_{x \to a} x = a$

1.2.1 THEOREM *Let a and k be real numbers.*

(a) $\lim_{x \to a} k = k$ (b) $\lim_{x \to a} x = a$ (c) $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ (d) $\lim_{x \to 0^+} \frac{1}{x} = +\infty$

BASIC TOOL FOR FINDING LIMITS ALGEBRAICALLY

- These look terrible, but I will explain them on the next slide and give examples after that.

1.2.2 THEOREM *Let a be a real number, and suppose that*

$$\lim_{x\to a} f(x) = L_1 \quad \textit{and} \quad \lim_{x\to a} g(x) = L_2$$

That is, the limits exist and have values L_1 and L_2, respectively. Then:

(a) $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2$

(b) $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L_1 - L_2$

(c) $\lim_{x\to a} [f(x)g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = L_1L_2$

(d) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L_1}{L_2}$, *provided* $L_2 \neq 0$

(e) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L_1}$, *provided $L_1 > 0$ if n is even.*

Moreover, these statements are also true for the one-sided limits as $x \to a^-$ or as $x \to a^+$.

INDETERMINATE FORM OF TYPE 0/0

- The following example is called indeterminate form of type 0/0 because if you do jump directly to substitution, you will get 0/0.

Find $\lim_{x \to -4} \frac{2x+8}{x^2+x-12} = \frac{2*-4+8}{(-4)^2 \pm 4-12} = \frac{0}{0}$ which is indeterminate form 0/0 so we must try something else

$= \lim_{x \to -4} \frac{2(x+4)}{(x+4)(x-3)}$ since x+4 is a common factor in numerator and denominator, it cancels out

$= \lim_{x \to -4} \frac{2}{x-3}$ now we can do substitution again

$= \frac{2}{-4-3}$ $= \frac{2}{-7}$

MORE INDETERMINATE FORM

- Sometimes, limits of indeterminate forms of type 0/0 can be found by algebraic simplification, as in the last example, but frequently this will not work and other methods must be used.

- One example of another method involves multiplying by the conjugate of the denominator (see example on next page).

EXAMPLE

Find $\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1} = \frac{1-1}{\sqrt{1}-1} = \frac{0}{0}$ which is indeterminate form 0/0 so we must try something else

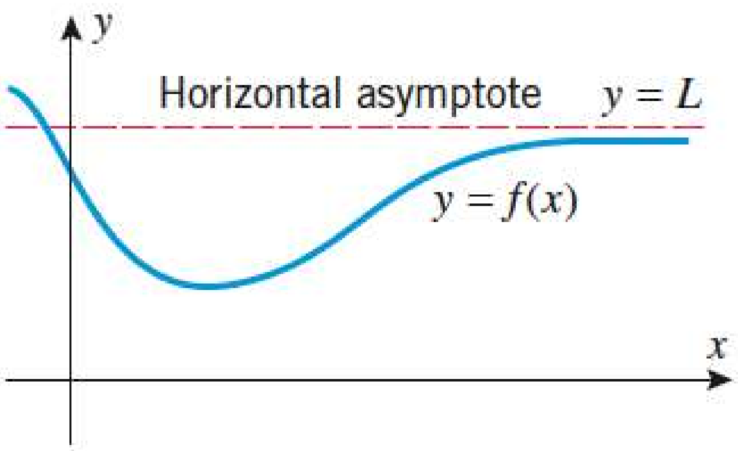
$= \lim_{x \to 1} \frac{(x-1)}{(\sqrt{x}-1)} * \frac{(\sqrt{x}+1)}{(\sqrt{x}+1)}$ try the new strategy of multiplying by the conjugate of the denominator

$= \lim_{x \to 1} \frac{(x-1)*(\sqrt{x}+1)}{x-1}$ since x-1 is a common factor in numerator and denominator, it cancels out

$= \lim_{x \to 1} \sqrt{x} + 1 = \sqrt{1} + 1 = 2$

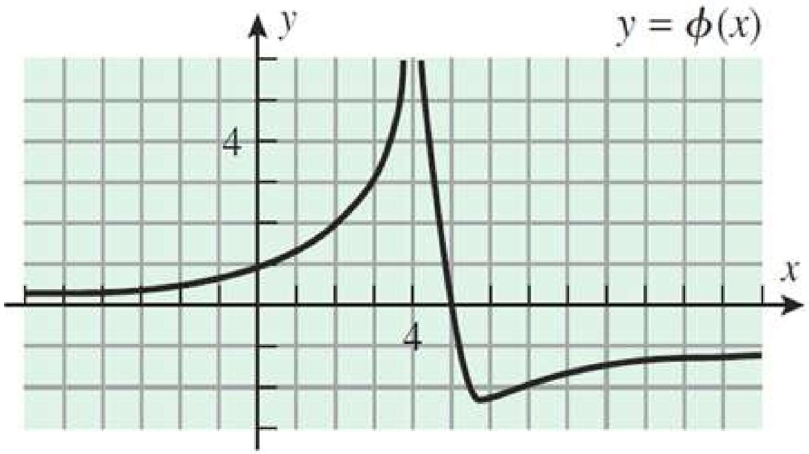
LIMITS AT INFINITY AND HORIZONTAL ASYMPTOTES

- We discussed infinite limits briefly on slides 12 & 13 in section 1.3.
- We will now expand our look at situations when the value of x increases or decreases without bound ("end behavior") and as a function approaches a horizontal asymptote.
- Below is a picture of a function with a horizontal asymptote and a related limit.
- The further you follow the graph to the right, the closer y values get to the asymptote. That is why the limit is L.



$$\lim_{x \to +\infty} f(x) = L$$

EXAMPLE #1



$\lim_{x\to-\infty} \emptyset(x) = 0$ since the curve approaches the x axis

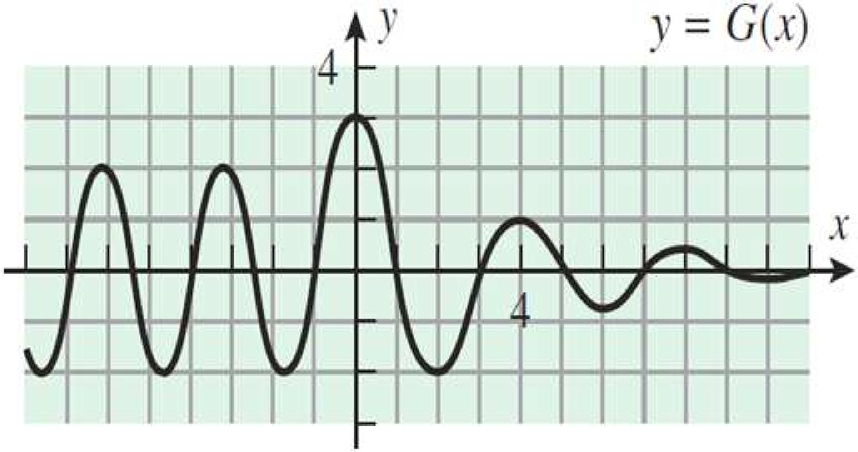
$\lim_{x\to+\infty} \emptyset(x) = -1$ since the curve appears to get closer and closer to the y-value -1

$\lim_{x\to 4-} \emptyset(x) = +\infty$ since the curve increases without bound to the left of 4

$\lim_{x\to 4+} \emptyset(x) = +\infty$ since the curve increases without bound to the right of 4

$\lim_{x\to 4} \emptyset(x) = +\infty$ since the one sided limits on each side of 4 are equal

EXAMPLE 2



$\lim_{x \to -\infty} G(x) = DNE$ Does not exist since the curve oscillates without getting closer to any y-value

$\lim_{x \to +\infty} G(x) = 0$ This side is a “dampened” oscillation which is getting closer to 0 as x increases.

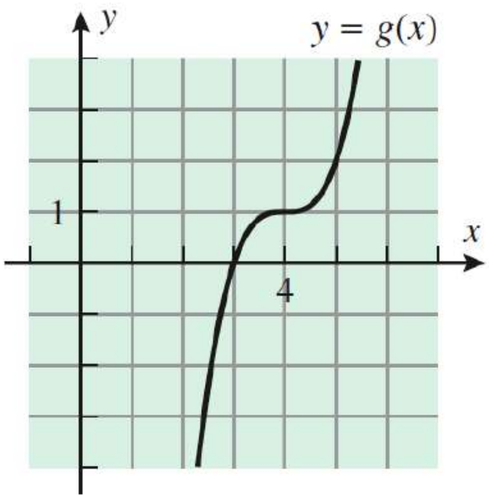
$\lim_{x \to 0} G(x) = 3$ since the curve approaches a y-value of 3 from both the left and right sides of 0

LIMITS OF POLYNOMIALS AS X APPROACHES +/- INFINITY

- The end behavior of a polynomial matches the end behavior of its highest degree term.

- From the Lead Coefficient Test (+1) and the degree (odd), we know that the end behavior of y= x^3 is down up, therefore, the same must be true for y = $x^3 - 12x^2 + 48x - 63$ which is $g(x)=(x-4)^3+1$ in graphing form.

- The graph of g(x) on the right does fall to the left and rise to the right, just as end behavior predicts.



HIGHER DEGREE EXAMPLE

$\lim_{x\to-\infty}(-4x^8+17x^3-5x+1) = \lim_{x\to-\infty} -4x^8 = -\infty$

Since the degree is even (8), the ends will match (like a parabola: up/up or down/down).

Since the lead coefficient is negative (-4), the graph will be upside down.

Therefore, the end behavior is down/down and both ends will decrease without bound, approaching $-\infty$.

It does not matter what is happening in the middle since we are evaluating the $\lim_{x\to-\infty}$.

LIMITS OF RATIONAL FUNCTIONS AS X APPROACHES +/- INFINITY

- One technique for determining the end behavior of a rational function is to divide each term in the numerator and denominator by the highest power of x that occurs in the denominator, then follow methods we already know.
- Example:

Find $\lim_{x\to+\infty}\frac{3x+5}{6x-8}$

$$= \lim_{x\to+\infty}\left(\frac{3x+5}{6x-8}\right)\frac{divide\ by\ x}{divide\ by\ x} = \lim_{x\to+\infty}\frac{3+5/x}{6-8/x}$$

Now apply the limit rules a-e from section 1.2

$$= \frac{\lim_{x\to+\infty} 3 + 5\lim_{x\to+\infty}\frac{1}{x}}{\lim_{x\to+\infty} 6 + 8\lim_{x\to+\infty}\frac{1}{x}} = \frac{3+5*0}{6+8*0} = \frac{1}{2}$$

Since the limit of a constant is a constant and

the graph of a hyperbola gives us the zeros.

INTRODUCTION

- A thrown baseball cannot vanish at some point and reappear someplace else to continue its motion. Thus, we perceive the path of the ball as an unbroken curve. In this section, we will define "unbroken curve" to mean <u>continuous</u> and include properties of continuous curves.

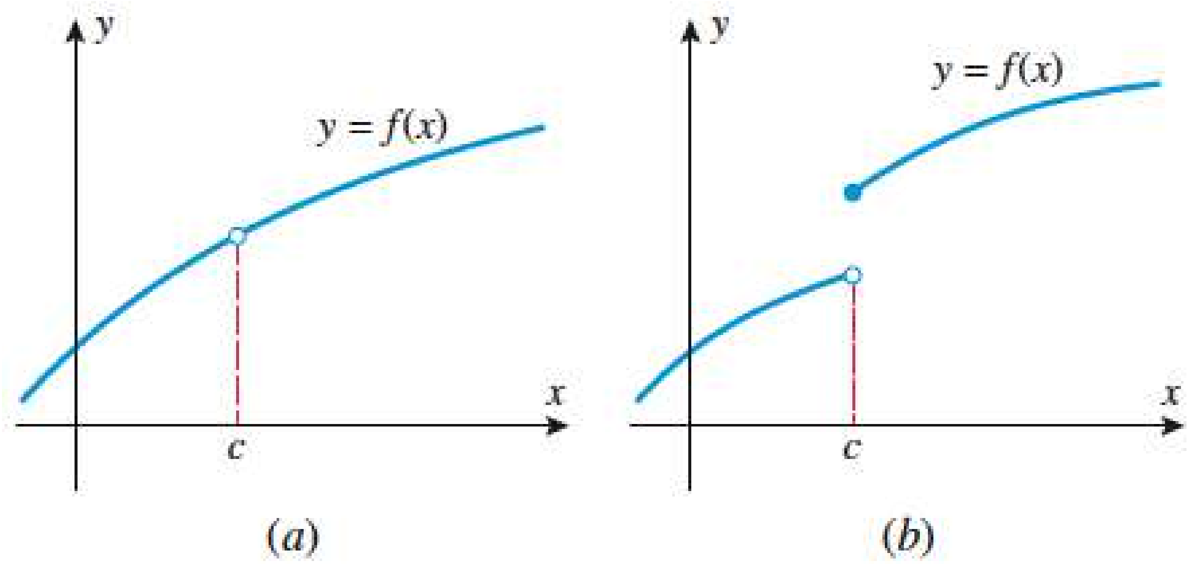
DEFINITION OF CONTINUITY

1.5.1 DEFINITION A function f is said to be *continuous at* $x = c$ provided the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

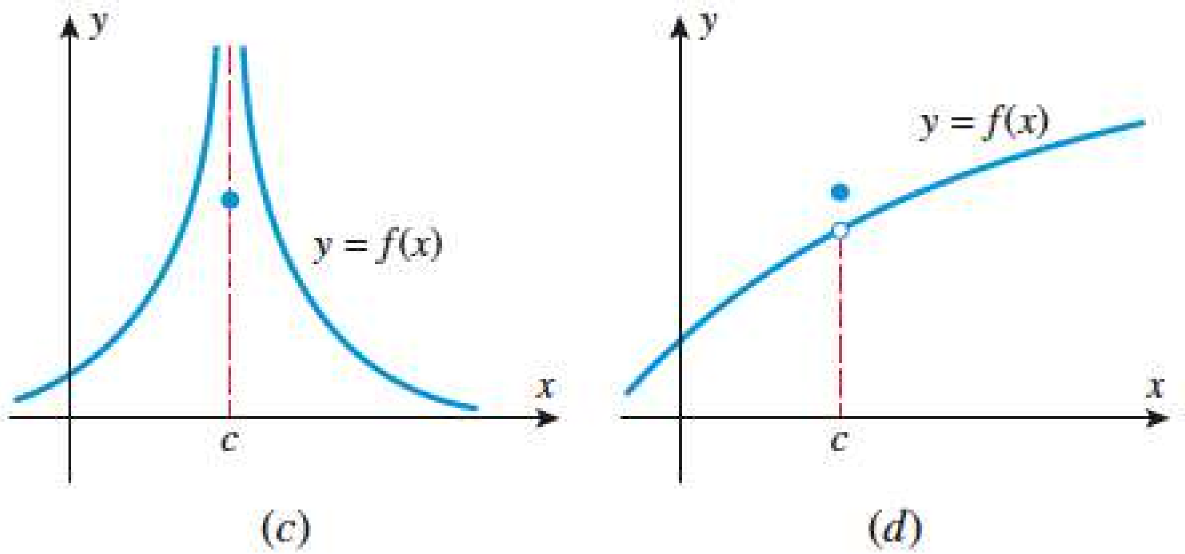
- #1 means that there cannot be an unfilled hole remaining at that value (c) where you are finding the limit.
- #2 means that the two one sided limits must be equal.
- #3 means that the limit and the point at that value (c) must be equal.

GRAPHING EXAMPLES THAT ARE NOT CONTINUOUS



- (a) has a hole, so it breaks rule #1.
- (b) has a limit that does not exist (DNE) at c because the two sided limits are not equal, so it breaks rule #2.

MORE GRAPHING EXAMPLES



- (c) & (d) both break rule #3. The two sided limit does exist, and it is defined at c, but the two values are not equal so they are not continuous.

FUNCTION EXAMPLE

h(x) = $\begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$ This is a piecewise function and the first piece needs to be simplified.

$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2}$ = x+2 (since x-2 is a common factor in numerator and denominator, it cancels out)

Therefore, $\lim_{x\to 2} h(x) = \lim_{x\to 2} (x+2) = 2+2 = 4$

The reason I chose to evaluate the limit as x approaches 2 is because that is the only number in the domain of the piecewise function where anything unusual is happening. Normally, there would be a hole there because the x-2 factored out. However, the second line of the piecewise function fills that hole in because it says that y = 4 when x = 2 and that is where the hole would have been.

CONTINUITY ON AN INTERVAL

- Continuity on an interval just means that we are testing for continuity only on a certain part of the graph, and the rules are very similar to the ones previously listed. You just have to be careful around the ends of the interval. See example 2 on page 112.

1.5.2 DEFINITION A function f is said to be *continuous on a closed interval* $[a, b]$ if the following conditions are satisfied:

1. f is continuous on (a, b).
2. f is continuous from the right at a.
3. f is continuous from the left at b.

CONTINUITY OF POLYNOMIALS, RATIONAL FUNCTIONS, AND ABSOLUTE VALUE

- Polynomials are continuous everywhere because their graphs are always smooth unbroken curves with no jumps breaks or holes which go on forever to the right and to the left.
- Rational functions are continuous at every point where the denominator is not zero because they are made up of polynomial functions which are continuous everywhere, but one cannot divide by zero. Therefore, they are only discontinuous where the denominator is zero.
- The absolute value of a continuous function is continuous.

EXAMPLE OF A RATIONAL FUNCTION

- I think you already know this, but it is worth making sure.
- Example:

For what values of x is there a discontinuity in the graph of $y = \frac{x^2-9}{x^2-5x+6}$?

This is a rational function because it has a polynomial in the numerator and denominator.

Therefore, it is continuous at every number where the denominator is not zero: $x^2 - 5x + 6 \neq 0$

$$(x-2)(x-3) \neq 0$$

$$x \neq 2, x \neq 3 \text{ is the domain}$$

which gives us discontinuities at x = 2 and x = 3.

CONTINUITY OF COMPOSITIONS

- A limit symbol can be moved through a function sign as long as the limit of the inner function exists and is continuous where you are calculating the limit.

1.5.5 THEOREM *If* $\lim_{x\to c} g(x) = L$ *and if the function* f *is continuous at* L, *then* $\lim_{x\to c} f(g(x)) = f(L)$. *That is,*

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right)$$

This equality remains valid if $\lim_{x\to c}$ *is replaced everywhere by one of* $\lim_{x\to c^+}$, $\lim_{x\to c^-}$, $\lim_{x\to +\infty}$, *or* $\lim_{x\to -\infty}$.

- Example: $\lim_{x\to 3}|5-x^2| = \left|\lim_{x\to 3}(5-x^2)\right| = |5-3^2| = |-4| = 4$

THE INTERMEDIATE-VALUE THEOREM

- We discussed this some last year, and we will continue to discuss it. It is more obvious than the theorem sounds.

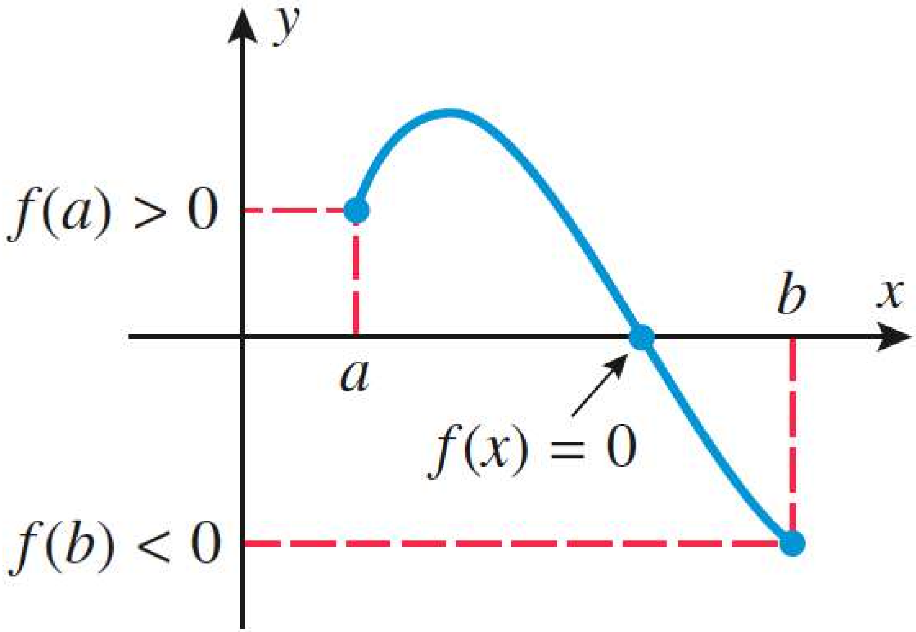
1.5.7 THEOREM (*Intermediate-Value Theorem*) *If f is continuous on a closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, inclusive, then there is at least one number x in the interval $[a, b]$ such that $f(x) = k$.*

- Often used to find the zeros of a function.

1.5.8 THEOREM *If f is continuous on $[a, b]$, and if $f(a)$ and $f(b)$ are nonzero and have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b).*

THE INTERMEDIATE-VALUE THEOREM GRAPHICALLY

- If two x values have different signs and the function is continuous (no jumps breaks or homes), then there will be a root/zero/x-intercept somewhere between those x values.



CONTINUITY OF TRIGONOMETRIC FUNCTIONS

- sin x and cos x are continuous everywhere.
- tan x, cot x, csc x, and sec x are continuous everywhere except at their asymptotes.

1.6.2 THEOREM *If f is a one-to-one function that is continuous at each point of its domain, then f^{-1} is continuous at each point of its domain; that is, f^{-1} is continuous at each point in the range of f.*

- Therefore, $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$ are only continuous on their own domains which are $(-\pi/2, \pi/2)$ for $\sin^{-1} x$ and $\tan^{-1} x$ and $(0,\pi)$ for $\cos^{-1} x$.

THE SQUEEZING THEOREM

- Theorem 1.6.5 on a later slide will give you the two most common uses.

1.6.4 THEOREM (*The Squeezing Theorem*) *Let f, g, and h be functions satisfying*

$$g(x) \leq f(x) \leq h(x)$$

for all x in some open interval containing the number c, with the possible exception that the inequalities need not hold at c. If g and h have the same limit as x approaches c, say

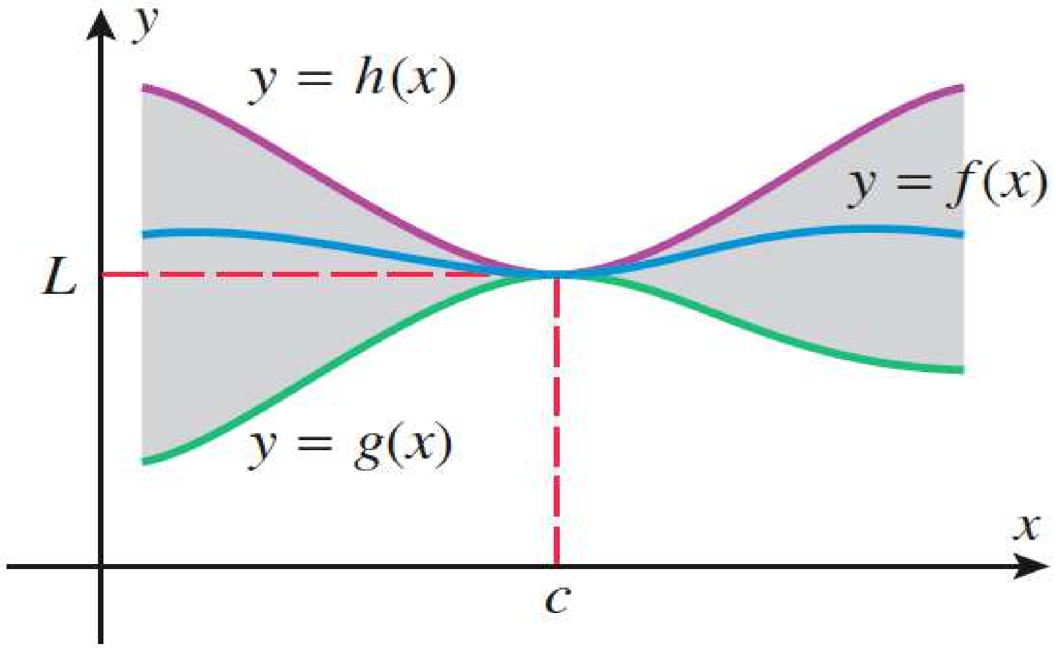
$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

then f also has this limit as x approaches c, that is,

$$\lim_{x \to c} f(x) = L$$

SQUEEZING THEOREM GRAPHICALLY

- There is no easy way to calculate some indeterminate type 0/0 limits algebraically so, for now, we will squeeze the function between two known functions to find its limit like in the graph below.



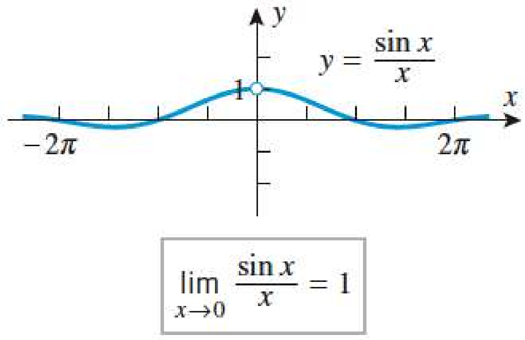
COMMON USES THIS YEAR

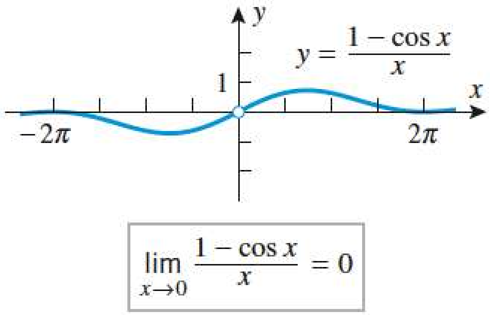
- These are the most common applications of the squeezing theorem for the first Calculus course.

1.6.5 THEOREM

$$(a)\quad \lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad (b)\quad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

- They may make more sense if you look at their graphs and find the limits that way.





PROOFS AND EXAMPLES

- Theorem 1.6.5 is proven on page 123 if you are interested in how it works.
- Example 1 of how it is used:

Find $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\frac{\sin x}{\cos x}}{x}$ — Use tan x quotient identity to rewrite

$= \lim_{x \to 0} \frac{\sin x}{\cos x} * \frac{1}{x} = \lim_{x \to 0} \frac{\sin x}{x} * \frac{1}{\cos x}$ — Multiply by the reciprocal and rearrange

$= \lim_{x \to 0} \frac{\sin x}{x} * \lim_{x \to 0} \frac{1}{\cos x}$ — Apply the limit of the product=product of the limits rule

$= (1) * \frac{1}{\cos 0}$ — By the squeezing theorem and substitution

$= 1 * \frac{1}{1} = 1$

ANOTHER EXAMPLE

- Example 2:

Find $\lim_{\theta \to 0} \frac{\sin 2\theta}{\theta} = \lim_{\theta \to 0} \frac{\sin 2\theta}{\theta} * \frac{2}{2}$ Multiply top and bottom by 2 so that we can apply theorem

$= \lim_{\theta \to 0} \frac{2 * \sin 2\theta}{2\theta} = 2 * \lim_{x \to 0} \frac{\sin 2\theta}{2\theta}$ Multiply, rearrange, and apply limit property

$= 2 * (1) = 2$ By the squeezing theorem

STRATEGIES FOR CALCULATING LIMITS

1. Graph
2. Substitution
3. Simplify, then substitute
4. Multiply numerator and denominator by conjugate of the denominator, then follow with step 3.
5. Analyze end behavior
6. Divide each term in the numerator and denominator by the highest power of x that occurs in the denominator, then follow with other steps.
7. Squeezing Theorems