

Chapter 9 FRQ Homework

1.

$$f(x) = \begin{cases} \frac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\frac{1}{2} & \text{for } x = 0 \end{cases}$$

The function f , defined above, has derivatives of all orders. Let g be the function defined by

$$g(x) = 1 + \int_0^x f(t) dt.$$

- (a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x = 0$.
- (b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.
- (c) Write the fifth-degree Taylor polynomial for g about $x = 0$.
- (d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x = 0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

(a) $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots$

$$f(x) = -\frac{1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + \dots + (-1)^{n+1} \frac{x^{2n}}{(2n+2)!} + \dots$$

$$3 : \begin{cases} 1 : \text{terms for } \cos x \\ 2 : \text{terms for } f \\ 1 : \text{first three terms} \\ 1 : \text{general term} \end{cases}$$

- (b) $f'(0)$ is the coefficient of x in the Taylor series for f about $x = 0$, so $f'(0) = 0$.

$$\frac{f''(0)}{2!} = \frac{1}{4!} \text{ is the coefficient of } x^2 \text{ in the Taylor series for } f \text{ about}$$

$$x = 0, \text{ so } f''(0) = \frac{1}{12}.$$

Therefore, by the Second Derivative Test, f has a relative minimum at $x = 0$.

$$2 : \begin{cases} 1 : \text{determines } f'(0) \\ 1 : \text{answer with reason} \end{cases}$$

(c) $P_5(x) = 1 - \frac{x}{2} + \frac{x^3}{3 \cdot 4!} - \frac{x^5}{5 \cdot 6!}$

$$2 : \begin{cases} 1 : \text{two correct terms} \\ 1 : \text{remaining terms} \end{cases}$$

(d) $g(1) \approx 1 - \frac{1}{2} + \frac{1}{3 \cdot 4!} = \frac{37}{72}$

Since the Taylor series for g about $x = 0$ evaluated at $x = 1$ is alternating and the terms decrease in absolute value to 0, we know

$$\left| g(1) - \frac{37}{72} \right| < \frac{1}{5 \cdot 6!} < \frac{1}{6!}.$$

$$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{explanation} \end{cases}$$

2.

The Maclaurin series for the function f is given by $f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1}$ on its interval of convergence.

(a) Find the interval of convergence for the Maclaurin series of f . Justify your answer.

(b) Show that $y = f(x)$ is a solution to the differential equation $xy' - y = \frac{4x^2}{1+2x}$ for $|x| < R$, where R is the radius of convergence from part (a).

$$(a) \lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{(n+1)-1} \cdot \frac{n-1}{(2x)^n} \right| = \lim_{n \rightarrow \infty} \left| 2x \cdot \frac{n-1}{n} \right| = \lim_{n \rightarrow \infty} \left| 2x \cdot \frac{n-1}{n} \right| = |2x|$$

$$|2x| < 1 \text{ for } |x| < \frac{1}{2}$$

Therefore the radius of convergence is $\frac{1}{2}$.

$$\text{When } x = -\frac{1}{2}, \text{ the series is } \sum_{n=2}^{\infty} \frac{(-1)^n (-1)^n}{n-1} = \sum_{n=2}^{\infty} \frac{1}{n-1}.$$

This is the harmonic series, which diverges.

$$\text{When } x = \frac{1}{2}, \text{ the series is } \sum_{n=2}^{\infty} \frac{(-1)^n 1^n}{n-1} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1}.$$

This is the alternating harmonic series, which converges.

The interval of convergence for the Maclaurin series of f is $\left(-\frac{1}{2}, \frac{1}{2}\right]$.

5 : {
 1 : sets up ratio
 1 : limit evaluation
 1 : radius of convergence
 1 : considers both endpoints
 1 : analysis and interval of convergence

$$(b) y = \frac{(2x)^2}{1} - \frac{(2x)^3}{2} + \frac{(2x)^4}{3} - \dots + \frac{(-1)^n (2x)^n}{n-1} + \dots$$

$$= 4x^2 - 4x^3 + \frac{16}{3}x^4 - \dots + \frac{(-1)^n (2x)^n}{n-1} + \dots$$

$$y' = 8x - 12x^2 + \frac{64}{3}x^3 - \dots + \frac{(-1)^n n(2x)^{n-1} \cdot 2}{n-1} + \dots$$

$$xy' = 8x^2 - 12x^3 + \frac{64}{3}x^4 - \dots + \frac{(-1)^n n(2x)^n}{n-1} + \dots$$

$$xy' - y = 4x^2 - 8x^3 + 16x^4 - \dots + (-1)^n (2x)^n + \dots$$

$$= 4x^2 (1 - 2x + 4x^2 - \dots + (-1)^n (2x)^{n-2} + \dots)$$

The series $1 - 2x + 4x^2 - \dots + (-1)^n (2x)^{n-2} + \dots = \sum_{n=0}^{\infty} (-2x)^n$ is a

geometric series that converges to $\frac{1}{1+2x}$ for $|x| < \frac{1}{2}$. Therefore

$$xy' - y = 4x^2 \cdot \frac{1}{1+2x} \text{ for } |x| < \frac{1}{2}.$$

4 : {
 1 : series for y'
 1 : series for xy'
 1 : series for $xy' - y$
 1 : analysis with geometric series

3.

The function f is defined by the power series

$$f(x) = 1 + (x+1) + (x+1)^2 + \dots + (x+1)^n + \dots = \sum_{n=0}^{\infty} (x+1)^n$$

for all real numbers x for which the series converges.

- (a) Find the interval of convergence of the power series for f . Justify your answer.
- (b) The power series above is the Taylor series for f about $x = -1$. Find the sum of the series for f .
- (c) Let g be the function defined by $g(x) = \int_{-1}^x f(t) dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ cannot be determined.
- (d) Let h be the function defined by $h(x) = f(x^2 - 1)$. Find the first three nonzero terms and the general term of the Taylor series for h about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

- (a) The power series is geometric with ratio $(x+1)$.
The series converges if and only if $|x+1| < 1$.
Therefore, the interval of convergence is $-2 < x < 0$.

OR

$$\lim_{n \rightarrow \infty} \left| \frac{(x+1)^{n+1}}{(x+1)^n} \right| = |x+1| < 1 \text{ when } -2 < x < 0$$

At $x = -2$, the series is $\sum_{n=0}^{\infty} (-1)^n$, which diverges since the

terms do not converge to 0. At $x = 0$, the series is $\sum_{n=0}^{\infty} 1$,

which similarly diverges. Therefore, the interval of convergence is $-2 < x < 0$.

- (b) Since the series is geometric,

$$f(x) = \sum_{n=0}^{\infty} (x+1)^n = \frac{1}{1-(x+1)} = -\frac{1}{x} \text{ for } -2 < x < 0.$$

- (c) $g\left(-\frac{1}{2}\right) = \int_{-1}^{-\frac{1}{2}} -\frac{1}{x} dx = -\ln|x| \Big|_{x=-1}^{x=-\frac{1}{2}} = \ln 2$

- (d) $h(x) = f(x^2 - 1) = 1 + x^2 + x^4 + \dots + x^{2n} + \dots$

$$h\left(\frac{1}{2}\right) = f\left(-\frac{3}{4}\right) = \frac{4}{3}$$

- 3 : $\begin{cases} 1 : \text{identifies as geometric} \\ 1 : |x+1| < 1 \\ 1 : \text{interval of convergence} \end{cases}$

OR

- 3 : $\begin{cases} 1 : \text{sets up limit of ratio} \\ 1 : \text{radius of convergence} \\ 1 : \text{interval of convergence} \end{cases}$

1 : answer

- 2 : $\begin{cases} 1 : \text{antiderivative} \\ 1 : \text{value} \end{cases}$

- 3 : $\begin{cases} 1 : \text{first three terms} \\ 1 : \text{general term} \\ 1 : \text{value of } h\left(\frac{1}{2}\right) \end{cases}$

4.

Let f be the function given by $f(x) = \frac{2x}{1+x^2}$.

- (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.
- (b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.
- (c) The derivative of $\ln(1+x^2)$ is $\frac{2x}{1+x^2}$. Write the first four nonzero terms of the Taylor series for $\ln(1+x^2)$ about $x = 0$.
- (d) Use the series found in part (c) to find a rational number A such that $\left|A - \ln\left(\frac{5}{4}\right)\right| < \frac{1}{100}$. Justify your answer.

(a)
$$\frac{1}{1-u} = 1 + u + u^2 + \dots + u^n + \dots$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots + (-x^2)^n + \dots$$

$$\frac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + \dots + (-1)^n 2x^{2n+1} + \dots$$

3 : $\begin{cases} 1 : \text{two of the first four terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

- (b) No, the series does not converge when $x = 1$ because when $x = 1$, the terms of the series do not converge to 0.

1 : answer with reason

(c)
$$\ln(1+x^2) = \int_0^x \frac{2t}{1+t^2} dt$$

$$= \int_0^x (2t - 2t^3 + 2t^5 - 2t^7 + \dots) dt$$

$$= x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \dots$$

2 : $\begin{cases} 1 : \text{two of the first four terms} \\ 1 : \text{remaining terms} \end{cases}$

(d)
$$\ln\left(\frac{5}{4}\right) = \ln\left(1 + \frac{1}{4}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 + \frac{1}{3}\left(\frac{1}{2}\right)^6 - \frac{1}{4}\left(\frac{1}{2}\right)^8 + \dots$$

Let $A = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4 = \frac{7}{32}$.

Since the series is a converging alternating series and the absolute values of the individual terms decrease to 0,

$$\left|A - \ln\left(\frac{5}{4}\right)\right| < \left|\frac{1}{3}\left(\frac{1}{2}\right)^6\right| = \frac{1}{3} \cdot \frac{1}{64} < \frac{1}{100}.$$

3 : $\begin{cases} 1 : \text{uses } x = \frac{1}{2} \\ 1 : \text{value of } A \\ 1 : \text{justification} \end{cases}$