Chapter 8

Mathematics: Investigating and Connecting, Grades Nine through Twelve

Contents

Note to reader: The use of the non-binary, singular pronouns *they*, *them*, *their*, *theirs*, *themself*, and *themselves* in this framework is intentional.

Introduction: A Need for Change in High School

In its most recent publication directed at high-school mathematics, the largest advocacy organization for improving mathematics instruction in the United States, the National Council of Teachers of Mathematics (NCTM), has called for lasting and impactful change to occur, at all levels, by all stakeholders. The purpose for this call is simple:

> The steady improvement in mathematics learning seen since 1990 at the elementary and middle school levels has not been shared at the high school level, underscoring the critical need for change in mathematics education at the high school level.
>
> *Catalyzing Change in High School Mathematics* (NCTM, 2018)

Among the various findings that support the need for a call to action, the National Association of Educational Progress (NAEP) and the Programme for International Student Assessment (PISA) provide the most compelling data. For the past 15 years, grade-twelve NAEP scores have changed little, with an average score of 150 in 2005, 153 in 2013 and 152 in 2015 (Gurria, 2016).

On a longer, trend level, PISA results were similar. Fifteen-year-olds show an increase, from an average of 474 in 2006 to 487 in 2009, followed by a precipitous drop to 470 in 2015. Thus, the gains that the United States achieved from 2006 to 2012 have disappeared by 2015.

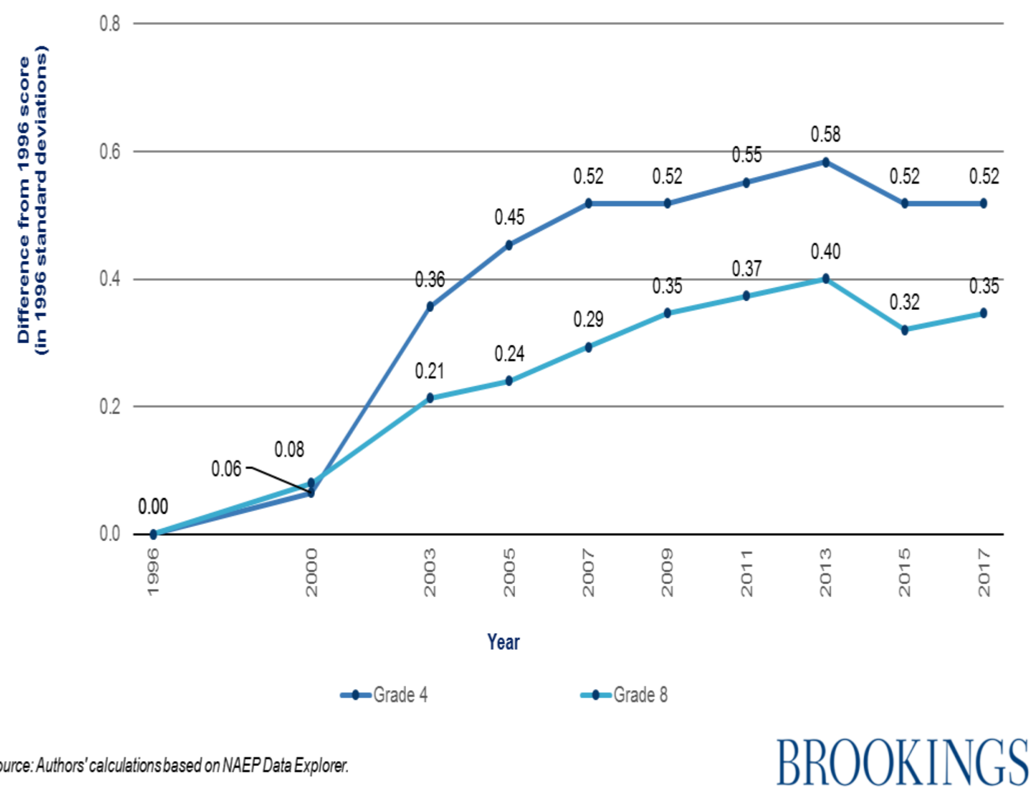
U.S. scores on PISA, 15-year-olds (2000–2015) TABLE 1-1

	2000	2003	2006	2009	2012	2015
Reading	504	495	—	500	498	497
Math		483	474	487	481	470
Science			489	502	497	496

Source: PISA 2015 Results (Volume I) Excellence and Equity in Education, Table I.4a (Reading); Table I.2.4a; Table I.5.4a (Math).

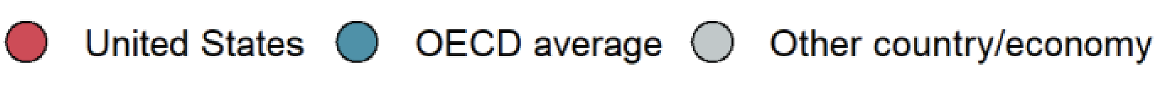
The NAEP results for grades 4 and 8 indicate a steady improvement trend since 1990, with a leveling out occurring in the past 10 years but not an overall decrease (NAEP, 2015).

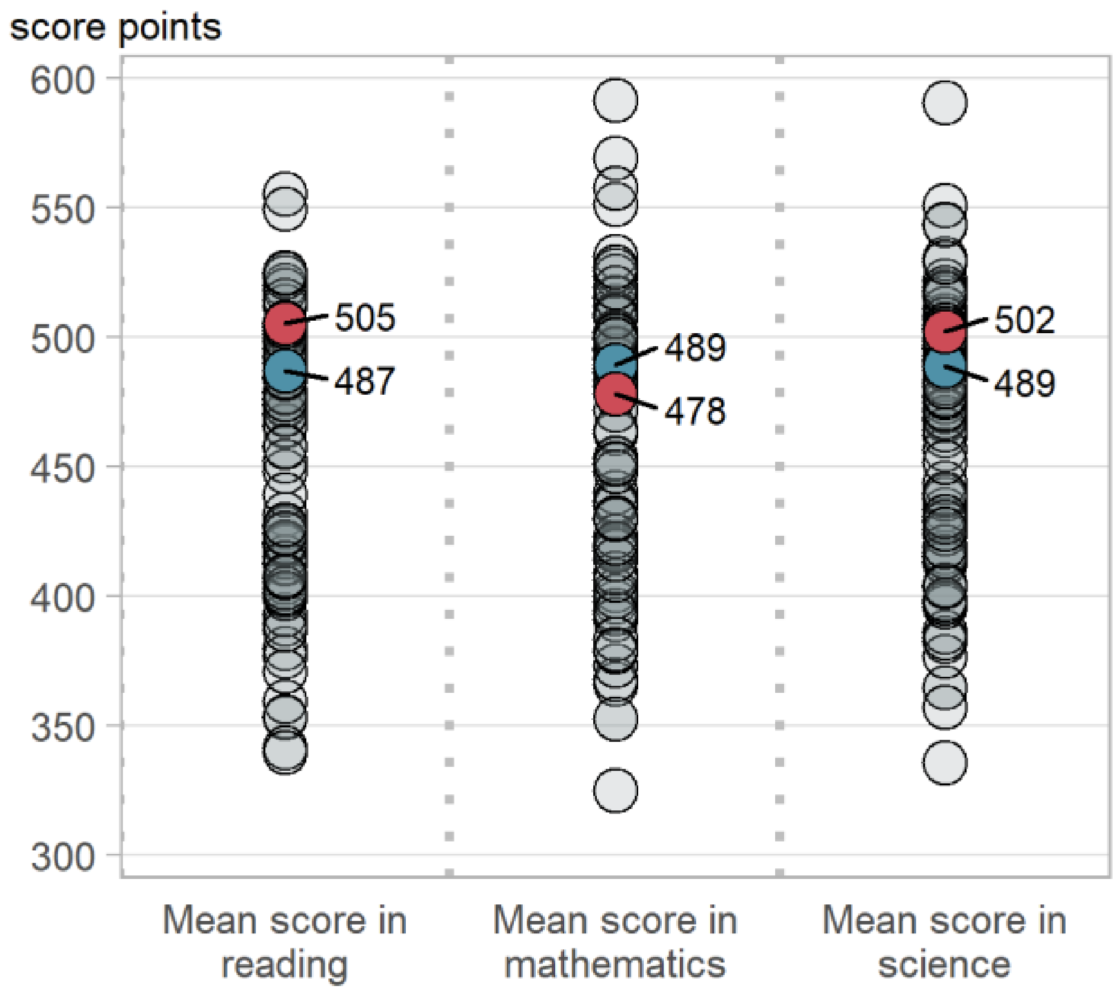
Figure 1: NAEP math score trends for grades 4 and 8



Source: Authors' calculations based on NAEP Data Explorer.

When compared to other countries, 15-year-olds from the U.S. achieved less than the global average of all participating countries (Schleicher, 2019). The graph below shows data from the United States and from all the countries that took part in the PISA tests—labelled OECD (Organization for Economic Co-operation and Development).





https://www.oecd.org/pisa/publications/PISA2018_CN_USA.pdf

Transition from Eighth Grade to High School

Ample research demonstrates the importance of grade nine for students’ future academic success. Finkelstein and Fong (2008) find that students who exit or do not receive the adequate support to remain on the college-preparatory track early in high school tend to fall farther behind and are less likely to complete a college-preparatory program as they progress through high school.

The grade-eight standards in the California Common Core State Standards for Mathematics (CA CCSSM) are significantly more rigorous than the previous Algebra I standards. The CA CCSSM for grade eight address the foundations of algebra by including content that was previously part of the Algebra I course—such as more in-depth study of linear relationships and equations, a more formal treatment of functions, and the exploration of irrational numbers. For example, by the end of the CA CCSSM for grade eight, students will have applied graphical and algebraic methods to analyze and solve systems of linear equations in two variables. The CA CCSSM for grade eight also include geometry standards that relate graphing to algebra in a new way—one that was not explored previously. Additionally, the statistics content in the CA CCSSM for grade eight are more sophisticated than those previously included in middle school and connect linear relations with the representation of bivariate data. (See Chapter Five for more discussion of this relationship.)

The CA CCSSM Mathematics I and Algebra I courses build on the CA CCSSM for grade eight and are therefore more advanced than they were prior to adoption of the CA CCSSM. Because many of the topics included in the former Algebra I course are in the CA CCSSM for grade eight, the Mathematics I and Algebra I courses typically start in ninth grade with more advanced topics, and include more in-depth work with linear functions and exponential functions and relationships, and go beyond the previous high-school standards for statistics. Since grade eight in CA CCSSM is designed to be integrated, Mathematics I builds directly on the CA CCSSM for grade eight, and provides a seamless transition of content through an integrated curriculum.

In order to support students to succeed in Mathematics I or Algebra I, schools have adopted a variety of approaches that have been more beneficial than remediating 8th grade mathematics over again. In 2017, Louisiana developed an Intensive Algebra I pilot in which students enrolled in two periods of Algebra I, with the same teacher for both periods, using curriculum that interwove foundational mathematics and algebra content together. The extended time, and additional supports for teachers were critical to the success of the project. Academic support courses for high school mathematics has been shown as effective in a number of studies (various studies described in https://www2.ed.gov/rschstat/eval/high-school/academic-support.pdf). The support courses are offered to provide additional time for: classroom instruction (as in the case of the Louisiana project above), homework support and supplemental assignments, emphasizing study skills and preparation in the core companion courses. There are a number of curricula that offer support course materials; for example, Illustrative Mathematics https://im.kendallhunt.com/HS/teachers/4/narrative.html.

Issues with Acceleration in Middle Grades

With knowledge of the rigor of the CA CCSSM for grade eight, educators must calibrate course sequencing to ensure students are able to learn the additional content. Specifically, students who previously may have been able to succeed in an Algebra I course in grade eight may find the new CA CCSSM for grade-eight content significantly more difficult. The CA CCSSM provides for strengthened conceptual understanding by encouraging students—even strong mathematics students—to take the grade-eight CA CCSSM course instead of opting for Algebra I or Mathematics I in grade eight.

Many students, parents, and teachers encourage acceleration in grade eight (or sooner in some cases) because of mistaken beliefs that Calculus is an important high school goal. This misinformation leads them to believe Algebra I must be taken in grade eight in order for the student to reach a calculus class in grade twelve. This framework clarifies these misunderstandings in three ways:

- First, because of the rigorous nature of the CA CCSSM grade-eight standards, a three-year high-school pathway can be sufficient preparation for a calculus class in grade twelve, as outlined in the pathway graphic on page x (to be decided by formatting)
- Second, the push to calculus in grade twelve is itself misguided. In Mathematical Association of America (MAA) and NCTM clarify that "...the ultimate goal of the K–12 mathematics curriculum should not be to get into and through a course of calculus by twelfth grade, but to have established the mathematical foundation that will enable students to pursue whatever course of study interests them when they get to college" (2012). The push to enroll more students in high school calculus often leads to shortchanging important content that does not lead directly to success in the advanced placement calculus syllabus, which is significantly procedural. "In some sense, the worst preparation a student heading toward a career in science or engineering could receive is one that rushes toward accumulation of problem-solving abilities in calculus while short-changing the

broader preparation needed for success beyond calculus" (Bressoud, Mesa, and Rasmussen 2015).

- Finally, the results do not support the push for more and more students to take calculus in high school: About half of the students taking Calculus I in college are repeating their high school course, and many others place into a *pre-calculus* course when they enter college (Bressoud, Mesa, and Rasmussen 2015).

The rapid expansion of calculus, at the expense of other important mathematics, reflects troubling realities of college admission, which colleges and universities are beginning to address partly in response to the MAA and NCTM joint statement (see for example, Mejia, Rodriguez, & Johnson, 2016). The UC/CSU systems also recognize a need for students to think more broadly, and positively, in mathematics. In the Statement of Competencies in Mathematics Expected for Entering College Students, students are expected to view mathematics as an endeavour which makes sense, demonstrate a willingness to work on problems requiring time and thought, communicate ideas with peers and build a "perception of mathematics as a unified field of study—students should see interconnections among various areas of mathematics, which are often perceived as distinct." p. 4 In addition, the need for students to engage in meaningful problem solving with unfamiliar problems to develop open, inquiring, and demanding minds with the confidence to approach novel situations with adaptability, insight, and creativity. (https://icas-ca.org/wp-content/uploads/2020/05/ICAS-Statement-Math-Competencies-2013.pdf)

Focusing on Essential Concepts

This framework draws on many sources that reflect the current state of high-school mathematics and research about effective practices. These include NCTM's *Catalyzing Change in High School Mathematics: Initiating Critical Conversations* (NCTM, 2018), and Just Equations' report on designing high school mathematics for equity, *Branching Out: Designing High School Math Pathways for Equity* (Daro & Asturias 2019).

NCTM (2018) advances four key recommendations with regard to effecting needed change at the high school level:

- Each and every student should learn the Essential Concepts (a focused set of 41 concepts for high school) in order to expand professional opportunities, understand and critique the world, and experience the joy, wonder, and beauty of mathematics.

Essential Concepts in High School Mathematics

Essential Concepts in Number

Essential Concepts in Algebra and Functions

Focus 1: Algebra

Focus 2: Connecting Algebra to Functions

Focus 3: Functions

Essential Concepts in Statistics and Probability

Focus 1: Quantitative Literacy

Focus 2: Visualizing and Summarizing Data

Focus 3: Statistical Inference

Focus 4: Probability

Essential Concepts in Geometry and Measurement

Focus 1: Measurement

Focus 2: Transformations

Focus 3: Geometric Arguments, Reasoning, and Proof

Focus 4: Solving Applied Problems and Modeling in Geometry

Source: www.nctm.org/uploadedFiles/Standards_and_Positions/executive%20summary.pdf

- High school mathematics should discontinue the practice of tracking teachers as well as the practice of tracking students into qualitatively different pathways or into courses that have no follow up.
- Classroom instruction should be consistent with research-informed and equitable teaching practices, such as those described in Chapter 2.

- High schools should offer continuous four-year mathematics pathways with all students studying mathematics each year, including two to three years of mathematics in a common shared pathway focusing on the Essential Concepts, to ensure the highest-quality mathematics education for all students.

Each of these is of critical importance in addressing the barriers to growth in math learning at California high schools. Practical, beautiful, and unifying ideas should be the drivers for each unit, lesson, and activity that students encounter. Tracking students into pathways for which they are unable to take, or even succeed in, other courses is a practice which must stop. And equitable teaching should utilize research-informed strategies, such as those recommended in Chapter 2.

NCTM’s last recommendation, that students transitioning from eighth grade to high school should expect to undertake four-year pathways which include multiple years of courses that are taken in common with their peers, is of paramount importance. The ninth-grade year is widely considered to be the most critical year of a student’s high school and beyond trajectory. Neild, Stoner-Eby, and Furstenberg (2008) conclude that the experience of the ninth-grade year contributes substantially to the probability of dropping out of high school, even after controlling for eighth grade academic performance and pre-high school attitudes and ambitions. If students are to be accelerated, then this should occur only after grade nine.

Similarly, in *Branching Out: Designing High School Math Pathways for Equity* (Daro & Asturias 2019), the authors call for multiple pathways in high school for students, rather than tracks for students with little opportunity to “jump tracks”. The report also challenges the notion of STEM vs Non-STEM as a useful binary paradigm for classifying career goals. There are many careers that do not fit this paradigm; according to the report, these are known as BRANCH fields, and include occupations such as “journalist, elected official, high school principal, marketing executive, attorney, game designer, first responder, movie producer, or stockbroker” (p. 8). (Note that while BRANCH itself is not an acronym, the all-capitals are used to indicate that these pathways are as rigorous as STEM pathways.) In designing new BRANCH math pathways, the report outlines goals:

1. STEM-interested students will be able to learn the mathematics that prepares them for STEM careers.
2. BRANCH-interested students will be able to learn the mathematics that prepares them for BRANCH careers without being blocked by irrelevant requirements.
3. Latinx and African American students will have ample opportunities to thrive in college, including in STEM fields, as will female students of all ethnicities.
4. Students who initially choose a BRANCH pathway will be able to switch to a STEM pathway during high school or college, and vice versa, if their interests change.

Exclusionary Math

In his 2020 book *Mathematics for Human Flourishing*, Francis Su describes experiences of exclusion from the mathematics community, in both school mathematics and the professional mathematics community.

> We are not educating ourselves as well as we should, and like most injustices, this especially harms the most vulnerable. Lack of access to mathematics and lack of welcome in mathematics have had devastating consequences. (Su, 2020)

The devastating consequences to which Su refers have particularly harmed students of color and those from low-income communities and other disadvantaged groups. PISA results corroborate Dr. Su's experiences and insight. In the PISA 2018 test, socio-economic status was a strong predictor of performance in mathematics in the United States. It explained 16 percent of the variation in mathematics performance in the United States versus 14 percent on average across all participating countries (PISA, 2018).

This raises the importance of mindset and belonging messages being given to high school students, especially those who have developed the idea that only some people are "math people" and that their brains are fixed (incapable of growth). It is crucial to share with students that struggle is the best time for brains and that they should embrace times of cognitive challenge. It is equally important to share that brains are not fixed and that any times of learning create opportunities for brain growth, connections, and strengthening of pathways. As mathematics has developed in such exclusive and elitist ways, it is also important to share with students examples of women and people of color who are successful mathematicians. Youcubed.org has many films that can be shared with students, sharing these messages and examples of people.

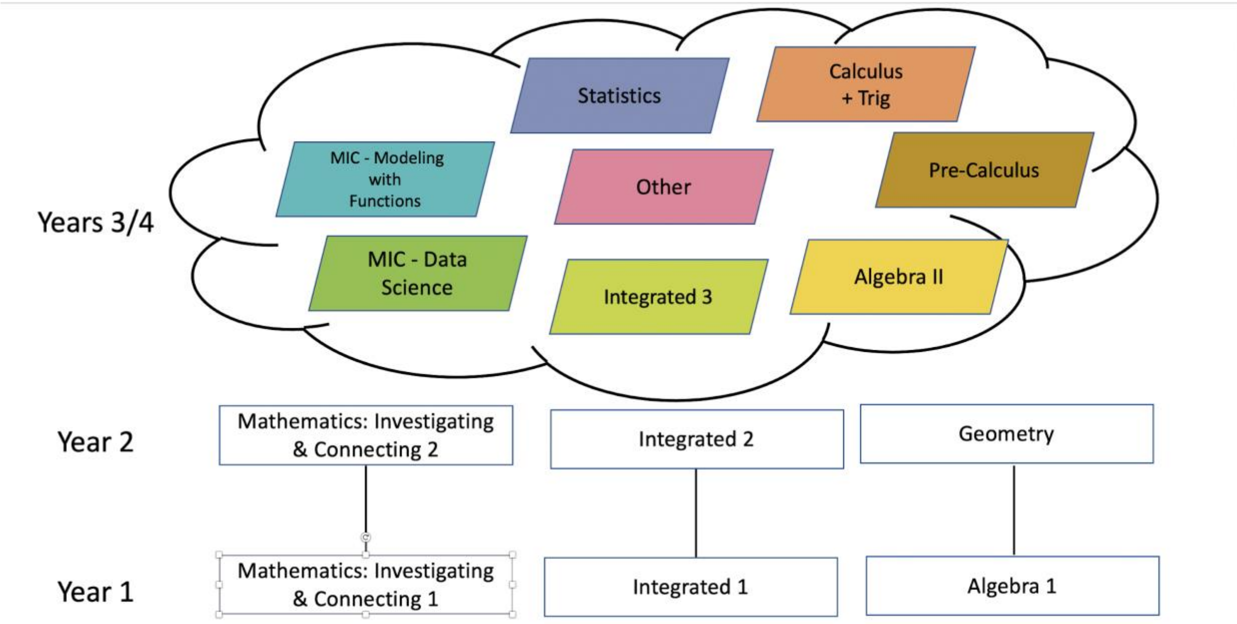
California schools must actively work to counteract the many forces that filter and exclude so many from mathematically-intense pursuits. It is well established that “much of what happens in the classroom is determined by a cultural code that functions, in some ways, like the DNA of teaching” (Stigler & Hiebert, 2009), and that changing what happens is remarkably difficult, even for teachers and departments that are committed to changing practice in order to right these historic injustices (Louie, 2017). Further, research has shown that when high school mathematics is taught in a narrow, procedural way students develop narrow and binary perceptions of both the curriculum (strongly like it or strongly dislike it), and of each other, leading to classroom inequalities (LaMar, Leshin & Boaler, 2020).

While the adoption of the CA CCSSM has provided a basis upon which to effect changes in equitable instruction TK–8, this change has been slower to come for high school. However, from 10 years’ experience with the CA CCSSM, California high schools are positioned to lead a movement towards greater inclusion and equity in mathematical sciences.

The word “inclusion” is used in this chapter to describe both a value and approaches to teaching (Roos, 2019). The value, that all California students deserve high-quality high school mathematics experiences that enable them to be powerful users of mathematics to understand and affect their world, is put into action by the approach to teaching—teaching methods, curricular materials, and approaches to mathematics that are designed to actively disrupt cultural patterns that perpetuate inequity, and to authentically engage students from all backgrounds.

Pathways in Grades 9–12

While the ninth-grade year has been shown to be of critical importance in establishing progress toward graduation, grades eleven and twelve are important as well. The graphic below indicates possible pathways for high school coursework, reflecting a common ninth and tenth grade experience, and a broader array of options in eleventh and twelfth grade.



By completing Mathematics: Investigating and Connecting 1 and 2, Mathematics I and II, or Algebra and Geometry, students will be satisfying the requirements of California Assembly Bill 220 that states that students complete two mathematics courses in order to receive a diploma of graduation from high school, with at least one course meeting the rigor of Algebra 1. Depending upon their post-secondary goals, students may choose different third- and fourth-year courses. For example, a student who is planning upon working in a fabrication shop upon high school graduation may choose to follow Mathematics I and II with a course in Modeling to help understand the mathematics of die-casting and 3-d printing. Or a student who is planning to study political science may choose a Data Science course in their third year and a Statistics course in their fourth year to understand the mathematics behind polling, apportionment and the implications of gerrymandering.

Should students decide to switch pathways (at high schools which offer multiple pathways), there is an increasing amount of flexibility afforded to those planning to enter a university upon graduation, in terms of which courses “count” for admission. In

October 2020, the University of California system updated the mathematics (area C) course criteria and guidelines for the 2021–22 school year and beyond. An exciting change in this update is the allowance of courses in data science to serve as the required third year of mathematics coursework. In the diagram above, Mathematics: Investigating and Connecting (MIC) – Data Science meets the criteria above and so fulfills the required third year, since MIC – Data substantially aligns with CACCSSM (+) standards. The MIC pathway is described later in this chapter. For additional information on Data Science, see Chapter 5.

Overall, the revisions are to:

- Clarify UC's expectation for college-prep mathematics courses that will help students acquire specific skills to master the subject's content and also gain proficiency in quantitative thinking and analysis;
- Support the efforts of high schools to develop and implement multiple college-prep mathematics course options for students;
- Encourage the submission of a broader range of advanced/honors math courses (e.g., Statistics, Introduction to Data Science) for area C approval.

Key highlights of the policy updates:

- Courses that substantially align with Common Core (+) standards (see chapters on *Higher Mathematics Courses: Advanced Mathematics* and *Higher Mathematics Standards by Conceptual Category* in Common Core Standards for Mathematical Practice (SMPs) https://www.cde.ca.gov/BE/st/ss/documents/ccssmathstandardaug2013.pdf), and are intended for 11th and/or 12th grade levels are eligible for area C approval and may satisfy the required third year or recommended fourth year of the mathematics subject requirement if approved as an advanced mathematics course.
    - Examples of such courses include, but are not limited to, applied mathematics, computer science, data science, pre-calculus, probability, statistics, and trigonometry.

- Courses eligible for UC honors designation must integrate, deepen, and support further development of core mathematical competencies. Such courses will address primarily the (+) standards of Common Core-aligned advanced mathematics (e.g., statistics, pre-calculus, calculus, or discrete mathematics).

The entire revised mathematics (area C) course criteria are located at https://hs-articulation.ucop.edu/guide/a-g-subject-requirements/c-mathematics/

The California State University (CSU) system has developed several courses for the fourth year of high school (and some for earlier grades) which meet the Area C (Mathematics) requirement for admission to the CSU. The CSU Bridge Courses page (http://cmrci.csu-eppsp.org/) lists mathematics/quantitative courses and projects working within the CSU system focused on supporting mathematics and quantitative reasoning readiness among K–12, CSU, and community-college educators. The courses developed have a variety of emphases, including modeling, inference, voting, informatics, financial decision making, introduction to basic calculus concepts, connections among topics, theory of games, cryptography, combinatorics, graph theory, and connecting statistics with algebra. These courses have been adopted throughout the state in coordination with district and school initiatives to increase the variety of rich high school mathematics coursework at the upper grade levels.

Note

The Just Equations Report *Branching Out: Designing High School Math Pathways for Equity* tackles several aspects of the traditional calculus pathway that has led to highly unequal opportunity for California students, and to very inequitable outcomes. The provision of alternative pathways is expected to broaden opportunities for students, increase interest in a wider range of students, and result in much more diverse participation in Science, Technology, Engineering, and Mathematics (STEM) pathways (LaMar, Leshin & Boaler, 2020).

Mathematics: Investigating and Connecting Pathway

Definition of Integration

There are multiple contexts for which the term “integrated” has been used in connection with mathematics education. In this chapter, “integrated” will refer to both the connecting of mathematics with students’ lives and their perspectives on the world, and to the connecting of mathematical concepts to each other. This reference to both can result in a more coherent understanding of mathematics. Integrated tasks, activities, projects, and problems are those which invite students to engage in both of these aspects of integration.

The integration of mathematical topics into authentic problems that draw from different areas of mathematics has been shown to increase engagement and achievement (Grouws et al, 2013). Some districts, in recent years, moved towards the integration of content by offering integrated courses but the textbooks they chose did not truly integrate mathematical concepts, instead interspersing chapters of algebra and geometry. This framework offers an approach that is conceptually integrated. The districts that moved to integrated courses—even when the content was not integrated—have course structures in place that will allow a smooth transition to this new, truly integrated approach, that is centered around broad ideas and meaningful engagement. Other districts teaching algebra and geometry may consider a move to the conceptually integrated approach that has been shown to increase engagement and understanding.

Children are naturally curious about the world in which they live, and this curiosity fuels their desire to wonder, describe, understand, and ask questions about their world. In a similar way, new mathematics is developed through attempts to describe, to understand, and to answer questions. Mathematics provides a set of lenses for viewing, describing, understanding, and analyzing phenomena, as well as solving problems, such as local issues related to environmental and social justice, through engineering design practices(CA NGSS HS-ETS1-2)—which might occur in the “real world” or in abstract settings such as within mathematics itself. For instance, finance, the environment, and science all offer phenomena, such as recurrent patterns or atypical cases, which are better understood through mathematical tools; such phenomena also arise *within* mathematics (see Chapter Four: *Exploring, Discovering, and Reasoning with and About Mathematics*, for instance).

However, mathematics is never developed in order to answer questions about which the explorer is *not* curious; and *learning* mathematics is not much different. By experiencing the ways in which mathematics can answer natural questions about their world, both in school and outside of it, a student’s perspectives on both mathematics and their world are integrated into a connected whole.

Motivation for Integration

Critique the effectiveness of your lesson, not by what answers students give, but by what questions they ask.

—Fawn Nguyen (2016), Mesa Union School District, junior high mathematics teacher

The Mathematics: Investigating and Connecting (MIC) pathway described here (implementing the content standards laid out in the CA CCSSM) emphasizes both aspects of integration: opportunities which are relevant to students and their experiences, and opportunities to connect different mathematical ideas. In keeping with the thrust of this framework, curriculum and instruction should take both of these into account. A guiding question for measuring these two aspects in classroom activities is, “Can I see evidence that students wonder about questions that will help to motivate learning of mathematics and that connect this learning to other knowledge?”

As has been mentioned previously, there are several studies which have documented the disproportionately negative impacts of mathematics on students of color when teaching approaches are largely procedural (e.g. Louie, 2017), and, more specifically, the negative impact 8th grade algebra has upon students of color (Domina, et al. 2015). Integrated approaches, such as Mathematics: Investigating and Connecting and the Integrated pathway, focused on the use of inclusive teaching practices, such as those described in Chapter 2, allow more equitable access to authentic mathematics for all students, and necessitate a view that mathematics is a beautiful and connected subject, both internally and to the greater world around it.

Designing Integration

The primary challenge for the design of any high-school pathway is to bridge the gap between the CA CCSSM's lists of critical content goals, on the one hand, and the difficult tasks facing teachers every day in helping their students to see mathematics as a subject of connected, meaningful ideas, and to become powerful users of mathematics to understand and affect their world. The *Mathematics: Investigating and Connecting* pathway presents one possible embedding of the CA CCSSM content into experience-based contexts designed to necessitate mathematics, so that mathematical content is experienced by students as tools for answering authentic questions.

The courses *Mathematics: Investigating and Connecting 1* and *Mathematics: Investigating and Connecting 2* are implementations of the Integrated Math I and Integrated Math II sample content outlines in the CA CCSSM (with some data clusters moved from Integrated Math III into MIC 1 and MIC 2). The Mathematics: Investigating and Connecting pathway has two options for advanced (years 3 and 4) courses: *Mathematics: Investigating and Connecting—Data Science* (MIC—Data) and *Mathematics: Investigating and Connecting—Functions and Modeling* (MIC—Modeling). MIC—Data and MIC —Modeling emphasize different types of investigations to frame student activities, and distribute student effort differently between the various Conceptual Categories of the CA CCSSM.

As described in Chapter 2: Teaching for Equity and Engagement, it is important that exploration and question-posing occur *prior to* teachers telling students about questions to explore, methods to use, or solution paths. A compelling experimental research study compared students who learned calculus actively, when they were given problems to explore before being shown methods, to students who received lectures followed by solving the same problems as the active learners (Deslauriers, McCarty, Miller, Callaghan, & Kestin, 2019). The students who explored the problems first learned significantly more (see also Schwartz & Bransford, 1998). However, despite their increased learning, the students believed that the lecture approach was more effective—as the active learning condition caused them to experience more challenge and uncertainty. The study not only showed the effectiveness of students exploring problems before being taught methods, but the value of sharing with students the importance of struggle and of thinking about mathematics problems deeply.

Other research examines beliefs and attitudes such as utility value (belief that mathematics is relevant to personal goals and to societal problems), and this research shows a severe drop-off in utility value during high school (Chouinard & Roy, 2008). However, teaching methods that increase connections between course content and students' lives, and that include careful focus on effective groupwork, can significantly increase utility value for students (Cabana, Shreve & Woodbury, 2014; Boaler, 2016a, 2016b, 2019; Hulleman, Kosovich, Barron, & Daniel, 2017; LaMar, Leshin & Boaler, 2020).

Driving Investigations and Connections

Since motivating students to care about the mathematics is crucial to forming meaningful content connections, the Mathematics: Investigating and Connecting pathway (abbreviated MIC below) identifies three **Drivers of Investigation**, which provide the "why" of learning mathematics, to pair with four categories of **Content Connections** (CCs), which provide the "how and what" mathematics (the high school CA CCSSM standards) to be learned in an activity. So, the DIs propel the learning of the content framed in the CCs.

Drivers of Investigation (DIs)

The Content Connections should be developed through investigation of questions in authentic contexts; these investigations will naturally fall into one or more of the following Drivers of Investigation. The DIs are meant to serve a purpose similar to that of the Crosscutting Concepts in the CA-NGSS, as unifying reasons that both elicit curiosity and provide the motivation for deeply engaging with authentic mathematics. In practical use, teachers can use these to frame questions or activities at the outset for the class period, the week, or longer; or refer to these in the middle of an investigation (perhaps in response to the "Why are we doing this again?" questions that often crop up), or circle back to these at the conclusion of an activity to help students see "why it all matters". Their purpose is to pique and leverage students' innate wonder about the world, the future of the world, and their role in that future, in order to foster a deeper understanding of the Content Connections and grow into a perspective that mathematics itself is a lively, flexible endeavor by which we can appreciate and understand so much of the inner workings of our world. The DIs are:

- DI 1: Making Sense of the World (Understand and Explain)
- DI 2: Predicting What Could Happen (Predict)
- DI 3: Impacting the Future (Affect)

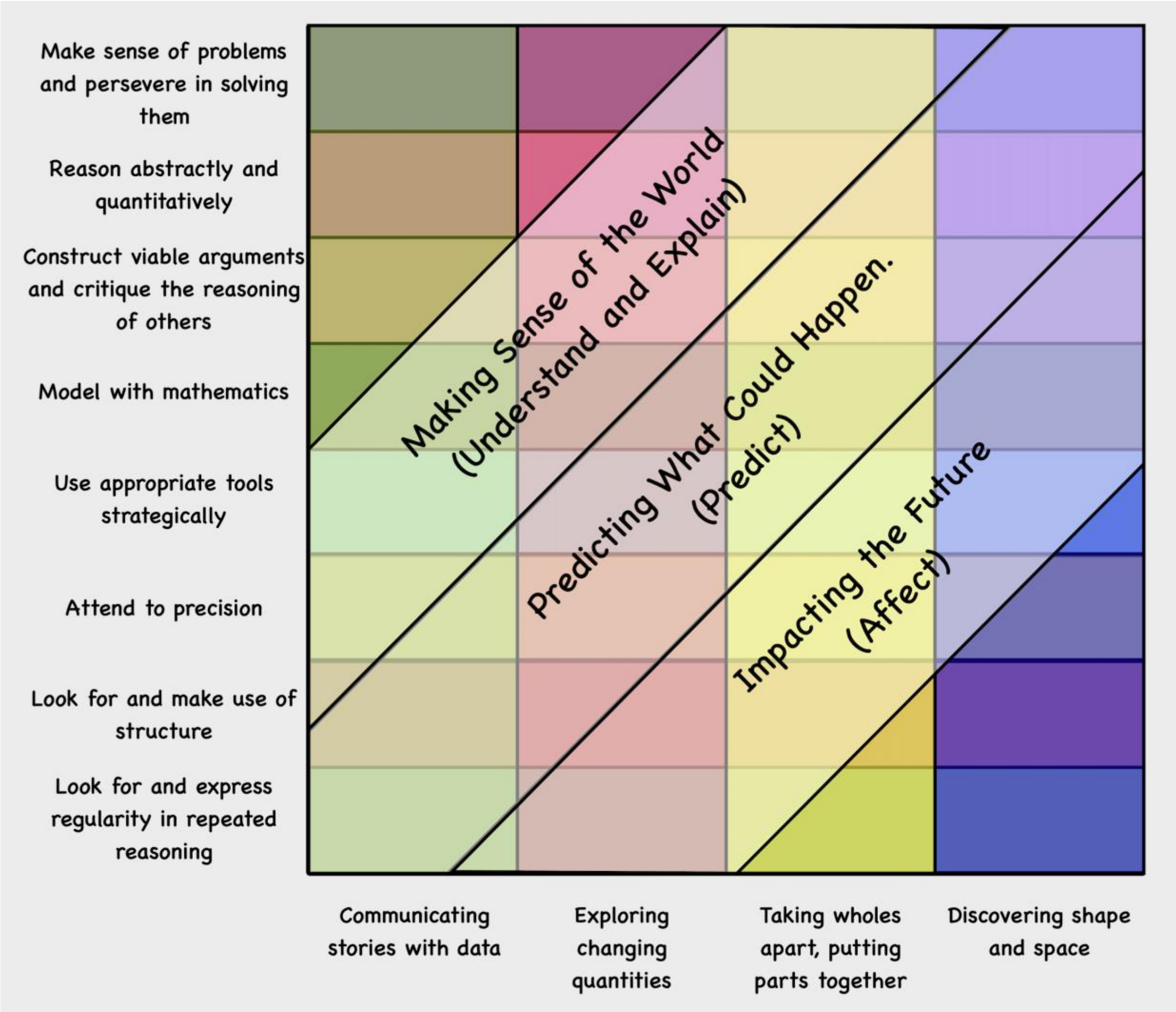
Content Connections (CCs)

The four Content Connections described in the framework organize content and provide mathematical coherence through the grades:

CC1: Communicating Stories with Data
CC2: Exploring Changing Quantities
CC3: Taking Wholes Apart, Putting Parts Together
CC4: Discovering Shape and Space

Big ideas that drive design of instructional activities will link one or more Content Connections, and SMPs, with a Driver of Investigation, so that students can Communicate Stories with Data *in order to* Predict What Could Happen, or Illuminate Changing Quantities *in order to* Impact the Future. The aim of the drivers of investigation is to ensure that there is always a reason to care about mathematical work -and that investigations allow students to make sense, predict, and/or affect the world. The following diagram is meant to illustrate the ways that the drivers of investigation relate to content connections and practices, as cross cutting themes. Any driver of investigation could go with any set of content and practices:

Figure 1: Content connections, Mathematical Practices and Drivers of Investigation



Instructional materials should primarily involve tasks that invite students to make sense of these big ideas, elicit wondering in authentic contexts, and necessitate mathematical investigation. Big ideas in math are central to the learning of mathematics, link numerous mathematical understandings into a coherent whole, and provide focal points for students' investigations. An authentic activity or problem is one in which students investigate or struggle with situations or questions about which they actually wonder. Lesson design should be built to elicit that wondering.

This framing helps teachers and curriculum writers to focus on big ideas (see Chapter 2 and Cabana, Shreve & Woodbury, 2014). It is similar to the way that the Next Generation Science Standards' seven Cross-cutting Concepts serve as themes which span multiple grades and are present in the various sciences.

Within each category, students' experiences should first emerge out of exploration or problems that incorporate student problem-posing (Cai & Hwang, 2019). Meaningful student engagement in identifying problems of interest helps increase engagement even in subsequent teacher-identified problems. Identifying contexts and problems *before* solution methods are known makes explorations more authentically problematic for students, as opposed to simply exercises to practice previously learned exercise-solving paths.

A well-known example of the difference between a stereotypical use of problems and the one assumed in this pathway is described in Dan Meyer's TED talk (Meyer, 2010): Meyer's considers a standard textbook problem about a cylindrical tank filling from a hose at a constant rate. The textbook provides several sub-steps (area of the base, volume of the tank), and the final question "How long will it take to fill the tank?" The task appears at the end of a chapter in which all the mathematical tools to solve the problem are covered; thus, students experience the task as an exercise, not an authentic problem.

In the problem-based technique advocated here, the tank-filling context is presented *prior* to any introduction of methods or a general class of problems, in some way that authentically raises the question, "how long will it take to fill?" and preferably in a way that has a meaningful answer available for a check (e.g., a video of the entire tank-filling process, as in the TED Talk). After the question has been raised (hopefully by students), students make some estimates, and then the development of the necessary mathematics is seen as having a purpose. Viewing the end of the video prompts meta-thinking about process—why is our answer different than the video shows?—much more effectively than a "check your work" prompt or a comparison with the answer in the back of the book. This tank-filling problem could occur in the "Exploring Changing Quantities" Content Connection of MIC I. Note that the problem integrates linear function and geometry standards.

As this example shows, the problem-embedded learning envisioned in this pathway does not imply a curriculum in which all learning takes place in the context of large, multi-week projects, though that is one approach that some curricula pursue. Problems and activities that emphasize an integrated approach as outlined here can also be incorporated into instruction in short time increments, such as 45-minute lessons or even in shorter routines such as Think-Pair-Share, or Math Talks (see Chapter 3). There are a number of lesson plan formats which take a problem-embedded approach, including one from LA Unified School District which adopts the Three-Phase Approach advocated by Dan Meyer. https://achieve.lausd.net/cms/lib/CA01000043/Centricity/domain/335/lessons/integrated%20math/integrated%20math%20pd/Three-PhaseLessonStructure.pdf

A more extensive investigation that cuts across several Content Connections is illustrated in this climate change vignette.

Vignette: Exploring Climate Change

Course: MIC1 / Integrated Math 1

Background Reading on Climate Change

With the beginning of the Industrial Revolution of the in the mid-1700s, the world began to see many changes in the production of goods, the work people did on a daily basis, the overall economy and, from an environmental perspective, the balance of the carbon cycle. The location and distribution of carbon began to shift as a result of the Industrial Revolution, and have continued to change over the last 250 years as a result of the growing consumption of fossil fuels, industrialization, and several other societal shifts. During this time, the distribution of carbon among Earth's principal reservoirs (atmosphere; the oceans; terrestrial plants; and rocks, soils, and sediments) has changed substantially. Carbon that was once located in the rock, soil, and sediment "reservoir," for example, was extracted and used as fossil fuels in the forms of coal and oil to run machinery, heat homes, and power automobiles, buses, trains, and tractors. [This provides a good opportunity for discussing and reinforcing California Environmental Principle IV. "The exchange of matter between natural systems and human societies affects the long-term functioning of both."] (Supporting materials are available in EEI Curriculum units *Britain Solves a Problem and Creates the Industrial*

Revolution and *The Life and Times of Carbon*, available at no charge from https://californiaeei.org/curriculum)

Before the Industrial Revolution, the input and output of carbon among the carbon reservoirs was more or less balanced, although it certainly changed incrementally over time. As a result of this balance, during the 10,000 years prior to industrialization, atmospheric CO_2 concentrations stayed between 260 and 280 parts per million (ppm). Over the past 250 years human population growth and societal changes have resulted in increased use of fossil fuels, dramatic increase in energy generation and consumption, cement production, deforestation and other land-use changes. As a result, the global average amount of carbon dioxide hit a new record high of 407.4 ppm in 2018—with the annual rate of increase over the past 60 years approximately 100 times faster than previously recorded natural increases.

The "greenhouse effect" impacts of rising atmospheric CO_2 concentrations are diverse and global in distribution and scale. In addition to melting glaciers and ice sheets that many people are becoming aware of, the impacts will include sea level rise, diminishing availability of fresh water, increased number and frequency of extreme weather events, changes to ecosystems, changes to the chemistry of oceans, reductions in agricultural production, and both direct and indirect effects on human health. [This offers a good opportunity to reinforce California Environmental Principle II. "The long-term functioning and health of terrestrial, freshwater, coastal and marine ecosystems are influenced by their relationships with human societies."]

You may visit https://www.climate.gov for more information.

Mathematics/Science/English Languages Arts/Literacy (ELA) Task:

Determine the relative contributions of each of the major greenhouse gases and which is the greatest contributor to the global greenhouse effect and, therefore, should be given the highest priority for policy changes and governmental action. Examine the growth patterns of related human activities and their relative contributions to release of the most influential greenhouse gas. Based on these factors, analyze the key components of the growth patterns and propose a plan that would reduce the human-source release of that greenhouse gas by at least 25–50%, and determine how that change would influence the rate of global temperature change.

Classroom Narrative:

Mathematics, science and language arts teachers met to co-plan this interdisciplinary task. They each felt that the task was challenging and authentic, requiring students to draw from different disciplines to forge a solution, just as we do in the real world. They developed a sequence of activities to get the students started, being careful not to over-scaffold the task or to give students too much guidance toward possible solutions pathways, but ensuring their work supplemented and supported the larger task.

Launch: Student teams are provided with the task and then read the article "Climate Change in the Golden State" (http://www.cde.ca.gov/ci/sc/cf/ch8.asp#link68) to gather evidence about the scale and scope of the effects of climate changes in California. As this is an extended text, the English language arts teacher provides an interactive note-taking guide for students to use. Students highlight parts that are not clear, they note important claims made by the authors, and formulate their own questions to share in groups. Students use their reading and research skills as basis for tackling the question of climate change.

Orienting Discussion: The class discusses three key questions:

1. Can the recent changes in California's climate be explained by natural causes?
2. If natural causes cannot explain the rising temperatures, what anthropogenic factors have produced these changes?

3. If temperatures in California's climate continue to rise, what effects will this have on humans and the state's natural systems?

Having read and processed the key article, students start to unpack these questions. Students look up the meaning of "anthropogenic, then rephrase the questions in their own words to see if they understand the meaning. Both the reading and the initial class discussion prepare students to push forward.

Motivated to help reduce climate change in California and globally, students decide to break down their task into more manageable pieces:

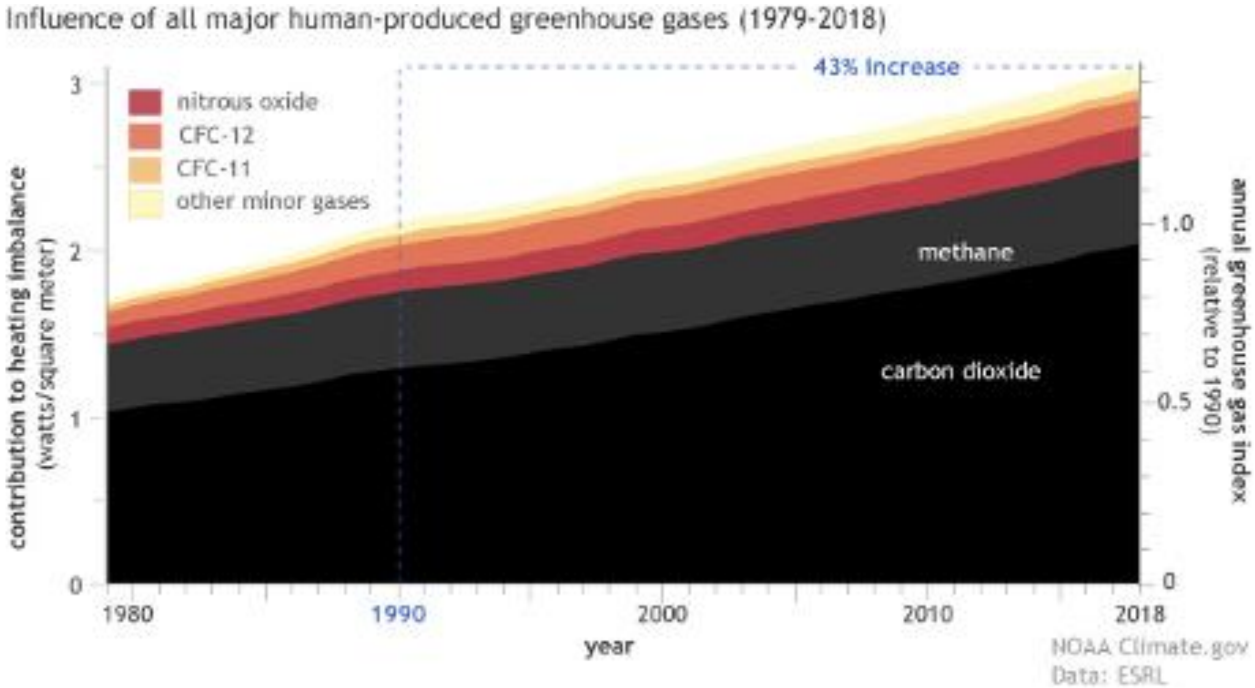
1. Determining the major greenhouse gases;
2. Analyzing the relative contributions of each gas and deciding which is the greatest contributor to global climate change and thus should be given the highest priority for policy changes and governmental action;
3. Collecting data on the human activities that cause increases to the release of the most influential greenhouse gas;
4. Analyzing the key components of the growth patterns of this gas;
5. Based on influences to the growth pattern, developing a plan to reduce the human-source release of that greenhouse gas by 25–50%; and,
6. Determining how their plan would influence the rate of global climate change.

Team Research

Students start researching online, considering the trustworthiness of the data sources.

They visit https://www.climate.gov and the California Air Resources Board (https://ww2.arb.ca.gov) to gather most of the data they need.

At https://www.climate.gov they discover a graph that shows the influence of the major human-produced greenhouse gases from 1980–2018.



Looking at the graph and prompted by the teacher's questions, "What do you notice? What do you wonder?" students wonder about various aspects and implications. They jot these wonderings down and then speak in small groups. They notice that all major contributing gases seem to be increasing over time, though some say CFC-11 isn't obviously increasing; and others note that CFC-12 seems to have leveled out around 1990. Some students question this, as both still look like they are "going up" on the graph; this disagreement and ensuing discussion helps all students make sense of the graph.

Through a process of collaboration, they work together to synthesize their questions into coherent and meaningful inquiries:

1. Why are there labels on both vertical axes? What do the three labeled axes represent?
2. Why is there a labeled 43-percent increase? An increase in what? Over what time frame? How was this calculated?
3. What does this data display suggest is the most important greenhouse gas?
4. How does the year-to-year growth change over these 38 years?

Most teams choose to focus their efforts on reducing CO_2 emissions based on the graph above. One team decides to work with methane because they believe that CO_2 emissions are harder to reduce, and they can make a bigger difference by reducing methane emissions. Students feel empowered since they have more autonomy to follow where their explorations lead them, which is not the usual way of learning in math, science or ELA. The teachers work with some groups that may struggle with the openness of the task. Teachers encourage students to build from and explore each other's ideas.

Each team researches the sources of human emissions of the gas they have chosen, uses their understanding of political and psychological opportunities and barriers to decide on most-likely policy shifts to achieve the desired 25–50% reduction in emissions, and prepares a presentation for the class outlining their solutions. The teaching team provides additional expertise to help interpret the complexity of the information students are collecting and synthesizing.

Team Presentations

As teams prepare for their presentations, they return to the driving question of the task. From all the data they collected, they must now distill the most important information to describe their analysis and recommendations. Part of each presentation is a version of the NOAA graph above, extended into the future with the assumed implementation of the team's proposal. Calculating the impact of their proposal on the rate of temperature change will require interpreting the left vertical axis label on the graph. The teaching team videotapes the presentations and reports to capture the range of practices that students are using such as quality of their research, analysis of data, effectiveness of their visuals, and clarity of their report, given audience, and purpose.

After all teams have presented, the final activity is to put all the pieces together to address the following big idea: What will be the impact on climate change if all the teams' proposals are implemented?

Following the common experience of MIC 1 and MIC 2, this framework presents two options for a MIC 3/4 course: *Mathematics: Investigating and Connecting—Data*

Science and *Mathematics: Investigating and Connecting—Functions and Modeling*. Both continue the MIC 1 and 2 emphasis on developing mathematical understanding in order to answer students' authentic questions. The two emphasize different types of investigations to frame student activities, and distribute student effort differently between the various Content Connections and the Conceptual Categories of the CA CCSSM.

The specifications for the MIC—Data and MIC—Modeling courses are consistent with the broad goals of the Integrated Math III guidance that is provided in the CA CCSSM: "It is in the Mathematics III course that students integrate and apply the mathematics they have learned from their earlier courses." Research and recommendations about high school pathways have added much to our understanding since the adoption of the CA CCSSM in 2010 (and postsecondary admission requirements have broadened the mathematics recognized as appropriate preparation, see Pathways in 9-12 section above), so the MIC—Data and MIC—Modeling courses are replacements for, rather than implementations of, the Integrated Math III content guidance in the CA CCSSM. The CA CCSSM foresaw this mechanism, pointing out that the framework "...will offer expanded explanations of the model courses and suggestions for additional courses."

Specifically, MIC implements the recommendation in (Daro & Asturias, 2019) that students have a common experience in ninth and tenth grades, with branching options in eleventh grade. This enables students to begin to explore mathematics in contexts that matter to them. An important caveat is that both MIC—Data and MIC—Modeling courses should offer a path to all twelfth-grade courses, so that students are not locked into a track with their MIC third year choice.

Mathematics: Investigating and Connecting—Functions and Modeling is designed around investigations centered in the Mathematical Modeling Conceptual Category (which might fit into any Content Connection), developing most content through these investigations. For more discussion of modeling, see Content Connection 2 on p. X. Mathematics: Investigating and Connecting—Data Science is designed around investigations centered in the Statistics and Probability Conceptual Category, and is explained in detail in Chapter 5: Data Science.

As indicated in the course diagram earlier in this chapter, additional advanced MIC courses are possible, as long as they are designed to situate mathematics learning in investigations of authentic contexts and problems, and offer a path to twelfth-grade courses offered by the school/district.

One example that is already offered by some districts (and is University of California A–G approved) is Financial Algebra, in which students engage in mathematical modeling in the context of personal finance. Through this modeling lens, they develop understanding of mathematical topics from advanced algebra, statistics, probability, precalculus, and calculus. Instead of simply incorporating a finance-focused word problem into each Algebra 2 lesson, this course incorporates the mathematics concept when it applies to the financial concept being discussed. For example, the concept of exponential functions is explored through the comparison of simple and compound interest; continuous compounding leads to a discussion of limits; and tax brackets shed light on the practicality of piecewise functions. In this way, the course ignites students' curiosity and ultimately their engagement. The scope of the course covers financial topics such as: taxes, budgeting, buying a car/house, (investing for) retirement, and credit, and develops algebra and modeling content wherever it is needed. “Never has mathematics seemed so relevant to students as it does in this course,” says one teacher.

Any of these advanced MIC courses could lead to a full range of fourth-year options as set out in the course diagram earlier in the chapter. The University of California and the California State Universities have approved courses in data science and statistics as valuable alternatives to calculus pathways. Research has shown that taking a precalculus class does not increase success in calculus (Sonnert & Sadler, 2014), and recent innovative approaches for students in California community colleges have shown that students who move from Algebra 2 to supported calculus classes are more successful than those who go through prerequisite courses (Mejia, Rodriguez, & Johnson, 2016). Thus, this Framework recommends that students be allowed to move from any advanced MIC course to any fourth-year course, including a calculus course or another advanced MIC course.

The four Content Connections are described and illustrated with a relevant vignette and with CA CCSSM content domains listed for each. See the CA CCSSM for the full language of standards in the domain. Note that almost all tasks and investigations will involve multiple domains, with a goal of building connections across multiple mathematical ideas.

The Content Connections

CC 1: Communicating Stories with data

Vignette: Whale Hunting

This Content Connection is covered in more depth in Chapter 5: Data Science. The Mathematics: Investigating and Connecting pathway gives prominence to reasoning about and with data, reflecting the growing importance of data as the source of most mathematical situations that students will encounter in their lives. Investigations in a data-driven context—data either generated/collected by students, or accessed from publicly-available sources—help maintain and build the integration of mathematics with students' lives (and with other disciplines such as science and social studies). Most investigations in this category also involve aspects of CC 2: *Illuminating changing quantities*.

Context:

In the 1970s the stock (or number) of bowhead whales in the Bering Sea was calculated to be as low as 600–2000 whales, mostly due to heavy commercial whaling. This was, of course, mightily concerning to environmentalists and thus the International Whaling Commission completely halted permissions to hunt whales hoping to restore the population. Commercial whaling had long been a known issue, and it was already restricted, but this really hurt native populations that hunt bowhead whales for subsistence. Note that this provides a good opportunity for discussing and reinforcing California Environmental Principle I, "The continuation and health of individual human lives and of human communities and societies depend on the health of the natural systems that provide essential goods and ecosystem services."

Here is some writing on the practice from an indigenous person from the region:

> “Subsistence whaling is a way of life for the Inupiat and Siberian Yupik people who inhabit the Western and Northern coasts of Alaska. From Gambell to Kaktovik, the bowhead whale has been our central food resource and the center of our culture for millennia, and remains so today.
>
> Our whale harvest brings us an average of approximately 1.1M to 2M pounds of food per year (12–20 tons x 45–50 whales), which our whaling captains and crews share freely throughout our whaling communities and beyond to relatives and other members of Alaska’s native subsistence community in other native villages. For perspective, replacing this highly nutritious food with beef would cost our subsistence communities approximately $11M – $30M per year.
>
> As important as whale is to keeping our bodies healthy, this subsistence harvest also feeds our spirit. The entire community participates in the activities surrounding the subsistence bowhead whale harvest, ensuring that the traditions and skills of the past are carried on by future generations. Portions of each whale are saved for celebration at Nalukataq (the blanket toss or whaling feast), Thanksgiving, Christmas, and potlucks held during the year. […] Sharing the whale is both an honor and an obligation.”

Over the years, the International Whaling Commission (IWC) has worked with the Inupiat and Siberian Yupik people to ensure their needs are met and whales are protected. Through this process, bowhead whale populations have bounced back. However, the IWC still establishes whaling quotas for the local indigenous folks to ensure the population remains strong.

The last ice-based abundance and Photo-ID-based surveys were conducted in 2011. The 2011 ice-based abundance estimate is 16,892 (within the range of 15,704 – 18,928). The rate of increase of the population, or trend, starting in 1979 was estimated to be 3.7 percent (within the range of 2.8–4.7 percent). These abundance and trend estimates show that the bowhead population is

> healthy and growing with a very low conservation risk under the current Aboriginal Subsistence Whaling management scheme."

[Source pending.]

Task:

The tribe has assembled a committee of tribal scientists and community members, along with outside scientific and economic advisors, to make a recommendation to the International Whaling Commission. The proposal will specify how many whales the Inupiat and Siberian Yupik people will hunt this year as part of the Aboriginal Subsistence Whaling management plan, while making sure the whale population continues its growing trend. As a member of the committee, it is your task to help create the proposal.

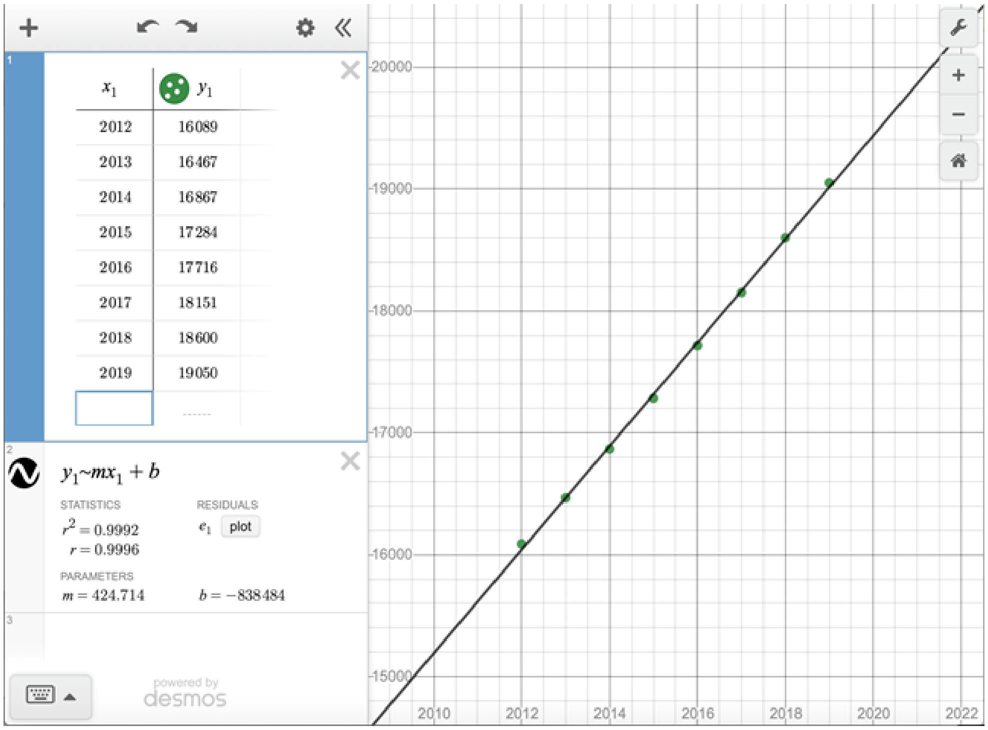
Student Vignette:

The group receives the task, and discusses what they were being asked for. They decide to break down the problem into more manageable pieces, so they make a checklist with three items:

1. Figure out what happened to whale population between 2011 and 2019.
2. Find out the current growth rate that should be maintained.
3. Calculate how many whales can be lost in 2020 so that the growth rate is maintained.

For point 1, they think they might be able to find more data online, so they look up whale hunt statistics in 2011–2019. They found a table in the IWC website that lists every whale catch between 1986 and 2018. It had a lot more information than they needed: different whale species and stocks from different oceans, but they reviewed the information and pulled out the data they needed. In order to estimate the whale stock in

2018, for each year between 2011 and 2018 they plan to use the equation:



(Number of whales in the year they're looking for) = (Number of whales in the year prior)*(growth rate per year) – (whales hunted that year)

They discuss with the whole group which numbers to use for growth rate and for the 2011 stock numbers, since they have the estimates but also the error ranges the experts gave. They decide that it's better to be safe than sorry, since whale overpopulation hardly seems like an issue, so they will use the lower end of the range for both numbers. Now comes a lot of number crunching, but computers can do that. They use Wolfram|Alpha to quickly complete the calculations and they estimate the 2019 stock at 19,050.

However, they know they need the stock for the beginning of 2020. They don't have the data for how many whales were hunted in 2019, so they estimate it by averaging the years they do have data for: 2011–2018. The average is 60.75, so they round it to 61 and use their equation to calculate the stock at the beginning of 2020 as 19,522.

Now they look at point 2: finding the rate at which the population is currently growing. They use Desmos to graph the population each year and map a line of best fit, which will show the target growth rate.

That leads them to point 3: how many whales can be killed to keep this target? They look back at the original growth equation, but now they solve it for how many whales can be hunted:

(whales hunted that year) = (Number of whales in the year prior)*(growth rate per year) – (Number of whales in the year they're looking for)

- That target growth line has the equation $y = 424.714x - 838{,}484$, so for $x = 2021$ (meaning, after the hunt in 2020), the population target would be 19,863, and they already know the growth rate they've been using, and their estimate for the 2020 population, so they can calculate the number of whales that can be hunted while maintaining the current growth and make a recommendation to the IWC.

  Note: This provides a good opportunity for discussing and reinforcing California Environmental Principle V, "Decisions affecting resources and natural systems are based on a wide range of considerations and decision-making processes." It demonstrates the importance of mathematical analysis in making policy recommendations and decisions about the conservation and management of organisms and the ecosystems they depend on. It also reinforces California Environmental Principle II, "The long-term functioning and health of terrestrial, freshwater, coastal and marine ecosystems are influenced by their relationships with human societies

The progression of CC1 through the courses

CC1 is the only Content Connection in which standards differ from those in the CA CCSSM Integrated Mathematics model course outlines. Given the rapidly increasing importance of data literacy, many Statistics and Probability standards that are in year 3 of the model course outlines are here addressed through all years of the MIC pathway.

The progression of data literacy is addressed in more detail in Chapter 5: Data Science. Briefly, in MIC 1, students should experience repeated random processes and keep track of the outcomes, to begin to develop a sense of the likelihood of certain types of events. They must have experience generating authentic questions that data might help to answer, and should have opportunities to gather some data to attempt to answer their questions. They should plot data on scatter plots, and informally fit linear and exponential functions when data appear in the plot to demonstrate a relationship (using physical objects like spaghetti or pipe cleaners, or online graphing technology).

In MIC 2, investigations should be designed to build students' understanding of probability as the basis for statistical claims. For functions modeling relationships between quantities, "strength of fit" is introduced (informally at first by comparing weak and strong associations with identical linear models) as a measure of how much of the observed variability is explained by the model; it measures predictive ability of the model.

MIC—Data has almost all student investigations driven by data, and requires extensive use of probability to make decisions. Students generate questions, design data collection, search for available existing data, analyze data, and represent data and results of analysis. Most content in other Content Connections is situated in stories told through data. See Chapter 5 for more detail.

Some MIC—Modeling investigations may be set in contexts where data leads to the mathematical model. Most investigations, however, will be based on a structural understanding of the context: A function to represent the height at time *t* seconds of a ball thrown at a given upward velocity; a model to represent the total cost of ownership of a car over *n* years based on sales price, fuel costs, and average maintenance costs. Data may play a bigger role in the validation stage of the modeling cycle (see below in CC2).

CA CCSSM domains by course

MIC 1: domains of emphasis for investigations in CC1 (from the CA CCSSM Mathematics I model course outline, augmented by additional Statistics and Probability standards):

- Number and Quantity
  - Quantities
- Algebra
  - Creating Equations
- Functions
  - Interpreting Functions
  - Building Functions (modeling a relationship)
  - Linear, Quadratic, and Exponential Models (linear and exponential in MIC 1)
- Statistics and Probability
  - Interpreting Categorical and Quantitative Data
  - Making Inferences and Justifying Conclusions (informally, emphasis on observing distributions resulting from random processes)

MIC 2: domains of emphasis for investigations in CC1 (from the CA CCSSM Mathematics II model course outline, augmented by additional Statistics and Probability standards):

- Algebra
  - Creating Equations
- Functions
  - Interpreting Functions
  - Building Functions (modeling a relationship)
  - Linear, Quadratic, and Exponential Models
- Statistics and Probability
  - Conditional Probability and the Rules of Probability
  - Using Probability to Make Decisions

MIC—Data: domains of emphasis for investigations in CC1:

- Statistics and Probability
  - Interpreting Categorical and Quantitative Data
  - Making Inferences and Justifying Conclusions
  - Using Probability to Make Decisions

- Algebra
  - Creating Equations
  - Reasoning with Equations and Inequalities
- Functions
  - Linear, Quadratic, and Exponential Models
  - Trigonometric Functions (model periodic phenomena)

MIC—Modeling: domains of emphasis for investigations in CC1:

- Statistics and Probability
  - Interpreting Categorical and Quantitative Data
  - Making Inferences and Justifying Conclusions
- Algebra
  - Creating Equations
  - Reasoning with Equations and Inequalities
- Functions
  - Linear, Quadratic, and Exponential Models

CC 2: Exploring Changing Quantities

Applications of mathematics in the 21st Century often require users to make sense of, keep track of, and connect a wide range of quantities. Quantities can represent vastly different—yet interrelated—components within a context, such as speed, weight, location, magnitude, and value, etc., and mathematicians must find ways to represent the relationships between these quantities in order to make sense of and model complex situations. To explore and make sense of changing quantities is an important skill that applies across mathematical contexts.

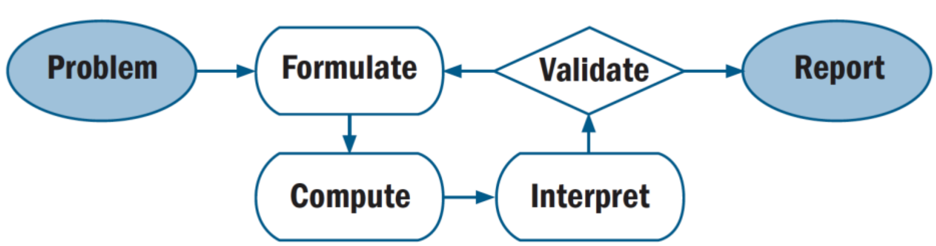
Through investigations in this Content Connection (CC), students build many concrete examples of functions to represent relationships between changing quantities in authentic contexts. The CC includes most modeling investigations. Specific, contextualized examples of functions are crucial precursors to students' work with *categories* of functions such as linear, exponential, quadratic, polynomial, rational, etc. and to the abstract notion of function. Notice that the name of the CC considers changing *quantities*, not changing *numbers*. Functions referring to authentic contexts gives students concrete representations that can serve as contexts for reasoning, providing multiple entry paths and reasoning strategies—as well as ample necessity to engage in SMP 2 (Reason abstractly and quantitatively). This embedding also maintains and builds connections between mathematical ideas and students' lives.

What is a Model?

Modeling, as used in the CACCSSM, is primarily about using mathematics to describe the world. In elementary mathematics, a model might be a representation such as a math drawing or a situation equation (operations and algebraic thinking), line plot, picture graph, or bar graph (measurement), or building made of blocks (geometry). In Grades 6–7, a model could be a table or plotted line (ratio and proportional reasoning) or box plot, scatter plot, or histogram (statistics and probability). In Grade 8, students begin to use functions to model relationships between quantities. In high school, modeling becomes more complex, building on what students have learned in K–8. Representations such as tables or scatter plots are often intermediate steps rather than the models themselves. The same representations and concrete objects used as models of real life situations are used to understand mathematical or statistical concepts. The use of representations and physical objects to understand mathematics is sometimes referred to as "modeling mathematics," and the associated representations and objects are sometimes called "models."

Taken from the K-12 Modeling Progression for the Common Core Math Standards (http://ime.math.arizona.edu/progressions/)

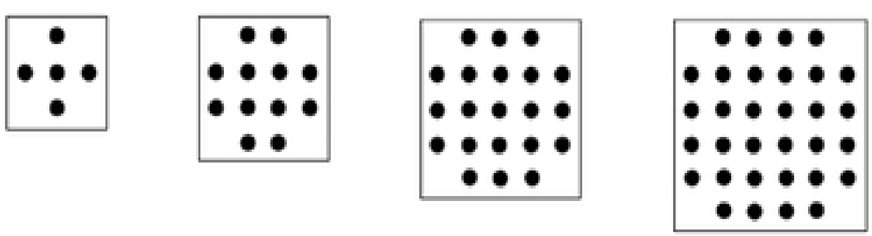
The Modeling Cycle



Mathematical modeling projects, large and small, provide many examples of such investigations. Mathematical modeling has also been shown to provide more equitably engaging mathematics for students (Boaler, Cordero & Dieckmann, 2019). The modeling cycle (graphic from the CA CCSSM shown here) includes many important aspects of doing mathematics that are dramatically underrepresented in traditional word problems in textbooks (essentially everything except “Compute” in the graphic):

- Identifying interesting questions
- Identifying questions amenable to mathematical formulation
- Making simplifying assumptions
- Formulating mathematical versions of questions and mathematical representations of relationships between quantities (“geometric, graphical, tabular, algebraic, or statistical representations”—CA CCSSM)
- Interpreting results in the original context
- Validating results by comparing with what is known about the context
- Deciding whether the results sufficiently represent the situation for the purpose at hand, or whether the model needs to be refined and the cycle repeated

While most investigations identified as modeling are set in empirical contexts, the important feature of the context for CC 3 investigations is not real-world versus made-up, but rather the concreteness of the context to the students engaging in the investigation. The context of the investigation must be sufficiently concrete for students to imagine questions, to identify changing quantities, to guess at what might happen, and to see enough structure to begin to describe relationships between the changing quantities.



Thus, a dot growth pattern such as the one here (*Illustrative Mathematics*, n.d.) can be a source for rich take apart/put together activities, as can larger-scale modeling problems such as predicting the effects of climate change over time in terms of several possible factors related to human activities (exhaust from cars, production of electricity, release of pollutants from factories, etc.)." [Note: This provides a good opportunity for discussing and reinforcing California Environmental Principle IV, "The exchange of matter between natural systems and human societies affects the long-term functioning of both."]

Vignette: Drone light show

Course: MIC3—Modeling with Functions (also Integrated Math 3)

CC 1: Exploring changing quantities

DI 3: Impacting the Future

Domains of Emphasis: HS.A-SSE, HS.A-CED, HS.F-BF, HS.F-TF, HS.G-GMD, HS,G-MG

SMPs: SMP 4, 5, 7

Source: Consortium for Mathematics and its Applications (COMAP), High School Mathematical Contest in Modeling (HiMCM)—2017 Problems.

Problem: Drone Clusters as Sky Light Displays

Intel© developed its Shooting Star TM drone and is using clusters of these drones for aerial light shows. In 2016, a cluster of 500 drones, controlled by a single laptop and one pilot, performed a beautifully choreographed light show

(https://youtu.be/aOd4-T_p5fA).

Our large city has an annual festival and is considering adding an outdoor aerial light show. The Mayor has asked your team to investigate the idea of using drones to create three possible light displays.

Part I – For each display:

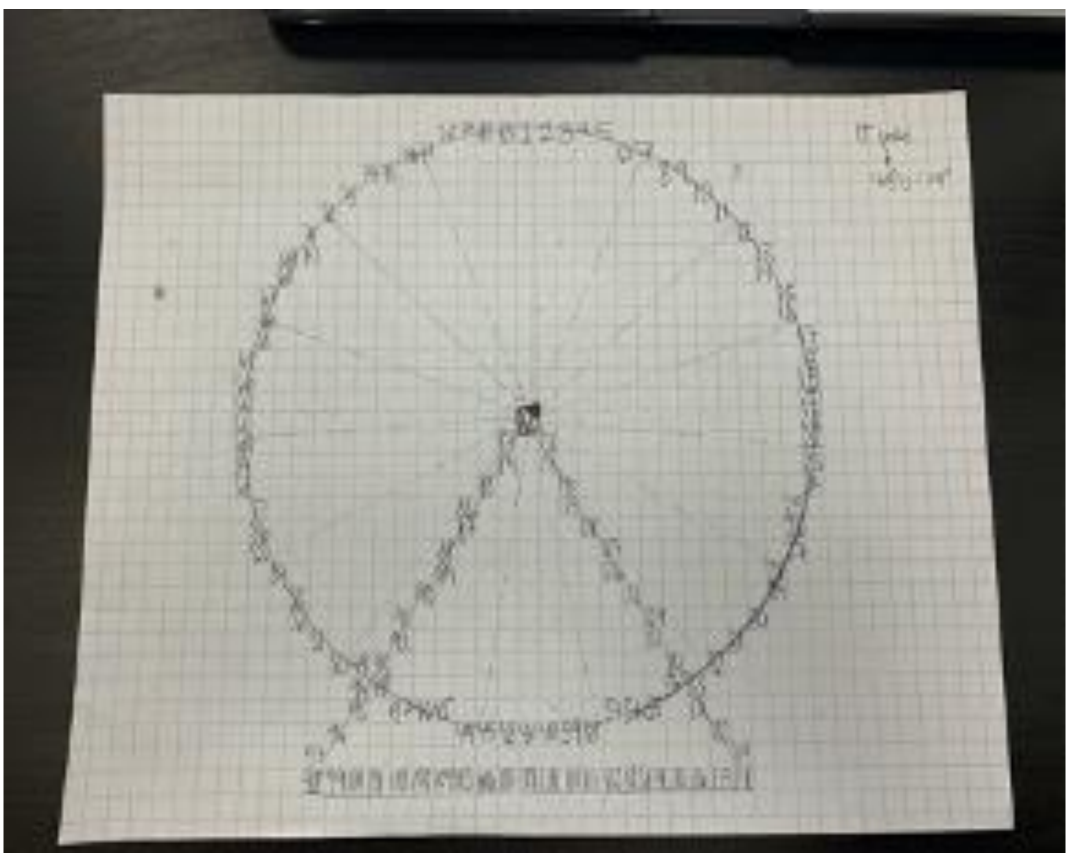
a) Determine the number of drones required and mathematically describe the initial location for each drone device that will result in the sky display (similar to a fireworks display) of a static image.
b) Determine the flight paths of each drone or set of drones that would animate your image and describe the animation. (Note that you do not have to actually write a program to animate the image, but you do need to mathematically describe the flight paths.)

Students are instructed to work together in three groups to design a solution to the problem. All three groups start out by reading the task and discuss the task. They are then given access to the video, which includes closed captioning, and then prompted to conduct a search for photos and clip art of Ferris wheels as a type of moving light system. Some groups want to watch the video several more times to be sure they understand. From experience, they know that this is not the kind of problem that allows them to find the answer in the back of the textbook. This kind of a problem can be approached in a variety of ways, and that the challenge of the openness of the problem is thrilling! Students will need to think about the math tools and processes they have already learned before and apply them to a new context. This can be understood as the “formulate” stage of the Modeling Cycle. The teacher notices three distinct strategies in her classroom, particularly in how each has decided to model the changing quantities within the problem—or the “compute,” “interpret,” and “validate” stages of the Modelling cycle.

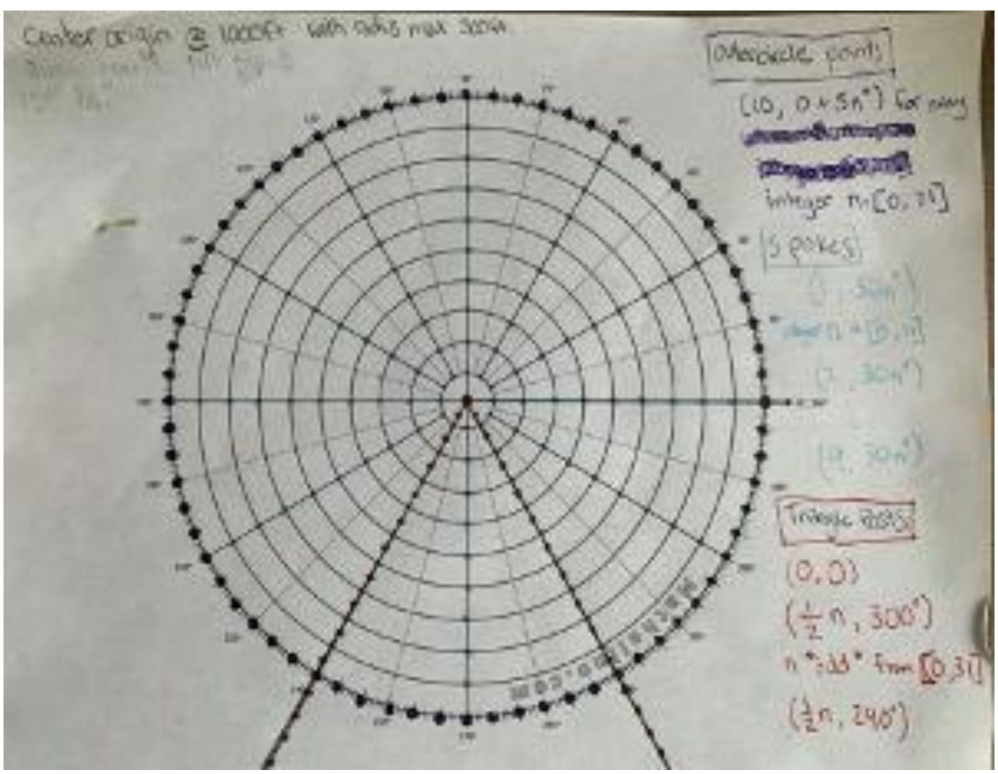
Over the course of the year, students have had several opportunities to engage in the math practice of modeling. Students know that math models help both to describe and predict real-world situations, and that models can be evaluated and improved. With every group member contributing to the brainstorm, students quickly start sketching as a way to visualize solution paths. As students are drawing, they explain and label their diagrams to show the “initial location,” for example. Some students are eager to get to Display 3, where they get to create their own design.

The teacher notices three unique approaches arising in the groups’ work, particularly in how they have decided to model the changing quantities within the problem. The teacher is pleased to see use of visuals and diagrams, as these are important ways of seeing and understanding mathematics and critical supports for students. As the teacher listens to the small group work, she acknowledges how well the groups are making space for everyone’s ideas. At first, the teacher notes that students are not writing much, but she has learned not to intervene too quickly. Instead, she allows their ideas to build, with the firm belief that her students will make progress.

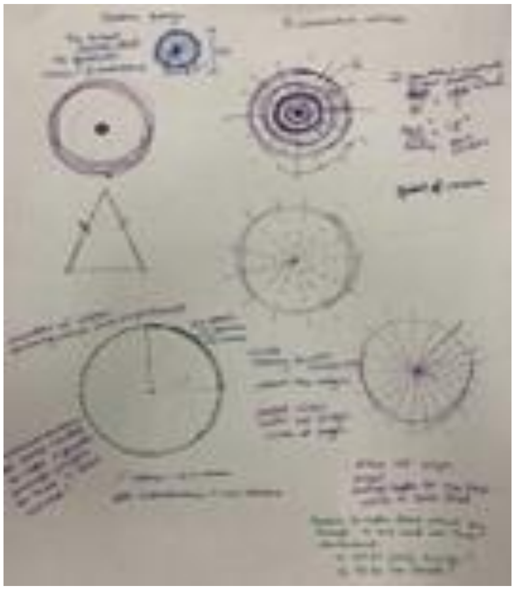
Group A: The students in this group have decided to model the problem on the idea of pixels in a grid that make up images on a tv screen. The team draws an image of a Ferris wheel on the grid, and numbers every “pixel” in their grid that will need to be lit up by a drone to represent the circumference of the Ferris wheel. Next, the group has decided to model the rotation of the wheel by programming some drones to stay in place and some to move in a particular pattern. They know the pixels for the triangle don’t move so these drones will be programmed to stay in place. And for the circle, it’s a loop.

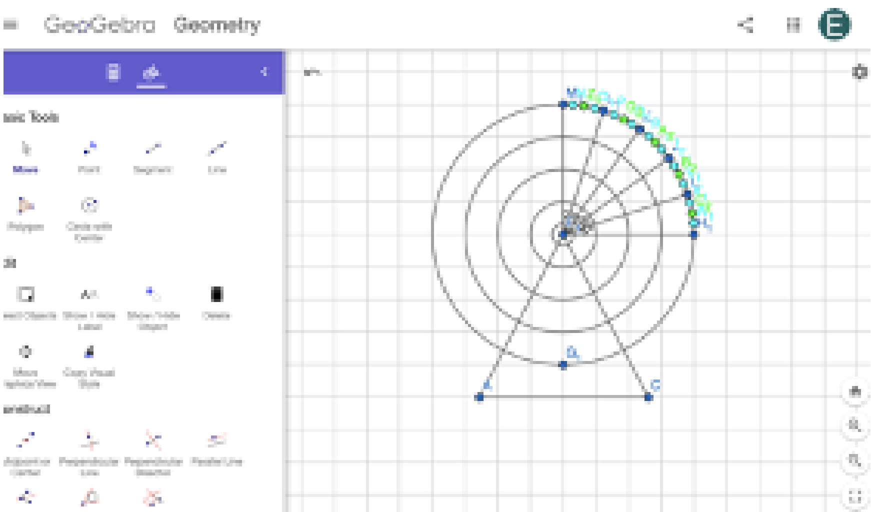


Group B: In this group, students have decided to model the Ferris wheel using polar coordinates. They decided that programming the coordinates (x,y) for the drones that make the circle of the Ferris wheel would require defining a unique x and y for every single drone! But, in polar coordinates (r,theta), the outer circle of the Ferris wheel can be thought of as many points in the plane sharing the same radius, which means that they would only need to change the theta for each drones coordinates and keep the r the same. The group determines with coordinates representing the wheel, spokes, and triangle posts of the Ferris wheel. To model the rotation of the wheel, the angle (theta) that each drone is programmed to will increase by 5° for a total of 72 moves of the circle to complete one full rotation of the wheel. To model the rotation of the spokes, the angle (theta) that each drone is programmed to will increase by 30° for a total of 12 moves, to complete one full rotation of the wheel. The drones placed to represent the base of the Ferris wheel are programmed to stay in place.



Group C: This group selected an image of the Great Seattle Wheel to use as their guide. They decided to model the image of the Ferris wheel using the equation of a circle in the cartesian plane, and various dilations of the outer circle to create inner circles that will model the spokes of the wheel. Finally the group decides to utilize online an graphing tool that will allow them to rotate the image within the plane to model the turn of the wheel. The group creates equations for 20 lines that start at the center of the circle, intersect each concentric circle, and end at the outer circle. While this is a slight modification to the 21 spokes on the Great Seattle Wheel, it allows the degrees of each arc length to be integer values, which the students agree will be easier to work with. These lines separate the circle into 20 equal sectors—each with an arc length of 18°. They decide to program a drone at each intersection of the circles and the lines to represent the spokes. A discussion ensues about the number of drones that must be placed between each spoke intersection on the outer circle to create an outline of the circle that looks smooth, the group decides on three for now because 18° is easily divided into three. Finally, the group decides to utilize an online graphing tool (GeoGebra) that will allow them to rotate the image within the plane to model the turn of the wheel. The group discusses the rate of rotation and degree of rotation that would be most appropriate to model the movement and speed of the Great Seattle Wheel.





After students have had time to work out the details of their models, each group gives prepares a presentation about their approach to the problem. Some students jot a few notes down to help them remember key ideas and terms. They prepare to describe their model and explain their choices to their peers. Students prepare a poster, using colors to highlight key features of their model. The teacher circles around and helps students who want to do a quick run-through of their presentation, giving students feedback to strengthen their work, supporting language learning by clarifying how content vocabulary can be used, and suggesting ways to better convey the information in presentation-worthy academic discourse as she does so. Each presentation is followed by a short question and answer session with the class. Each presentation poster is displayed at the front of the class, clearly showing a wide range of methods and approaches.

Following these presentations, the teacher conducts a Gallery Walk, allowing smaller groups of students to spend a few minutes viewing the posters up close. This activity is followed by a whole-class discussion on the different strategies taken by each group, including a discussion about the affordances and challenges presented by each choice for modeling the changing quantities in the problem. Throughout this process, the teacher is taking notes on feedback, including areas of strength and where possible improvement is needed as students engage with the modeling cycle. She will use this information in responding to the students’ presentations during evaluation, and framing the next modeling task.

Disciplinary Language Development

This task provides extended opportunity to deepen in the area of mathematical modeling within an authentic context. The challenging nature of this task encourages collaboration, building on one another’s ideas and key skills using students’ mathematical language. In groups, students make use of the full array of mathematical resources to construct their models, effecting utilizing prior mathematics learning. The visual nature of the task, along with the video, and their presentation posters expand the modalities in mathematics, supporting the guidelines in Universal Design for Learning (UDL), which move beyond the more typical confined to calculations and symbols. Here, the visuals are not support for their models, they are the models themselves.

The progression of CC2 through the courses

Investigations that develop the mathematical content of CC2: Exploring Changing Quantities should span the range of the DIs, with particular attention paid to culturally relevant activities in DI 2 and DI 3, since these types of activities most easily help students experience mathematics as a useful lens for their lives.

In MIC 1, tasks and explorations in this CC should focus mostly on quantities that change with respect to time or “step number.” Relationships should be primarily linear and exponential, with some other relationships explored only informally (for example, predicting using a plot of known points and a pipe cleaner for interpolating or extrapolating). Quantities should include linear measurement (length and distance), population growth (e.g., bacteria), and interest (both deposits and debts), among many other contexts that generate linear and exponential growth. Most questions begin with “When will…?” or “At this time, what will…?” Students must generate many of the questions for exploration, and even some of the contexts for questioning. For example, “What are some things that affect your life, that change over the course of the school year?” can generate contexts to explore.

In MIC 1, quantities should include linear measurement (length and distance), population growth (e.g., bacteria), and interest (both deposits and debts), among many other contexts that generate linear and exponential growth. Typically, students will approach these situations recursively at first, seeing either a constant additive (linear growth: same amount added each time period) or constant multiplicative (exponential growth: quantity grows by the same factor or percent each time period). Most of the mathematical work emerges from attempts to find or predict the value of the changing quantity at a point in the future or at a point in between known values; then to express the value of the quantity at an arbitrary point in time. Verbal, graphical, and symbolic representations should all appear as appropriate, with emphasis on the connections between them and the features of the relationship between quantities that each representation helps to make clear.

Beginning in MIC 1 and continuing through MIC 2, the general notion of function should be developed and synthesized through this CC, typically building from different situations that generate the same linear or exponential relationship, then noting the similarities, and discussing function notation as a way to capture multiple situations at once. (See the discussion of abstraction in the “Rigor” section in Chapter 1: *Introduction*.) Problems framed in terms of abstract functions (that is, functions given as formulas, graphs, or tables without an accompanying context) should frequently include prompts to “invent a context that this function (or equation or expression) might represent.” This prompt helps maintain the connection between mathematics and students’ lives that is so important in order for students to see mathematics as having value.

In MIC 2, measured and observed quantities that change relative to other quantities besides time or step number should be investigated, in addition to the time/step relationships in MIC 1. Relationships modeled should expand to include quadratic, in addition to linear and exponential relationships explored in MIC 1. The general idea of function should be further developed as an abstraction of repeated efforts to understand, describe, and use relationships in particular contexts.

In MIC—Data, the focus is the creation of function models for relationships that are observed through data, and the use and interpretation of those models. At first, these models should be guided by student-generated ad-hoc methods, such as:

- We used a yardstick on the graph and moved it around until it was as close as possible to all the dots.
- We measured the distance the car went when we raised the high end of the ramp to different heights. When we graphed it, it looked sort of like a line going up. On average, raising the ramp by 1 inch increased the car's distance by 3¼ inches, so we decided to try 3.25 as the slope for our line.
- We used Desmos to graph the area for different scale factors, and it curved upward. So we first tried graphing exponential functions to see if they would match up, but none of them looked right. Then we tried quadratic functions and just played around with the numbers until they looked right with our dots.

Such ad-hoc methods should lead to discussions about what makes one proposed function "fit" the data better than another, and activities and should develop a conceptual idea (not by-hand computational skill) that the "best fit" function minimizes the total distance of all the data points from the function—while pointing out that it is actually *vertical* distances that are minimized, and that most software systems minimize the sum of the *squared* vertical distances, not the sum of the (absolute) vertical distances.

Later, students use appropriate technological tools to generate "best fit" functions, and use those functions as models for the relationships, in order to predict one quantity given the other. Extrapolating beyond known data should be contrasted with interpolating within.

In MIC—Modeling, functional models will be driven by understood or theorized underlying structure governing the relationship between quantities, rather than by data about the relationship. For instance, the notion that speed of a vehicle changes at a constant rate if a constant force is applied is consistent with many students' experience (within a reasonable range and with some important simplifying assumptions!). Given this, a relationship between time and distance traveled can be developed and used to answer questions about the context. Data points can then be used to select the parameters (constants) of the model. (The mathematics of this example has been used in one of California's longest court cases over a speeding ticket: https://www.pressdemocrat.com/article/news/gps-or-not-teen-must-pay-190-speeding-ticket/).

In all courses, investigations should include situations requiring solving equations and systems of equations. Such questions as these will necessitate such solutions:

- When will one quantity reach a fixed value?
- When will two different quantities that change over time be equal?
- When will one be greater than the other?
- At a fixed time, what is the rank order of the quantities?
- What value of (one quantity) corresponds to (a) specified value(s) of (other quantity(ies))?

CA CCSSM Content in CC2

CC 2: *Exploring changing quantities* includes much of the content of the CA CCSSM Conceptual Categories below:

- Functions
- Modeling
- Algebra

Modeling and Algebra are also heavily represented in CC3: Taking Wholes Apart, Putting Parts Together. In addition, CC2 includes some CA CCSSM domains from other Conceptual categories. Also note that many investigations in CC1: *Telling Stories with*

Data will involve extensive work in CC2 content. The specific domains that should be emphasized in CC2 investigations are highlighted by course below.

CA CCSSM domains by course

MIC 1: domains of emphasis for investigations in CC2 (from the CA CCSSM Mathematics I model course outline):

- Number and Quantity
    - Quantities
- Algebra
    - Creating Equations
    - Reasoning with Equations and Inequalities
- Functions
    - Interpreting Functions
    - Building Functions (modeling a relationship)
    - Linear, Quadratic, and Exponential Models (linear and exponential in MIC 1)
- Statistics and Probability
    - Interpreting Categorical and Quantitative Data (interpret linear models)

MIC 2: domains of emphasis for investigations in CC2 (from the CA CCSSM Mathematics II model course outline):

- Algebra
    - Creating Equations
    - Reasoning with Equations and Inequalities
- Functions
    - Interpreting Functions
    - Building Functions (modeling a relationship)
    - Linear, Quadratic, and Exponential Models

MIC—Data: domains of emphasis for investigations in CC2:

- Statistics and Probability
    - Interpreting Categorical and Quantitative Data

    - Making Inferences and Justifying Conclusions
- Algebra
    - Creating Equations
    - Reasoning with Equations and Inequalities
- Functions
    - Interpreting Functions
    - Building Functions
    - Linear, Quadratic, and Exponential Models
    - Trigonometric Functions (model periodic phenomena)

MIC—Modeling: domains of emphasis for investigations in CC2:

- Statistics and Probability
    - Interpreting Categorical and Quantitative Data
    - Making Inferences and Justifying Conclusions
- Algebra
    - Creating Equations
    - Reasoning with Equations and Inequalities
- Functions
    - Interpreting Functions
    - Building Functions
    - Linear, Quadratic, and Exponential Models
    - Trigonometric Functions (model periodic phenomena)

CC 3: Taking Wholes Apart, Putting Parts Together

Students enter high school with many experiences of taking wholes apart and putting parts together:

- Decomposing numbers by place value
- Assembling sub-products in an area representation of two-digit by two-digit multiplication
- Finding area of a plane figure by decomposing into rectangular or triangular pieces

- Exploring polygons and polyhedra in terms of faces, edges, vertices, and angles

Breaking down challenges and ideas into manageable pieces, and assembling understanding of smaller parts into understanding of a larger whole, are fundamental aspects of learning, doing, and using mathematics. Often these processes are closely tied with SMP 7 (Look for and make use of structure). This Content Connection spans and connects many typically-separate content clusters in algebra and geometry. Plane figures in geometry, for example, are made up of points, lines/line segments and circles/circular arcs (and perhaps other curves); angles, lengths, and areas are some parts that can be measured or calculated. Decomposing an area computation into parts can lead to an algebraic formulation as a quadratic expression, in which the terms in the expression have actual geometric meaning for students.

It is common to hear teacher stories of students who “know how to do all the parts, but they can’t put them together.” Mathematics textbooks often handle this challenge by doing the intellectual work of breaking down wholes and of assembling parts *for* the students (perhaps assuming that by reading repeated examples, students will eventually be able to replicate). Word problems in which exactly the mathematically relevant information is included, sub-problems that lay out intermediate calculations and all reasoning, and references to almost-identical worked examples, are all ways of avoiding—rather than developing—the ability to assemble understanding.

Situations that are presented with insufficient or (mathematically) extraneous information, investigations requiring students to decide how to split up the workload (and thus needing to assemble understanding at the conclusion), and problems requiring piecing together factors affecting behavior (such as the function assembly problems in the high school section of Chapter Four) are all ways to engage in this CC.

This Content Connection can serve as a vehicle for student exploration of larger-scale problems and projects, many of which will intersect with other CCs as well. Investigations in this CC will require students to decompose challenges into manageable pieces, and assemble understanding of smaller parts into understanding of a larger whole. When an investigation is included in this CC, it is crucial that decomposing and assembly is a *student* task, not one that is taken on by teacher or text.

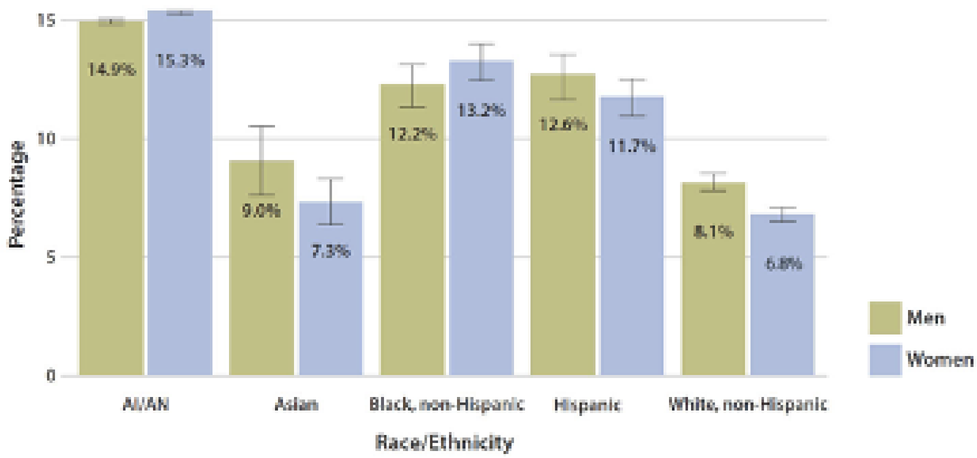
Vignette: Blood Insulin levels

Grade level: MIC I/Integrated Math 1/Algebra I

Ms. Alfie loved science and all things mathematics. She found that her Mathematics I students came to her from various backgrounds and experiences and they did not feel the same as she did about STEM subjects. She was excited to teach Integrated Mathematics I using Core Plus with the goal of exciting her students about the role mathematics plays in the world around them.

Ms. Alife was midway through the first year of IMI and felt her students were ready for a math investigation that included medicine, coming from Core Plus 1. In her materials she found several examples that included the concept of half-life and she wondered how she could use a medicine context to introduce exponential functions. She also wondered how students would embrace the topic, knowing that fractions and number sense were not topics students felt confident about. The activities they had completed around linear functions earlier in the year had helped them learn to interpret slope as a fraction and interpreting slopes within the context of the problem. For example, Ms. Alife's students were happy to consider an equation in the form $y = 3/4x + 5$ as starting at the y intercept, (0,5) and increasing 3/4 of a unit vertically for every horizontal step. They also thought about it as 3 steps up and 4 steps right for every unit. She wanted to challenge and extend her students' thinking about rates of change that were not constant, for example exponential decay in context, i.e., every 60-minute increase in time the amount of drug might decrease by 50 percent in the body.

Ms. Alife began the unit by doing a graph talk, using real world data from the Centers for Disease Control (CDC). A graph talk is a math routine where students were asked to study the graph and be ready to share what they notice and wonder (see also https://www.youcubed.org/resource/data-talks/). Ms. Alfie purposefully left the title of the graph off and asked students to brainstorm what the data was about. This is analogous to students reading a news article and having to develop a “headline” that captures the main idea.



https://www.cdc.gov/media/releases/2017/p0718-diabetes-report-infographic.html

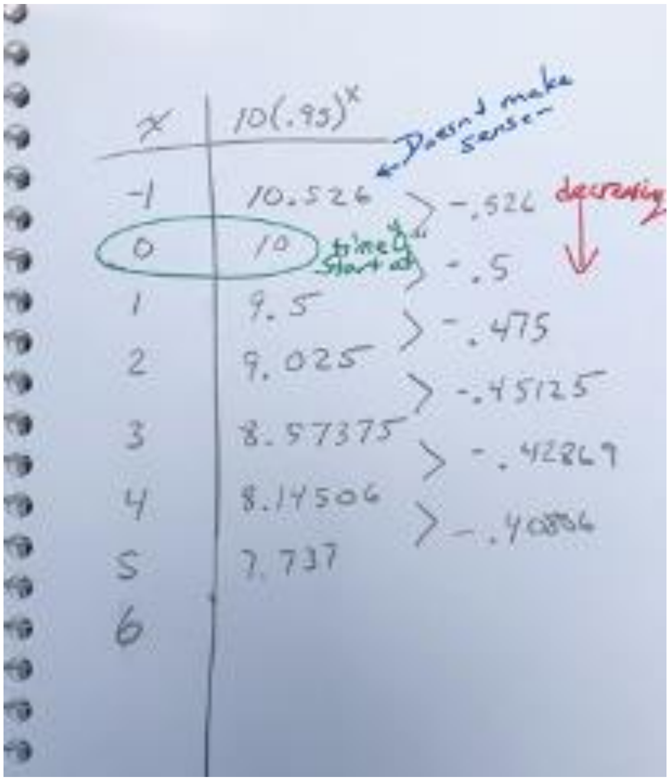
As students discussed the graph and the information they wondered if the graph showed participation in sports, academic clubs, or favorite television shows. Her students did not come close to the actual story (a way of creating a narrative to express what is being communicated) of the graph which shows data of the estimated age-adjusted prevalence of diagnosed diabetes cases in the U.S. for adults in 2013 to 2015. But Ms. Alfie knows that with more experiences with interpreting graphs and other visual display of data, her students would learn to identify the main themes.

The activity was supported by Ms. Alfie’s collaboration with a teacher who provided designated ELD instruction to the English learners in her class. ELD support included helping the students to understand and develop the critical language and grammatical structures necessary for successful engagement in this activity. The students were prepared when, after the data talk and the story reveal, Ms. Alfie asked the class to spend 20 minutes in small groups looking up information on diabetes. Each group had three types of roles: the recorder, the searcher/investigator, and brainstormers. Ms. Alfie was aware that for many students in the community, diabetes was not any medical condition, but one that affected family members deeply. She framed the investigation around using math and data science more specifically to understand the prevalence and treatments of diabetes. This was a mathematical investigation of a real-world problem, and it relied on scaffolding the context with specific medical vocabulary. On this language foundation, the first step in understanding a real-world phenomenon is to gather information. She asked each group to share the research they had found and as a class the discussion continued about the disease as well as the use of prescription drugs to improve the health and well-being of people living with the disease. Ms. Alfie then asked students to look for more information about diabetes and the hormone, insulin, and the role it plays in the body. Information was not just limited to online research. The community clinic also had pamphlets and health advice about diabetes. The students discussed the difference between public information (in the form of a pamphlet) can differ from online internet searches and sources. Ms. Alfie used these different texts to focus students as they looked closer at issues around the dosing of insulin, as it is a common therapy for diabetes.

First Ms. Alife shared with students the function: $y = 10(0.95)x$. She explained to students that the body metabolizes drugs in an interesting way and while different bodies process drugs differently we can model the metabolism of a drug with a function. Her EL students had worked with the science vocabulary in the lesson, and helped support her when other students needed support with understanding the meaning of “metabolize.” Students looked up varying definitions and came to understand that it means to “break down” over time in this context. (Assess the EL students’ understanding of phrasal verbs such as “break down,” or for that matter, “look up” as well, and do a mini-lesson on these linguistic structures, if necessary.) And it turns out that different medicines break down at different rates in our bodies. Although it seems like a straight-forward definition, many students could possibly do all computations without ever understanding this central idea.

Ms. Alfie returned to the idea of representing data in the form of a story. She told students the equation told a story of insulin metabolism and she asked students to use DESMOS to illustrate and study the function. In groups, students were asked to study the graph and make a table of values where *x* represented time and y represented the units of insulin that were injected at t=0. Together, they brainstormed responses to the question: What story does the function illustrate? Or put another way, how does the function behave?

Students worked together graphing the function and thinking about what the values meant in the table as well as the values that were in the function. Students did not always agree on how to interpret the graph or the values of the function. When they disagreed, members took turns explaining their reasoning, and responding to questions from their peers. To explain more clearly and avoid unnecessary confusion, they decided to label their axes, agree on phrases such as, “When *x* is 20, *y* is [blank],” and so on. They discussed as a class how the function was decreasing and how the output was decreasing in a way that was not linear.



Ms. Alfie asked students to think using various forms of mathematical representations beyond graphs. She introduced the table above to stimulate more thinking.

She posed the following questions:

- What is the initial amount of insulin administered?
- How much time has passed when the amount of insulin is 50 percent?
- When does the amount of insulin reach zero?

As the lesson continued students asked questions about how often a drug should be administered and why some types of medicine say one time per day, two times per day and three times per day. The lesson continued with students analyzing different equations for drug metabolism such as penicillin, where the half-life is about 1.4 hours.

As a way of wrapping up the investigation, the teacher asked students to connect what they had learned about how insulin metabolizes in the body over time with the broader theme of diabetes awareness and treatment in the community. This reinforced the use of mathematics, as well as the terms and language acquired in the lesson, and helped students solidify their understanding. Some students still had lingering questions, such as: Do people have different metabolic rates? Why do some people take different dosages of insulin? Why do some take it at different times of the day? From the students work and conversation, Ms. Alfie knew that the lesson had sparked solid mathematical thinking about variables. She wondered if a representative from the community health center could come speak with her class about these questions.

The progression of CC3 through the courses

In MIC 1, students interpret the structure of expressions by connecting parts of an expression (terms, factors, coefficients) with their meaning in the given context (primarily in linear expressions and in exponential expressions with integer exponents). They build new functions from existing ones—for instance, a constant term plus a proportional term, or a constant multiple of $f(x) = x^3$—and examine the effect of these combinations of known functions, and the meaning of these effects in terms of the quantities represented. In plane geometry, they experiment to see that, and then demonstrate why, a combination (composition) of rigid transformations is another rigid transformation, and build up rigid motions as compositions in order to demonstrate congruence of different figures. Steps in geometric constructions are understood as ways to build additional structure that can be used to produce a desired result (such as a copy of a segment or angle, or an equilateral triangle).

MIC 2 uses CC3 investigations to explore properties of the real numbers as ways in which real numbers can be combined, and to extend these properties to new numbers (e.g. extending properties of exponents to rational exponents). Investigating the structure of expressions by understanding the contributions of different parts to the whole expression continues from MIC 1. Equivalent expressions, and arithmetic with polynomials and rational expressions, are explored as different ways to put parts together, in order to highlight different features. Composing functions is a new way to build new functions from old, and frames the exploration of graph transformations such as replacing $f(x)$ by $f(kx)$, $kf(x)$, or $f(x + k)$ for specific values of k. Finally, explorations of probabilistic events made up of smaller events drives the ideas of independence and conditional probability.

In MIC—Data, investigations begin by searching for or gathering data about students' authentic questions, with the aim of exploring the effects of one or more quantity(ies) on another quantity of interest, and exploring the way that those effects combine. Thus, functional models developed to represent relationships between quantities may have parts (such as terms, factors, coefficients) corresponding to different factors influencing the quantity of interest. Thus, understanding the structure of polynomial and rational functions is a means to explaining observed relationships, and writing equivalent expressions helps to explain different characteristics of those observed relationships. Geometric measurement and dimension, and modeling with geometry, serve to build models of systems that generate the data being explored. For example, gathering data on leaf surface area of a species of plant as a function of some linear measurement (e.g. height or stem/trunk diameter), and then attempting to use that data to estimate leaf surface area for a larger specimen, will require that students wrestle with questions of dimension (does leaf surface area grow more like the surface area of the trunk or like the volume of the trunk?).

In MIC—Modeling, students may investigate features of quadratic functions (assembled from x^2, x, and constant terms) that lead to two real zeros, one real zero, and no real zeros; the latter leads to complex roots and a demonstration of the Fundamental Theorem of Algebra for quadratics, as well as to understanding the relationship between zeros and factors of polynomials. Polynomials up to degree 3 can be developed to meet building design challenges involving scaling (How much paint? How much trim? What capacity is needed for the heating system?), emphasizing the meaning in context of each term.

CA CCSSM Content in CC3

CC3: *Taking Wholes Apart, Putting Parts Together* includes much of the content of the CA CCSSM Conceptual Categories below:

- Modeling
- Algebra

Modeling and Algebra are also heavily represented in CC2: *Exploring Changing Quantities*. In addition, CC3 includes some CA CCSSM domains from other Conceptual categories. The specific domains that should be emphasized in CC3 investigations are highlighted by course below.

CA CCSSM domains by course

MIC 1: domains of emphasis for investigations in CC3 (from the CA CCSSM Mathematics I model course outline):

- Algebra
  - Seeing Structure in Expressions
- Functions
  - Building Functions (from existing functions)
- Geometry
  - Congruence (rigid motions, geometric constructions)

MIC 2: domains of emphasis for investigations in CC3 (from the CA CCSSM Mathematics II model course outline):

- Number and Quantity
  - The Real Number System
  - The Complex Number System
- Algebra
  - Seeing Structure in Equations
  - Arithmetic with Polynomials and Rational Expressions
- Functions
  - Building Functions (from existing function)
- Statistics and Probability
  - Conditional Probability and the Rules of Probability

MIC—Data: domains of emphasis for investigations in CC3:

- Algebra
  - Seeing Structure in Expressions
  - Arithmetic with Polynomials and Rational Expressions

- Geometry
    - Geometric Measurement and Dimension
    - Modeling with Geometry

MIC—Modeling: domains of emphasis for investigations in CC2:

- Number and Quantity
    - The Complex Number System
- Algebra
    - Seeing Structure in Expressions
    - Arithmetic with Polynomials and Rational Expressions
- Geometry
    - Geometric Measurement and Dimension
    - Modeling with Geometry

CC4: Discovering Shape and Space

Developing mathematical tools to explore and understand the physical world should continue to motivate explorations in shape and space. As in other areas, maintaining connection to concrete situations and authentic questions is crucial and this content area could be investigated in any of the ways—to understand, predict or affect.

Geometric situations and questions encourage different modes of thought than do numerical, algebraic, and computational work. It is important to realize that “visual thinking” or “geometric reasoning” is as legitimate as algebraic or computational thinking; and geometric thinking can provide access more readily to rich mathematical work for some students (Driscoll et al., 2007). The CA CCSSM supports this visual thinking by defining congruence and similarity in terms of dilations and rigid motions of the plane, and through its emphasis on physical models, transparencies, and geometry software.

As emphasized throughout this framework, flexibility in moving between different representations and points of view brings great mathematical power. Students should not experience geometry primarily as a way to formalize visual thinking into algebraic or numerical representations. Instead, they should have occasion to gain insight into situations presented numerically or algebraically by transforming them into geometric representations, as well as the more common algebraic or numerical representations of geometric situations. For example, students can use similar triangles to explore questions about integer-coordinate points on a line presented algebraically (Driscoll et al., 2017).

In grades three through five, students develop many foundational notions of two- and three-dimensional geometry, such as area (including surface area of three-dimensional figures), perimeter, angle measure, and volume. Shape and space work in grades six through eight is largely about connecting these notions to each other, to students’ lives, and to other areas of mathematics.

In grade six, for example, two-dimensional and three-dimensional figures are related to each other via nets and surface area (6.G.4), two-dimensional figures are related to algebraic representation via coordinate geometry (6.G.3), and volume is connected to fraction operations by exploring the size of a cube that could completely pack a shoebox with fractional edge lengths (6.G.2). In grade seven, relationships between angle or side measurements of two-dimensional figures and their overall shape (7.G.2), between three-dimensional figures and their two-dimensional slices (7.G.3), between linear and area measurements of two-dimensional figures (7.G.4), and between geometric concepts and real-world contexts (7.G.6) are all important foci.

In grade eight, two important relationships between different plane figures are defined and explored in depth (congruence and similarity), and used as contexts for reasoning in the manner discussed in Chapter 4: Exploring, Discovering, and Reasoning With and About Mathematics, the Pythagorean Theorem is developed as a relationship between an angle measure in a triangle and the area measures of three squares (8.G.6). Also, in grade eight, several clusters in the Expressions and Equations domain should sometimes be approached from a geometric point of view, with algebraic representations coming later: In an investigation, proportional relationships between quantities can be first encountered as a graph, leading to natural questions about points of intersection (8.EE.7, 8.EE.8) or the meaning of slope (8.EE.6).Mathematicians often need to employ a variety of points of view in a situation in order to gain fuller understanding. This can be literal: It is much easier to understand a three dimensional geometric solid if one can look at it from many directions. But there are many other settings in which looking at the same mathematical scene in different ways provides insight.

Vignette: Finding the Volume of a Complex Shape.

Course: Integrated 2/MIC 2/MIC—Modeling with Functions

Marina Lopez is preparing to teach her integrated high-school mathematics class 3, with a group-based interactive task that will help prepare students for learning calculus. She is using an approach that gives students the opportunity to explore a mathematics problem before being taught formal content that might help them solve it (Deslauriers et al, 2019). Her plan is to ask students to consider ways to find the volume of a complex shape, specifically a lemon. Prior to doing this, activity Marina has spent time in her class building and reinforcing group-work norms and she has previously made use of a structured approach to group work known as Complex Instruction (Cohen and Lotan, 2014) and specifically assigning roles in groups. She continues to use this because of the ways it makes authentic use of different roles to reinforce the fact that students are important resources for each other.

She opens the task on the first day holding up a lemon and asks the class, “How can we find the volume of a lemon?” While a few hands are immediately raised she does not call on anyone but tells the group they will have an opportunity over the next two days of class to answer the question using lemons and various resources. As students work in groups to tackle this problem, they will review what volume is and how it is measured, and how it relates to other measures of shapes such as surface area.

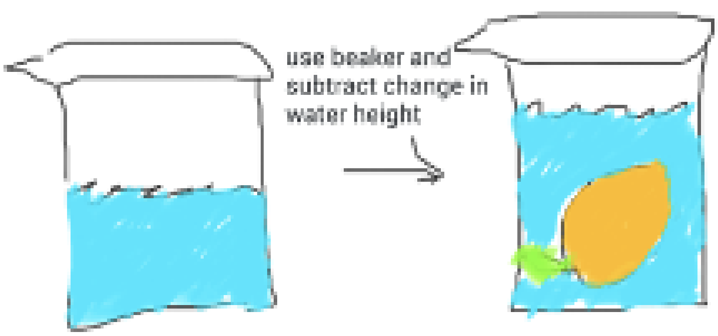
Marina knows that concrete materials are not just for elementary students. Mathematicians use models, illustrations, and visual representations to explore ideas, strategies that are highlighted in guidelines of UDL. When students visualize they bring important brain pathways into their learning of mathematics. Prior to class Marina has setup a table at the back with different supplies including different colors of modeling clay, vases, knives and cutting boards, pipe cleaners, scissors and a few other materials. Groups are free to choose from the assortment of materials provided. To facilitate the use of materials, students are instructed that only the resource manager is allowed to get up to get supplies from the resource table and they can only have 3 supplies out at one time. During the early weeks of her class Marina helped her class develop a set of group work norms and has previously used roles for groupwork so students are used to these structures and have been working on engaging productively in groups (see also Cabana, Shreve & Woodbury, 2014).

Image of supply table.

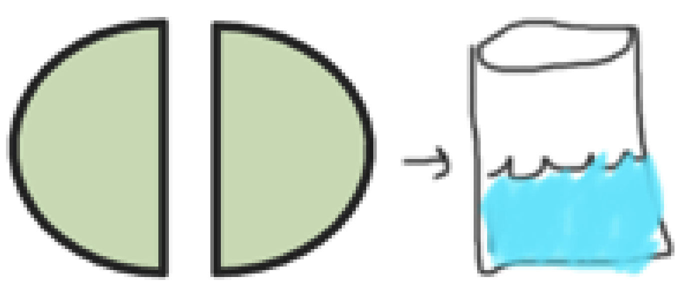
Animated noise begins to fill the room as students start talking in their groups and sharing their ideas. With much experience in group work, students exhaust the brainstorm process to collect as many ideas as possible and invite each group member to share their ideas. When ideas are not clear, they ask clarifying questions posted on the wall that promote justification and help students understand. Students also take one idea as a spark and build off it, elaborating and extending in new ways. Over time, these ideas become the group’s ideas, not just the ideas from one person. They have been given one lemon for today but have also been told they will be able to get a second lemon tomorrow, so they have some freedom to play and even mess up their lemons.

As groups begin to dig into the problem, Marina reminds students to capture their ideas with notes, drawing, and sketches so that they don’t lose track of their thinking. Students know not to worry about “complete sentences or perfect spelling” since they are just trying out ideas. Marina listens closely to discussion in each group, making quick notes of what she hears students saying. Their language is exploratory and imaginative at this stage of the lesson, e.g., “Would peeling the lemon help?” and “What about squeezing the lemon first?” and, “Is this a good way to cut it up?” Some of the students in class are multilingual and are designated at different levels of English development. As designed, these students not only have access to the task, but also multiple opportunities to use language to explore their ideas and share their mathematical thinking. The concrete materials, small-group work, and structured group presentations all provide key supports in language developments.

One group decided to use a bowl and water from the drinking fountain to see how the height of the water changes once the lemon is under the water. They draw a quick sketch to describe their idea (below). The students decide to use a marker to mark up the bowl like a beaker and begin filling it with water.



Another group has selected modeling clay and is attempting to make a mold of the lemon. They record their plan and describe that they will carefully fill the mold with water, and then find a way to measure the amount of water the mold holds.



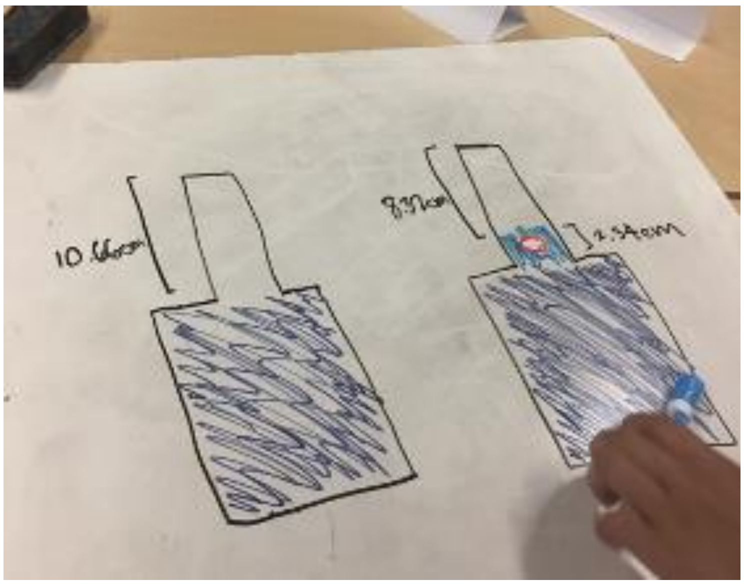
A third group has opted to use a knife and cutting board. They have decided that the shape of the lemon is very close to that of a sphere, so they can use the volume of a sphere formula to approximate the volume. To measure the lemons diameter and radius, they will cut the lemon in half, as shown in their diagram:

As this first period nears its end, Marina reminds students that they will be getting new lemons tomorrow so if they want to consider using the knives and cutting boards provided now would be the time. She also reminds them to be sure to document the work they did today and where they want to start tomorrow. They should plan to keep discussing and working as homework so they can be ready to create posters and present on day two.

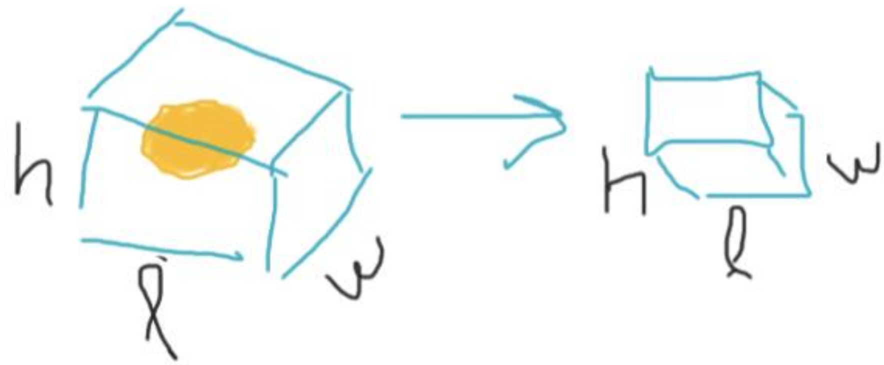
For the second day of the project, students pick up where their work the previous day ended. One group finalizes its ideas and begins creating a poster to share their strategies with the class. Adam and Andres' group managed to try two ideas, but they engage in a debate over the best ways to present their work. Marina reminds her students that the group's Reporter should take the lead in the creation of the poster, but that other roles in the group should be ready to share-out later in class. She says this as she walks among groups handing out additional lemons.

Marina knows that this is a group-worthy task because it draws on many aspects of mathematical thinking. Students are making connections to science and ideas of measurement through displacement, and to surface area, and still others groups are using a sort of "decomposition" approach by forming small cylinders. As she continues to circulate Marina, eyes the different strategies she sees groups using to document their progress, and starts noting how she can sequence the group presentations so they meet specific learning targets she wants to highlight with this lesson.

After the 15 minutes pass, Marina calls her students back together and asks a group who attempted to use a water displacement method (but was not able to finish) to share first. As they share, she writes key phrases and words on the board that highlight their creative problem solving and calls on a second group that got further using a similar method. Marina asks this group to share their thinking and build on the work of the first group. Marina refers to her notes capturing what she heard during the groupwork as a way to highlight examples of mathematical language they were using. As this second group wraps up, Julio questions the group by wondering how the displacement method (shown below) might relate to his group's method of negative space.



Marina invites Julio's group to present next. This group presents a solution using modeling clay surrounding the lemon and molded into the shape of a rectangular prism. First, they found the volume of their prism with the lemon inside, then they explained that they removed the lemon from the modeling clay and reformed it in the shape of a rectangular prism and found the volume again. They explained that the difference between the two volumes had to be the same as the volume of the lemon.



Other students in the class respond to this group's idea with enthusiasm, excited by their creativity. One student from the team that used a displacement approach raised her hand and connected with the idea that this team's method was kind of like an "opposite" of what her team did. Several students nodded in agreement. The fact that students intuited the idea of "opposite" indicates that they paying attention to the relationship among methods, namely their inverse relationship which they cannot yet define completely. This is cognitively complex work which develops over time, and students are reaching into their mathematics to find words that convey their ideas.

Finally, Marina asks a fourth group to share their explanation. Silvia explains that the group tried many things, but their favorite method involved slicing up the lemon into many pieces. The group decided that each slice could be thought of like a very short cylinder. So, the group found the volume of each slice using the formula for the volume of a cylinder and then added them all together.

As Silvia explains her groups work, several other students appear to be taking notes and multiple hands are immediately raised to ask questions.

A whole class discussion ensues around the various strategies that groups utilized. Marina is careful not to rush the discussion, and to unpack students' comments and questions that she does not understand at first. At times, other students rephrase for one another to see if the idea is clearer. Marina poses the questions:

- "What are the strengths and challenges to these approaches?"
- Which approach would you say is most accurate?"
- How do you know?"

This metacognitive part of the lesson helps students move beyond just the lemon itself, towards noticing the methods they use in their analysis. The students take turns commenting on and comparing each other's strategies. Marina closes the class period by acknowledging the various mathematical practices that students engaged with and highlights the multiple dimensions of content that students utilized.

The progression of CC4 through the courses

For a more detailed description of the content in progression, see the Geometry, 7–8, High School progression (Common Core State Standards Writing Team, 2016).

Shape and space are explored in several parallel and connected strands: Properties of geometric figures and the logical connections between them, geometric measurement, and coordinate geometry.

Coordinate geometry is first introduced in fifth grade, and is an important way that geometry can be connected to algebra, in ways that make clear the usefulness of algebraic tools and that illuminate meaning in many algebraic representations. In MIC 1 and 2, students use coordinates to prove simple geometric theorems, motivated by noticing features that seem to be true, and then trying to answer "Will that always be true? How can we know for sure?" In MIC 2, they switch between geometric and algebraic (equation) descriptions of conic sections, when such different points of view are helpful to answering authentic questions about a context.

Geometric measurement is a strand that extends across the full K–12 grade range. In MIC 2, students use dissection and transformation arguments to informally justify formulas for circumference and area of circles and volume formulas for various 3-dimensional figures. They explore the effect of scaling all linear measurements on area and volume measurements. All of these can be developed and used in the context of investigations that generate authentic questions for students: I wonder how much…; I wonder how long… etc. In MIC—Data and MIC—Modeling, geometric models of physical objects help to build models for data-driven or model-driven investigations.

While exploration of shape and space should be one of the easiest areas to motivate through investigations generating authentic questions, many students do not experience high school geometry this way. The strand that is the exploration of properties of geometric figures and the logical connections between them is the biggest culprit. One challenge is that *proving things that students consider obvious is not motivating*. As in most areas, much of the work of instructional designers (whether designing instructional materials or creating lesson plans) is to design activities in which students experience questions as authentic: that is, something they actually wonder about. After all, the mathematics of proof was originally developed to answer questions about which people were actually curious, and “it is useful for individuals to experience intellectual perturbations that are similar to those that resulted in the discovery of new knowledge” (Fuller, Rabin, & Harel, 2011). Thus, the mathematical activity of exploration of a context and deciding what might be true (by noticing patterns from examples) needs to be far more heavily represented in geometry class than is typical.

Middle school notions of congruence and similarity for plane figures are informal, based on work with transparencies or other tools that enable direct comparison. Experimentation with transformations continues in MIC 1, while definitions are made more precise. Congruence is defined in terms of rigid motions of the plane, and—because precisely finding and using rigid motions can be tedious—students show that triangles can be shown to be congruent using measurement instead. Triangle congruence criteria, demonstrated in terms of the rigid motion definition of congruence, need to answer an authentic question, perhaps as simple as “what’s the least information you can give your partner about your triangle, so that they can create a triangle that you are both certain is congruent to your original?” Similarly for geometric constructions: they must answer a wonder—“I wonder if…” or “I wonder how….”

MIC 2 introduces similarity, by adding dilations to the rigid transformations that define congruence. Students prove a variety of geometric theorems, with a focus on understanding reasoning and not on a rigid form of proof. As mentioned in CC2, the relationship between lengths of corresponding sides of similar right triangles gives rise to the fact that their ratios are constant, and thus to names for those ratios (trigonometric functions).

As MIC—Data and MIC—Modeling are both based in real-world-generated contexts, they do not include standards about exploration of shape in plane geometry, though some explorations may make use of and reinforce understanding developed in MIC 1 and 2. For instance, design challenges in MIC—Modeling might have design constraints that call on plane geometry results.

CA CCSSM Content in CC4

CC4: *Discovering Shape and Space* includes primarily the content of the CA CCSSM Conceptual Category *Geometry*. Investigations in CC4 will often involve quantities that change in related ways (e.g. lengths of sides in similar triangles) and will often require consideration of relationships between parts and wholes (e.g. the effect of scaling linear dimensions on area and volume measurements); thus, many investigations will pair CC4 with CC2 or CC3. The specific domains that should be emphasized in CC4 investigations are highlighted by course below.

CA CCSSM domains by course
MIC 1: domains of emphasis for investigations in CC4 (from the CA CCSSM Mathematics I model course outline):

- Geometry
    - Congruence
    - Expressing Geometric Properties with Equations

MIC 2: domains of emphasis for investigations in CC4 (from the CA CCSSM Mathematics II model course outline):

- Functions

    - Trigonometric Functions
- Geometry
    - Congruence
    - Similarity, Right Triangles, and Trigonometry
    - Circles
    - Expressing Geometric Properties with Equations
    - Geometric Measurement and Dimension

MIC—Data: domains of emphasis for investigations in CC4:

- Functions
    - Trigonometric Functions (for modeling periodic phenomena)
- Geometry
    - Expressing Geometric Properties with Equations
    - Geometric Measurement and Dimension
    - Modeling with Geometry

MIC—Modeling: domains of emphasis for investigations in CC4:

- Functions
    - Trigonometric Functions (for modeling periodic phenomena)
- Geometry
    - Expressing Geometric Properties with Equations
    - Geometric Measurement and Dimension
    - Modeling with Geometry

The Integrated Mathematics Pathway

Many schools and districts in California have implemented an "Integrated Mathematics Pathway" according to the course outlines in the CA CCSSM. In recognition of this investment, this Framework continues to support these pathways, as the field strives to develop truly integrated approaches (in the sense of the *Definition of Integration* above) to the teaching and learning of higher mathematics content. The standards for the Integrated Pathway, by course, begin on p. 85 of the CA CCSSM. (https://www.cde.ca.gov/be/st/ss/documents/ccssmathstandardaug2013.pdf)

These courses are described here.

Integrated Math I

The fundamental purpose of the Mathematics I course is to formalize and extend students' understanding of linear functions and their applications. The critical topics of study deepen and extend understanding of linear relationships—in part, by contrasting them with exponential phenomena and, in part, by applying linear models to data that exhibit a linear trend. Mathematics I uses properties and theorems involving congruent figures to deepen and extend geometric knowledge gained in prior grade levels. The courses in the Integrated Pathway follow the structure introduced in the K–8 grade levels of the California Common Core State Standards for Mathematics (CA CCSSM); they present mathematics as a coherent subject and blend standards from different conceptual categories.

The standards in the integrated Mathematics I course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, Geometry, and Statistics and Probability. The content of the course is explained in the addendum according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences.

What Students Learn in Mathematics I

Students in Mathematics I continue their work with expressions and modeling and analysis of situations. In previous grade levels, students informally defined, evaluated, and compared functions, using them to model relationships between quantities. In Mathematics I, students learn function notation and develop the concepts of domain and range. Students move beyond viewing functions as processes that take inputs and yield outputs and begin to view functions as objects that can be combined with operations (e.g., finding). They explore many examples of functions, including sequences. They interpret functions that are represented graphically, numerically, symbolically, and verbally, translating between representations and understanding the limitations of various representa- tions. They work with functions given by graphs and tables, keeping in mind that these representations are likely to be approximate and incomplete, depending upon the context. Students’ work includes functions that can be described or approximated by formulas, as well as those that cannot. When functions describe relationships between quantities arising from a context, students reason with the units in which those quantities are measured. Students build on and informally extend their understanding of integer exponents to consider exponential functions. They compare and contrast linear and exponential functions, distinguishing between additive and multiplicative change. They also interpret arithmetic sequences as linear functions and geometric sequences as exponential functions.

Students who are prepared for Mathematics I have learned to solve linear equations in one variable and have applied graphical and algebraic methods to analyze and solve systems of linear equations in two variables. Mathematics I builds on these earlier experiences by asking students to analyze and explain the process of solving an equation and to justify the process used in solving a system of equations. Students develop fluency in writing, interpreting, and translating between various forms of linear equations and inequalities and using them to solve problems. They master solving linear equations and apply related solution techniques and the laws of exponents to the creation and solving of simple

exponential equations. Students explore systems of equations and inequalities, finding and interpreting solutions. All of this work is based on understanding quantities and the relationships between them.

In Mathematics I, students build on their prior experiences with data, developing more formal means of assessing how a model fits data. Students use regression techniques to describe approximately linear relationships between quantities. They use graphical representations and knowledge of the context to make judgments about the appropriateness of linear models. With linear models, they look at residuals to analyze the goodness of fit.

In previous grade levels, students were asked to draw triangles based on given measurements. They also gained experience with rigid motions (translations, reflections, and rotations) and developed notions about what it means for two objects to be congruent. In Mathematics I, students establish triangle congruence criteria based on analyses of rigid motions and formal constructions. They solve problems about triangles, quadrilaterals, and other polygons. They apply reasoning to complete geometric constructions and explain why the constructions work. Finally, building on their work with the Pythagorean Theorem in the grade-eight standards to find distances, students use a rectangular coordinate system to verify geometric relationships, including properties of special triangles and quadrilaterals and slopes of parallel and perpendicular lines.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (SMPs) apply throughout each course and, together with the Standards for Mathematical Content, prescribe that students experience mathematics as a coherent, relevant, and meaningful subject. The SMPs represent a picture of what it looks like for students to do mathematics and, to the extent possible, content instruction should include attention to appropriate practice standards.

The CA CCSSM call for an intense focus on the most critical material, allowing depth in learning, which is carried out through the SMPs. Connecting practices and content happens in the context of working on problems; the very first SMP is to make sense of problems and persevere in solving them. Table XX gives examples of how students can engage in the SMPs in Mathematics I.

Table XX. Standards for Mathematical Practice—Explanation and Examples for Mathematics

Standards for Mathematical Practice	Explanation and Examples

SMP.1 Make sense of problems and persevere in solving them.	Students persevere when attempting to understand the differences between linear and exponential functions. They make diagrams of geometric problems to help make sense of the problems.
SMP.2 Reason abstractly and quantitatively.	Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.
SMP.3 Construct viable arguments and critique the reasoning of others. Students build proofs by induction and proofs by contradiction. CA 3.1 (for higher mathematics only).	Students reason through the solving of equations, recognizing that solving an equation involves more than simply following rote rules and steps. They use language such as "If_____, then______" when explaining their solution methods and provide justification for their reasoning.
SMP.4 Model with mathematics.	Students apply their mathematical understanding of linear and exponential functions to many real-world problems, such as linear and exponential growth. Students also discover mathematics through experimentation and by examining patterns in data from real-world contexts.
SMP.5 Use appropriate tools strategically.	Students develop a general understanding of the graph of an equation or function as a representation of that object, and they use tools such as graphing calculators or graphing software to create graphs in more complex examples, understanding how to interpret the results.
SMP.6 Attend to precision.	Students use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure and labeling axes to clarify the correspondence with quantities in a problem.

SMP.7 Look for and make use of structure.	Students recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects.
SMP.8 Look for and express regularity in repeated reasoning.	Students see that the key feature of a line in the plane is an equal difference in outputs over equal intervals of inputs, and that the result of evaluating the expression $\frac{y_2 - y_1}{x_2 - x_1}$ for points on the line is always equal to a certain number m. Therefore, if (x, y) is a generic point on this line, the equation $m = \frac{y - y_1}{x - x_1}$ will give a general equation of that line.

SMP.4 holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a significant place in instruction. Some standards are marked with a star () symbol to indicate that they are modeling standards—that is, they may be applied to real-world modeling situations more so than other standards. In the description of the Mathematics I content standards that follow, Modeling is covered first to emphasize its importance in the higher mathematics curriculum.

Integrated Math II

The Mathematics II course focuses on quadratic expressions, equations, and functions and on comparing the characteristics and behavior of these expressions, equations, and functions to those of linear and exponential relationships from Mathematics I. The need for extending the set of rational numbers arises, and students are introduced to real and complex numbers. Links between probability and data are explored through conditional probability and counting methods and involve the use of probability and data in making and evaluating decisions.

The study of similarity leads to an understanding of right-triangle trigonometry and connects to quadratics through Pythagorean relationships. Circles, with their quadratic algebraic representations, finish out the course.

The courses in the Integrated Pathway follow the structure introduced in the K–8 grade levels of the California Common Core State Standards for Mathematics (CA CCSSM); they present mathematics as a coherent subject and blend standards from different conceptual categories.

The standards in the integrated Mathematics II course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, Geometry, and Statistics and Probability. The course content is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences.

What Students Learn in Mathematics II

In Mathematics II, students extend the laws of exponents to rational exponents and explore distinctions between rational and irrational numbers by considering their decimal representations. Students learn that when quadratic equations do not have real solutions, the number system can be extended so that solutions exist, analogous to the way in which extending whole numbers to negative numbers allows $x + 1 = 0$ to have a solution. Students explore relationships between number systems: whole numbers, integers, rational numbers, real numbers, and complex numbers. The guiding principle is that equations with no solutions in one number system may have solutions in a larger number system.

Students consider quadratic functions, comparing the key characteristics of quadratic functions to those of linear and exponential functions. They select from these functions to model phenomena. Students learn to anticipate the graph of a quadratic function by interpreting various forms of quadratic expressions. In particular, they identify the real solutions of a quadratic equation as the zeros of a related quadratic function. Students also learn that when quadratic equations do not have real solutions, the graph of the related quadratic function does not cross the horizontal axis. Additionally, students expand their experience with functions to include more specialized functions—absolute value, step, and other piecewise-defined functions.

Students in Mathematics II focus on the structure of expressions, writing equivalent expressions to clarify and reveal aspects of the quantities represented. Students create and solve equations, inequalities, and systems of equations involving exponential and quadratic expressions.

Building on probability concepts introduced in the middle grades, students use the language of set theory to expand their ability to compute and interpret theoretical and experimental probabilities for compound events, attending to mutually exclusive events, independent events, and conditional probability. Students use probability to make informed decisions, and they should make use of geometric probability models whenever possible.

Students apply their earlier experience with dilations and proportional reasoning to build a formal understanding of similarity. They identify criteria for similarity of triangles, use similarity to solve problems, and apply similarity in right triangles to understand right-triangle trigonometry, with particular attention to special right triangles and the Pythagorean Theorem. In Mathematics II, students develop facility with geometric proof. They use what they know about congruence and similarity to prove theorems involving lines, angles, triangles, and other polygons. They also explore a variety of formats for writing proofs.

In Mathematics II, students prove basic theorems about circles, chords, secants, tangents, and angle measures. In the Cartesian coordinate system, students use the distance formula to write the equation of a circle when given the radius and the coordinates of its center, and the equation of a parabola with a vertical axis when given an equation of its horizontal directrix and the coordinates of its focus. Given an equation of a circle, students draw the graph in the coordinate plane and apply techniques for solving quadratic equations to determine intersections between lines and circles, between lines and parabolas, and between two circles. Students develop informal arguments to justify common formulas for circumference, area, and volume of geometric objects, especially those related to circles.

Examples of Key Advances from Mathematics I

Students extend their previous work with linear and exponential expressions, equations, and systems of equations and inequalities to quadratic relationships.

- A parallel extension occurs from linear and exponential functions to quadratic functions: students begin to analyze functions in terms of transformations.
- Building on their work with transformations, students produce increasingly formal arguments about geometric relationships, particularly around notions of similarity.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (SMPs) apply throughout each course and, together with the Standards for Mathematical Content, prescribe that students experience mathematics as a coherent, relevant, and meaningful subject. The SMPs represent a picture of what it looks like for students to do mathematics and, to the extent possible, content instruction should include attention to appropriate practice standards.

The CA CCSSM call for an intense focus on the most critical material, allowing depth in learning, which is carried out through the SMPs. Connecting content and practices happens in the context of working on problems, as is evident in the first SMP ("Make sense of problems and persevere in solving them"). Table XX offers examples of how students can engage in each mathematical practice in the Mathematics II course.

Table XX. Standards for Mathematical Practice—Explanation and Examples for Mathematics II

Standards for Mathematical Practice	Explanation and Examples
SMP.1 Make sense of problems and persevere in solving them.	Students persevere when attempting to understand the differences between quadratic functions and linear and exponential functions studied previously. They create

	diagrams of geometric problems to help make sense of the problems.
SMP.2 Reason abstractly and quantitatively.	Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.
SMP.3 Construct viable arguments and critique the reasoning of others. Students build proofs by induction and proofs by contradiction. CA 3.1 (for higher mathematics only).	Students construct proofs of geometric theorems based on congruence criteria of triangles. They understand and explain the definition of *radian measure.*
SMP.4 Model with mathematics.	Students apply their mathematical understanding of quadratic functions to real-world problems. Students also discover mathematics through experimentation and by examining patterns in data from real-world contexts.
SMP.5 Use appropriate tools strategically.	Students develop a general understanding of the graph of an equation or function as a representation of that object, and they use tools such as graphing calculators or graphing software to create graphs in more complex examples, understanding how to interpret the result.
SMP.6 Attend to precision.	Students begin to understand that a *rational number* has a specific definition and that *irrational numbers* exist. When deciding if an equation can describe a function, students make use of the definition of *function* by asking, “Does every input value have exactly one output value?”
SMP.7 Look for and make use of structure.	Students apply the distributive property to develop formulas such as $(a \pm b)^2 = a^2 \pm 2an + b^2$. They see that the expression $5 + (x-2)^2$ takes the form of “5 plus ‘something’ squared,” and therefore that expression can be no smaller than 5.
SMP.8 Look for and express regularity in repeated reasoning.	Students notice that consecutive numbers in the sequence of squares 1, 4, 9, 16, and 25 always differ by an odd number. They use polynomials to represent this interesting finding by expressing it as .

SMP.4 holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a significant place in instruction. Some standards are marked with a star (★) symbol to indicate that they are modeling standards—that is, they may be applied to real-world modeling situations more so than other standards. Modeling in higher mathematics centers on problems that arise in everyday life, society, and the workplace. Such problems may draw upon mathematical content knowledge and skills articulated in the standards prior to or during the Mathematics II course.

Integrated Math III

In the Mathematics III course, students expand their repertoire of functions to include polynomial, rational, and radical functions. They also expand their study of right-triangle trigonometry to include general triangles. And, finally, students bring together all of their experience with functions and geometry to create models and solve contextual problems. The courses in the Integrated Pathway follow the structure introduced in the K–8 grade levels of the California Common Core State Standards for Mathematics (CA CCSSM); they present mathematics as a coherent subject and blend standards from different conceptual categories.

The standards in the integrated Mathematics III course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, Geometry, and Statistics and Probability. The course content is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences.

What Students Learn in Mathematics III

In Mathematics III, students understand the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. They connect multiplication of polynomials with multiplication of multi-digit integers and division of polynomials with long division of integers. Students identify zeros of polynomials and make connections between zeros of polynomials and solutions of polynomial equations. Their work on polynomial expressions culminates with the Fundamental Theorem of Algebra. Rational numbers extend the arithmetic of integers by allowing division by all numbers except 0. Similarly, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial. A central theme of working with rational expressions is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

Students synthesize and generalize what they have learned about a variety of function families. They extend their work with exponential functions to include solving exponential equations with logarithms. They explore the effects of transformations on graphs of diverse functions, including functions arising in an application, in order to abstract the general principle that transformations on a graph always have the same effect, regardless of the type of the underlying functions.

Students develop the Laws of Sines and Cosines in order to find missing measures of general (not necessarily right) triangles. They are able to distinguish whether three given measures (angles or sides) define 0, 1, 2, or infinitely many triangles. This discussion of general triangles opens up the idea of trigonometry applied beyond the right triangle—that is, at least to obtuse angles. Students build on this idea to develop the notion of radian measure for angles and extend the domain of the trigonometric functions to all real numbers. They apply this knowledge to model simple periodic phenomena.

Students see how the visual displays and summary statistics they learned in previous grade levels or courses relate to different types of data and to probability distributions.

They identify different ways of collecting data—including sample surveys, experiments, and simulations—and recognize the role that randomness and careful design play in the conclusions that may be drawn.

Finally, students in Mathematics III extend their understanding of modeling: they identify appropriate types of functions to model a situation, adjust parameters to improve the model, and compare models by analyzing appropriateness of fit and by making judgments about the domain over which a model is a good fit. The description of modeling as "the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions" (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010e) is one of the main themes of this course. The discussion about modeling and the diagram of the modeling cycle that appear in this chapter should be considered when students apply knowledge of functions, statistics, and geometry in a modeling context.

Examples of Key Advances from Mathematics II

- Students begin to see polynomials as a system analogous to the integers that they can add, subtract, multiply, and so forth. Subsequently, polynomials can be extended to rational expressions, which are analogous to rational numbers.
- Students extend their knowledge of linear, exponential, and quadratic functions to include a much broader range of classes of functions.
- Students begin to examine the role of randomization in statistical design.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (SMP) apply throughout each course and, together with the Standards for Mathematical Content, prescribe that students experience mathematics as a coherent, relevant, and meaningful subject. The SMPs represent a picture of what it looks like for students to do mathematics and, to the extent possible, content instruction should include attention to appropriate practice standards. The Mathematics III course offers ample opportunities for students to engage with each SMP; table XX offers some examples.

Table XX. Standards for Mathematical Practice—Explanation and Examples for Mathematics III

Standards for Mathematical Practice	Explanation and Examples
SMP.1 Make sense of problems and persevere in solving them.	Students apply their understanding of various functions to real-world problems. They approach complex mathematics problems and break them down into smalle problems, synthesizing the results when presenting solutions.
SMP.2 Reason abstractly and quantitatively.	Students deepen their understanding of variables—for example, by understanding that changing the values of the parameters in the expression has consequences for the graph of the function. They interpret these parameters in a real-world context.
SMP.3 Construct viable arguments and critique the reasoning of others. Students build proofs by induction and proofs by contradiction. CA 3.1 (for higher mathematics only).	Students continue to reason through the solution of an equation and justify their reasoning to their peers. Students defend their choice of a function when modeling a real-world situation.
SMP.4 Model with mathematics.	Students apply their new mathematical understanding to real-world problems, making use of their expanding repertoire of functions in modeling. Students also discover mathematics through experimentation and by examining patterns in data from real-world contexts.
SMP.5 Use appropriate tools strategically.	Students continue to use graphing technology to deepen their understanding of the behavior of polynomial, rational, square root, and trigonometric functions.
SMP.6 Attend to precision.	Students make note of the precise definition of *complex number,* understanding that real numbers are a subset o complex numbers. They pay attention to units in real-world problems and use unit analysis as a method for verifying their answers.
SMP.7 Look for and make use of structure.	Students understand polynomials and rational numbers as sets of mathematical objects that have particular operations and properties. They understand the periodicity of sine and cosine and use these functions to model periodic phenomena.

SMP.8 Look for and express regularity in repeated reasoning.	Students observe patterns in geometric sums—for example, that the first several sums of the form $\sum_{k=0}^{n} 2^k$ can be written as follows: $1 = 2^1 - 1$ $1 + 2 = 2^2 - 1$ $1 + 2 + 4 = 2^3 - 1$ $1 + 2 + 4 + 8 = 2^4 - 1$ Students use this observation to make a conjecture about any such sum.

The Traditional High School Pathway

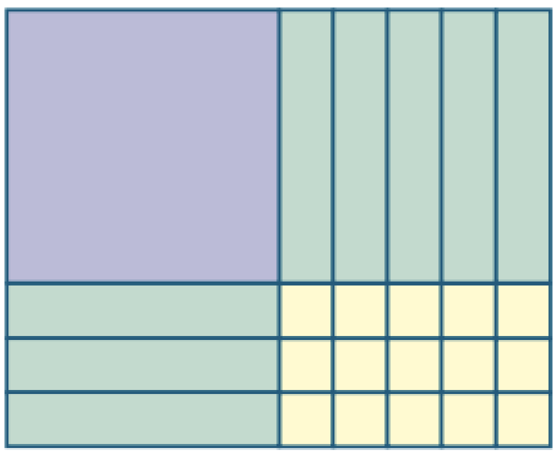
Most of us are familiar with the Algebra I–geometry–Algebra II sequence of high school mathematics courses, as it has been the most common pathway for decades. The six conceptual categories for the CA CCSSM at the high school level are Number and Quantity, Algebra, Functions, Modeling, Geometry, and Statistics and Probability. In the Traditional Pathway described in the CA CCSSM, the standards from these conceptual categories have been organized into the three courses of Algebra I, Geometry, and Algebra II. Despite having a new set of standards, as of 2013, the outline of the courses has not changed significantly, so the outlines below will look familiar to many. The standards for the Traditional Pathway, by course, begin on p. 59 of the CA CCSSM. (https://www.cde.ca.gov/be/st/ss/documents/ccssmathstandardaug2013.pdf)

Note that “Traditional Pathway” refers to the organization of content, not to teaching practices. Although these courses are traditional in their content, they should be taught through active student engagement, as set out in the Mathematics: Investigating and Connecting pathway, and whenever possible students should see and work on content that is conceptually integrated.

Algebra I

The main purpose of Algebra I is to develop students’ fluency with linear, quadratic, and exponential functions. The critical areas of instruction involve deepening and extending students’ understanding of linear and exponential relationships by comparing and contrasting those relationships and by applying linear models to data that exhibit a linear trend. In addition, students engage in methods for analyzing, solving, and using exponential and quadratic functions. Some of the overarching elements of the Algebra I course include the notion of *function,* solving equations, rates of change and growth patterns, graphs as representations of functions, and modeling.

Figure A1-2. Algebra Tiles



The rectangle above has height $(x+3)$ and base $(x+5)$. The total area represented, the product of these binomials, is seen to be $x^2+5x+3x+15=x^2+8x+15$.

For the Traditional Pathway, the standards in the Algebra I course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, and Statistics and Probability. The course content is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not simply topics to be checked off from a list during isolated units of instruction; rather, they represent content that should be present throughout the school year in rich instructional experiences.

Table A1-1. Standards for Mathematical Practice—Explanation and Examples for Algebra I

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	Students learn that patience is often required to fully understand what a problem is asking. They discern between useful and extraneous information. They expand their repertoire of expressions and functions that can be used to solve problems.
MP.2 Reason abstractly and quantitatively.	Students extend their understanding of slope as the rate of change of a linear function to comprehend that the average rate of change of any function can be computed over an appropriate interval.
MP.3 Construct viable arguments and critique the reasoning of others. **Students build proofs by induction and proofs by contradiction. CA 3.1** (for higher mathematics only).	Students reason through the solving of equations, recognizing that solving an equation involves more than simply following rote rules and steps. They use language such as "If ________, then ________" when explaining their solution methods and provide justification for their reasoning.
MP.4 Model with mathematics.	Students also discover mathematics through experimentation and by examining data patterns from real-world contexts. Students apply their new mathematical understanding of exponential, linear, and quadratic functions to real-world problems.
MP.5 Use appropriate tools strategically.	Students develop a general understanding of the graph of an equation or function as a representation of that object, and they use tools such as graphing calculators or graphing software to create graphs in more complex examples, understanding how to interpret results. They construct diagrams to solve problems.
MP.6 Attend to precision.	Students begin to understand that a *rational number* has a specific definition and that *irrational numbers* exist. They make use of the definition of *function* when deciding if an equation can describe a function by asking, "Does every input value have exactly one output value?"
MP.7 Look for and make use of structure	Students develop formulas such as $(a \pm b)^2 = a^2 \pm 2ab + b^2$ by applying the distributive property. Students see that the expression $5+(x-2)^2$ takes the form of 5 plus "something squared," and because "something squared" must be positive or zero, the expression can be no smaller than 5.
MP.8 Look for and express regularity in repeated reasoning.	Students see that the key feature of a line in the plane is an equal difference in outputs over equal intervals of inputs, and that the result of evaluating the expression $\frac{y_2 - y_1}{x_2 - x_1}$ for points on the line is always equal to a certain number m. Therefore, if (x, y) is a generic point on this line, the equation $m = \frac{y - y_1}{x - x_1}$ will give a general equation of that line.

What Students Learn in Algebra I

In Algebra I, students use reasoning about structure to define and make sense of rational exponents and explore the algebraic structure of the rational and real number systems. They understand that numbers in real-world applications often have units attached to them—that is, the numbers are considered *quantities*.

Student work with numbers and operations throughout elementary and middle school leads them to an understanding of the structure of the number system; in Algebra I, students explore the structure of algebraic expressions and polynomials. They see that certain properties must persist when they work with expressions that are meant to represent numbers—which they now write in an abstract form involving variables. When two expressions with overlapping domains are set as equal to each other, resulting in an equation, there is an implied solution set (be it empty or non-empty), and students not only refine their techniques for solving equations and finding the solution set, but they can clearly explain the algebraic steps they used to do so.

Students began their exploration of linear equations in middle school, first by connecting proportional equations to graphs, tables, and real-world contexts, and then moving toward an understanding of general linear equations ($y = mx + b$, $m \neq 0$) and their graphs. In Algebra I, students extend this knowledge to work with absolute value equations, linear inequalities, and systems of linear equations. After learning a more precise definition of *function* in this course, students examine this new idea in the familiar context of linear equations—for example, by seeing the solution of a linear equation as solving for two linear functions.

Students continue to build their understanding of functions beyond linear types by investigating tables, graphs, and equations that build on previous understandings of numbers and expressions. They make connections between different representations of the same function. They also learn to build functions in a modeling context and solve problems related to the resulting functions. Note that in Algebra I the focus is on linear, simple exponential, and quadratic equations.

Finally, students extend their prior experiences with data, using more formal means of assessing how a model fits data. Students use regression techniques to describe approximately linear relationships between quantities. They use graphical representations and knowledge of the context to make judgments about the appropriateness of linear models. With linear models, students look at residuals to analyze the goodness of fit.

Examples of Key Advances from Kindergarten Through Grade Eight

- Having already extended arithmetic from whole numbers to fractions (grades four through six) and from fractions to rational numbers (grade seven), students in grade eight encountered specific irrational numbers such as 5 and ð. In Algebra I, students begin to understand the real number *system*. See Chapter Three: Number Sense for a detailed progression of how students' understanding of numbers develops through the grades.
- Students in middle grades worked with measurement units, including units obtained by multiplying and dividing quantities. In Algebra I (conceptual category N–Q), students apply these skills in a more sophisticated fashion to solve problems in which reasoning about units adds insight.
- Algebraic themes beginning in middle school continue and deepen during high school. As early as grades six and seven, students began to use the properties of operations to generate equivalent expressions (standards 6.EE.3 and 7.EE.1). By grade seven, they began to recognize that rewriting expressions in different forms could be useful in problem solving (standard 7.EE.2). In Algebra I, these aspects of algebra carry forward as students continue to use properties of operations to rewrite expressions, gaining fluency and engaging in what has been called "mindful manipulation."
- Students in grade eight extended their prior understanding of proportional relationships to begin working with functions, with an emphasis on linear functions. In Algebra I, students learn linear and quadratic functions. Students encounter other kinds of functions to ensure that general principles of working with functions are perceived as applying to all functions, as well as to enrich the range of quantitative relationships considered in problems.
- Students in grade eight connected their knowledge about proportional relationships, lines, and linear equations (standards 8.EE.5–6). In Algebra I, students solidify their understanding of the analytic geometry of lines. They understand that in the Cartesian coordinate plane: the graph of any linear

equation in two variables is a line; any line is the graph of a linear equation in two variables.

- As students acquire mathematical tools from their study of algebra and functions, they apply these tools in statistical contexts (e.g., standard S-ID.6). In a modeling context, they might informally fit a quadratic function to a set of data, graphing the data and the model function on the same coordinate axes. They also draw on skills first learned in middle school to apply basic statistics and simple probability in a modeling context. For example, they might estimate a measure of center or variation and use it as an input for a rough calculation.
- Algebra I techniques open an extensive variety of solvable word problems that were previously inaccessible or very complex for students in kindergarten through grade eight. This expands problem solving dramatically.

Example: Exponential Growth **F-BF.1–2**

When a quantity grows with time by a multiplicative factor greater than 1, it is said the quantity grows exponentially. Hence, if an initial population of bacteria, P_0, doubles each day, then after t days, the new population is given by $P(t) = P_0 2^t$. This expression can be generalized to include different growth rates, r, as in $P(t) = P_0 r^t$. A more specific example illustrates the type of problem that students may face after they have worked with basic exponential functions:

On June 1, a fast-growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way that the area covered by the algae doubles every day. If the algae continue to grow unabated, the lake will be totally covered, and the fish in the lake will suffocate. Based on the current rate at which the algae are growing, this will happen on June 30.

Possible Questions to Ask:

a. When will the lake be covered halfway?

b. Write an equation that represents the percentage of the surface area of the lake that is covered in algae, as a function of time (in days) that passes since the algae were introduced into the lake.

Solution and Comments:

a. Since the population doubles each day, and since the entire lake will be covered by June 30, this implies that half the lake was covered on June 29.

b. If $P(t)$ represents the *percentage* of the lake covered by the algae, then we know that $P(29) = P_0 2^{29} = 100$ (note that June 30 corresponds to $t = 29$). Therefore, we can solve for the initial percentage of the lake covered, $P_0 = \frac{100}{2^{29}} \approx 1.86 \times 10^{-7}$. The equation for the percentage of the lake covered by algae at time t is therefore $P(t) = (1.86 \times 10^{-7})2^t$.

Adapted from Illustrative Mathematics 2013i.

Geometry

The fundamental purpose of the geometry course is to introduce students to formal geometric proofs and the study of plane figures, culminating in the study of right-triangle trigonometry and circles. Students begin to formally prove results about the geometry of the plane by using previously defined terms and notions. Similarity is explored in greater detail, with an emphasis on discovering trigonometric relationships and solving problems with right triangles. The correspondence between the plane and the Cartesian coordinate system is explored when students connect algebra concepts with geometry concepts. Students explore probability concepts and use probability in real-world situations. The major mathematical ideas in the geometry course include geometric transformations, proving geometric theorems, congruence and similarity, analytic geometry, right-triangle trigonometry, and probability.

Table G-1. Standards for Mathematical Practice—Explanation and Examples for Geometry

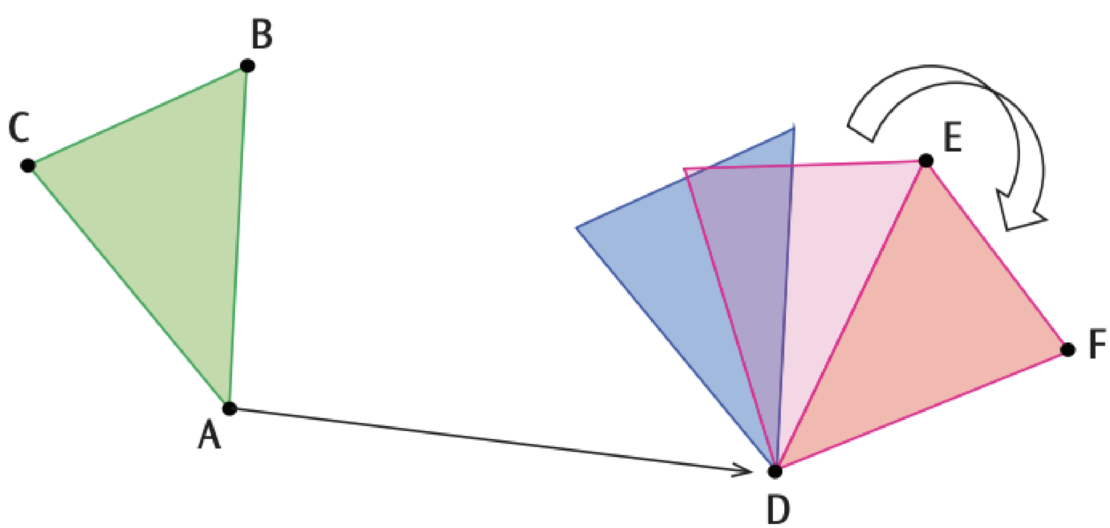
Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	Students construct accurate diagrams of geometry problems to help make sense of them. They organize their work so that others can follow their reasoning (e.g., in proofs).
MP.2 Reason abstractly and quantitatively.	Students understand that the coordinate plane can be used to represent geometric shapes and transformations, and therefore they connect their understanding of number and algebra to geometry.
MP.3 Construct viable arguments and critique the reasoning of others. **Students build proofs by induction and proofs by contradiction. CA 3.1** (for higher mathematics only).	Students reason through the solving of equations, recognizing that solving an equation involves more than simply following rote rules and steps. They use language such as "If ________, then ________" when explaining their solution methods and provide justification for their reasoning.
MP.4 Model with mathematics.	Students apply their new mathematical understanding to real-world problems. They learn how transformational geometry and trigonometry can be used to model the physical world.
MP.5 Use appropriate tools strategically.	Students make use of visual tools for representing geometry, such as simple patty paper, transparencies, or dynamic geometry software.
MP.6 Attend to precision.	Students develop and use precise definitions of geometric terms. They verify that a particular shape has specific properties and justify the categorization of the shape (e.g., a rhombus versus a quadrilateral).
MP.7 Look for and make use of structure.	Students construct triangles in quadrilaterals or other shapes and use congruence criteria of triangles to justify results about those shapes.
MP.8 Look for and express regularity in repeated reasoning.	Students explore rotations, reflections, and translations, noticing that some attributes of shapes (e.g., parallelism, congruency, orientation) remain the same. They develop properties of transformations by generalizing these observations.

The standards in the traditional geometry course come from the following conceptual categories: Modeling, Geometry, and Statistics and Probability. The content of the course is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences.

What Students Learn in Geometry

Although there are many types of geometry, school mathematics is devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates). In the higher mathematics courses, students begin to formalize their geometry experiences from elementary and middle school, using definitions that are more precise and developing careful proofs. The standards for grades seven and eight call for students to see two-dimensional shapes as part of a generic plane (i.e., the Euclidean plane) and to explore transformations of this plane as a way to determine whether two shapes are congruent or similar.

Figure G-2. Illustration of the Reasoning That Congruent Corresponding Parts Imply Triangle Congruence



Point A is translated to D, the resulting image of ΔABC is rotated so as to place B onto E, and the image is then reflected along line segment DE to match point C to F.

These concepts are formalized in the geometry course, and students use transformations to prove geometric theorems. The definition of congruence in terms of rigid motions provides a broad understanding of this means of proof, and students explore the consequences of this definition in terms of congruence criteria and proofs of geometric theorems.

Students investigate triangles and decide when they are similar—and with this newfound knowledge and their prior understanding of proportional relationships, they define trigonometric ratios and solve problems by using right triangles. They investigate circles and prove theorems about them. Connecting to their prior experience with the coordinate plane, they prove geometric theorems by using coordinates and describe shapes with equations. Students extend their knowledge of area and volume formulas to those for circles, cylinders, and other rounded shapes. Finally, continuing the development of statistics and probability, students investigate probability concepts in precise terms, including the independence of events and conditional probability.

Examples of Key Advances from Previous Grade Levels or Courses

- Because concepts such as rotation, reflection, and translation were treated in the grade-eight standards mostly in the context of hands-on activities and with an emphasis on geometric intuition, the geometry course places equal weight on precise definitions.
- In kindergarten through grade eight, students worked with a variety of geometric measures: length, area, volume, angle, surface area, and circumference. In geometry, students apply these component skills in tandem with others in the course of modeling tasks and other substantial applications (MP.4).
- The skills that students develop in Algebra I around simplifying and transforming square roots will be useful when solving problems that involve distance or area and that make use of the Pythagorean Theorem.
- Students in grade eight learned the Pythagorean Theorem and used it to determine distances in a coordinate system (8.G.6–8). In geometry, students build on their understanding of distance in coordinate systems and draw on their growing command of algebra to connect equations and graphs of circles (G-GPE.1).
- The algebraic techniques developed in Algebra I can be applied to study analytic geometry. Geometric objects can be analyzed by the algebraic equations that

give rise to them. Algebra can be used to prove some basic geometric theorems in the Cartesian plane.

Example: Defining Rotations **G-CO.4**

Mrs. B wants to help her class understand the following definition of a *rotation*:

> A *rotation* about a point P through angle α is a transformation $A \mapsto A'$ such that (1) if point A is different from P, then $PA = PA'$ and the measure of $\angle APA' = \alpha$; and (2) if point A is the same as point P, then $A' = A$.

Mrs. B gives her students a handout with several geometric shapes on it and a point, P, indicated on the page. In pairs, students copy the shapes onto a transparency sheet and rotate them through various angles about P; then they transfer the rotated shapes back onto the original page and measure various lengths and angles as indicated in the definition.

While justifying that the properties of the definition hold for the shapes given to them by Mrs. B, the students also make some observations about the effects of a rotation on the entire plane. For example:

- Rotations preserve lengths.
- Rotations preserve angle measures.
- Rotations preserve parallelism.

In a subsequent exercise, Mrs. B plans to allow students to explore more rotations on dynamic geometry software, asking them to create a geometric shape and rotate it by various angles about various points, both part of the object and not part of the object.

Algebra II

Algebra II course extends students' understanding of functions and real numbers and increases the tools students have for modeling the real world. Students in Algebra II extend their notion of number to include complex numbers and see how the introduction of this set of numbers yields the solutions of polynomial equations and the Fundamental Theorem of Algebra. Students deepen their understanding of the concept of function and apply equation-solving and function concepts to many different types of functions. The system of polynomial functions, analogous to integers, is extended to the field of rational functions, which is analogous to rational numbers. Students explore the relationship between exponential functions and their inverses, the logarithmic functions. Trigonometric functions are extended to all real numbers, and their graphs and properties are studied. Finally, students’ knowledge of statistics is extended to include under- standing the normal distribution, and students are challenged to make inferences based on sampling, experiments, and observational studies.

For the Traditional Pathway, the standards in the Algebra II course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, and Statistics and Probability. The course content is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not simply topics to be checked off from a list during isolated units of instruction; rather, they represent content that should be present throughout the school year in meaningful and rigorous instructional experiences.

Table A2-1. Standards for Mathematical Practice—Explanation and Examples for Algebra II

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	Students apply their understanding of various functions to real-world problems. They approach complex mathematics problems and break them down into smaller problems, synthesizing the results when presenting solutions.
MP.2 Reason abstractly and quantitatively.	Students deepen their understanding of variables—for example, by understanding that changing the values of the parameters in the expression $A\sin(Bx+C)+D$ has consequences for the graph of the function. They interpret these parameters in a real-world context.
MP.3 Construct viable arguments and critique the reasoning of others. **Students build proofs by induction and proofs by contradiction. CA 3.1** (for higher mathematics only).	Students continue to reason through the solution of an equation and justify their reasoning to their peers. Students defend their choice of a function when modeling a real-world situation.
MP.4 Model with mathematics.	Students apply their new mathematical understanding to real-world problems, making use of their expanding repertoire of functions in modeling. Students also discover mathematics through experimentation and by examining patterns in data from real-world contexts.
MP.5 Use appropriate tools strategically.	Students continue to use graphing technology to deepen their understanding of the behavior of polynomial, rational, square root, and trigonometric functions.
MP.6 Attend to precision.	Students make note of the precise definition of *complex number*, understanding that real numbers are a subset of complex numbers. They pay attention to units in real-world problems and use unit analysis as a method for verifying their answers.
MP.7 Look for and make use of structure.	Students see the operations of complex numbers as extensions of the operations for real numbers. They understand the periodicity of sine and cosine and use these functions to model periodic phenomena.
MP.8 Look for and express regularity in repeated reasoning.	Students observe patterns in geometric sums—for example, that the first several sums of the form $\sum_{k=0}^{n} 2^k$ can be written as follows: $1 = 2^1 - 1$ $1+2 = 2^2 - 1$ $1+2+4 = 2^3 - 1$ $1+2+4+8 = 2^4 - 1$ Students use this observation to make a conjecture about any such sum.

What Students Learn in Algebra II

Building on their work with linear, quadratic, and exponential functions, students in Algebra II extend their repertoire of functions to include polynomial, rational, and radical functions.

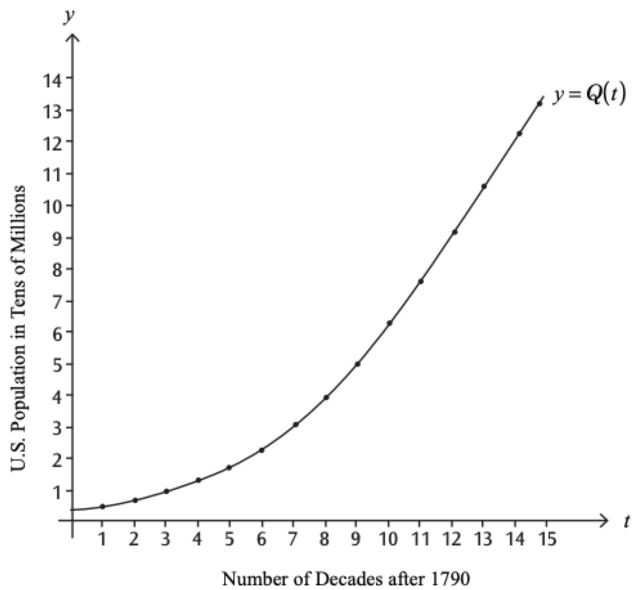
Students work closely with the expressions that define the functions and continue to expand and hone their abilities to model situations and to solve equations, including solving quadratic equations over the set of complex numbers and solving exponential equations using the properties of logarithms. Based on their previous work with functions, and on their work with trigonometric ratios and circles in geometry, students now use the coordinate plane to extend trigonometry to model periodic phenomena. They explore the effects of transformations on graphs of diverse functions, including functions arising in applications, in order to abstract the general principle that transformations on a graph always have the same effect regardless of the type of underlying function. They identify appropriate types of functions to model a situation, adjust parameters to improve the model, and compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit.

Example: Population Growth — **F-BF.1**

The approximate population of the United States, measured each decade starting in 1790 through 1940, can be modeled with the following function:

$$P(t) = \frac{(3,900,000 \times 200,000,000)e^{0.31t}}{200,000,000 + 3,900,000\,(e^{0.31t} - 1)}$$

In this function, t represents the number of decades after 1790. Such models are important for planning infrastructure and the expansion of urban areas, and historically accurate long-term models have been difficult to derive.



Questions:

a. According to this model, what was the population of the United States in the year 1790?

b. According to this model, when did the U.S. population first reach 100,000,000? Explain your answer.

c. According to this model, what should the U.S. population be in the year 2010? Find the actual U.S. population in 2010 and compare with your result.

d. For larger values of t, such as $t = 50$, what does this model predict for the U.S. population? Explain your findings.

Solutions:

a. The population in 1790 is given by $P(0)$, which is easily found to be 3,900,000 because $e^{0.31(0)} = 1$.

b. This question asks students to find t such that $P(t) = 100,000,000$. Dividing the numerator and denominator on the left by 100,000,000 and dividing both sides of the equation by 100,000,000 simplifies this equation to

$$\frac{3.9 \times 2 \times e^{0.31t}}{200 + 3.9\left(e^{0.31t} - 1\right)} = 1.$$

Algebraic manipulation and solving for t result in $t \approx \frac{1}{0.31} \ln 50.28 \approx 12.64$. This means that after 1790, it would take approximately 126.4 years for the population to reach 100 million.

c. Twenty-two (22) decades after 1790, the population would be approximately 190,000,000, which is far less (by about 119,000,000) than the estimated U.S. population of 309,000,000 in 2010.

d. The structure of the expression reveals that for very large values of t, the denominator is dominated by $3{,}900{,}000e^{0.31t}$. Thus, for very large values of t,

$$P(t) \approx \frac{3{,}900{,}000 \times 200{,}000{,}000 \times e^{0.31t}}{3{,}900{,}000e^{0.31t}} = 200{,}000{,}000.$$

Therefore, the model predicts a population that stabilizes at 200,000,000 as t increases.

2308 Adapted from Illustrative Mathematics 2013m.

Students see how the visual displays and summary statistics learned in earlier grade levels relate to different types of data and to probability distributions. They identify different ways of collecting data—including sample surveys, experiments, and simulations—and the role of randomness and careful design in the conclusions that can be drawn.

Examples of Key Advances from Previous Grade Levels or Courses

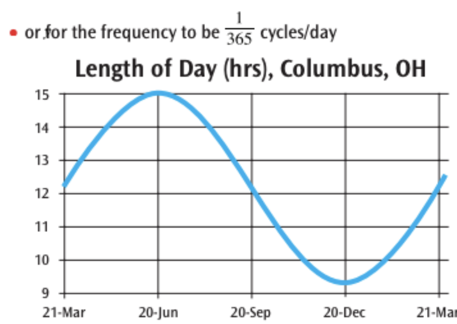
- In Algebra I, students added, subtracted, and multiplied polynomials. Students in Algebra II divide polynomials that result in remainders, leading to the factor and remainder theorems. This is the underpinning for much of advanced algebra, including the algebra of rational expressions.
- Themes from middle-school algebra continue and deepen during high school. As early as grade six, students began thinking about solving equations as a process of reasoning (6.EE.5). This perspective continues throughout Algebra I and Algebra II (A-REI). “Reasoned solving” plays a role in Algebra II because the equations students encounter may have extraneous solutions (A-REI.2).
- In Algebra I, students worked with quadratic equations with no real roots. In Algebra II, they extend their knowledge of the number system to include complex numbers, and one effect is that they now have a complete theory of quadratic equations: Every quadratic equation with complex coefficients has (counting multiplicity) two roots in the complex numbers.
- In grade eight, students learned the Pythagorean Theorem and used it to determine distances in a coordinate system (8.G.6–8). In the geometry course, students proved theorems using coordinates (G-GPE.4–7). In Algebra II, students build on their understanding of distance in coordinate systems and draw on their growing command of algebra to connect equations and graphs of conic sections (for example, refer to standard G-GPE.1).
- In geometry, students began trigonometry through a study of right triangles. In Algebra II, they extend the three basic functions to the entire unit circle.
- As students acquire mathematical tools from their study of algebra and functions, they apply these tools in statistical contexts (for example, refer to standard S-ID.6). In a modeling context, students might informally fit an exponential function

to a set of data, graphing the data and the model function on the same coordinate axes (Partnership for Assessment of Readiness for College and Careers 2012).

Example: Modeling Daylight Hours **F-TF.5**

By looking at data for length of days in Columbus, Ohio, students see that the number of daylight hours is approximately sinusoidal, varying from about 9 hours, 20 minutes on December 21 to about 15 hours on June 21. The average of the maximum and minimum gives the value for the midline, and the amplitude is half the difference of the maximum and minimum. Approximations of these values are set as $A = 12.17$ and $B = 2.83$. With some support, students determine that for the period to be 365 days (per cycle), or for the frequency to be $\frac{1}{365}$ cycles per day, $C = \frac{2\pi}{365}$, and if day 0 corresponds to March 21, no phase shift would be needed, so $D = 0$.

Thus, $f(t) = 12.17 + 2.83\sin\left(\frac{2\pi t}{365}\right)$ is a function that gives the approximate length of day for t, the day of the year from March 21. Considering questions such as when to plant a garden (i.e., when there are at least 7 hours of midday sunlight), students might estimate that a 14-hour day is optimal. Students solve $f(t) = 14$ and find that May 1 and August 10 mark this interval of time.



Students can investigate many other trigonometric modeling situations, such as simple predator–prey models, sound waves, and noise-cancellation models.

Source: UA Progressions Documents 2013c, 19.

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