Chapter 6
Mathematics: Investigating and Connecting, Transitional Kindergarten through Grade Five

Contents

Note to reader: The use of the non-binary, singular pronouns *they*, *them*, *their*, *theirs*, *themself*, and *themselves* in this framework is intentional.

Why investigating and connecting mathematics?

The goal of the California Common Core State Standards for Mathematics (CA CCSSM) at every grade is for students to make sense of mathematics. To achieve this in grades TK–5, students must experience rich mathematical investigations that offer frequent opportunities for students to engage with one another in connecting big ideas in mathematics.

Frequent opportunities for mathematical discourse, e.g., “math talk,” make way for mathematical investigations, which promote understanding (Sfard, 2007), language for communicating (Moschkovich, 1999) about mathematics, and mathematical identities (Langer-Osuna & Esmonde, 2017). Mathematical discourse can center student thinking by offering, explaining, and justifying mathematical ideas and strategies, as well as attending to, making sense of, and responding to the mathematical ideas of others. Mathematical discourse includes communicating about mathematics with words, gestures, drawings, manipulatives, representations, symbols, and other tools that make sense to and are helpful for learning. In the early grades, students might, for example, explore geometric shapes, investigate ways to compose and decompose them, and reason with peers about attributes of objects. Teachers’ orchestration of mathematical discussions (Stein & Smith, 2011) involves modeling mathematical thinking and communication, noticing and naming students' mathematical strategies, and orienting students to one another's ideas. Opportunities for mathematical discourse can emerge throughout the school day, even for the youngest learners. Pencils are regularly needed at each table of students (How many at each table? What is the total number of pencils needed?). More milk cartons are needed from the cafeteria (How many more?). Other questions arise: How many minutes before lunch time? How can you tell? How many more cotton balls are needed for this activity? How do you know? Solving these and other problems in classroom conversation allows children to see how mathematics is an indelible aspect of daily living. As students progress through the elementary and into the middle grades, authentic opportunities for mathematical discourse increase and deepen. Engaging and meaningful mathematical activities (described in Chapter 2) encourage students to explore and make sense of number, data, and space, and to think mathematically about the world around them. Teachers can support their students in investigating mathematics through classroom discourse in ways that foster the development of positive mathematical identities by acknowledging students' histories and cultural backgrounds.

Equitable instruction also means that students are ensured access to rich mathematics and are well prepared for the pathways they choose. Tracking—which manifests early, often in the elementary grades—can occur through the practice of ability grouping and limiting options for students by restricting development. Instead, teachers should focus on heterogeneous grouping (see Complex Instruction, Cohen & Lotan, 1997; Featherstone, et al, 2011), as well as guidance throughout this document to support the participation of all learners in rich mathematical activity.

The grade four vignette below, based on research on supporting English learners (ELs) in mathematical activities, highlights ways that teachers can build on students' existing knowledge and support their developing understandings.

Vignette: Comparing Numbers and Place Value Relationships – Grade 4, Integrated ELD

(Note: This Integrated ELD and Mathematics Instruction Vignette was created by the Tulare County Office of Education under the Creative Commons Attribution-Non Commercial-Share Alike 4.0 International License, http://creativecommons.org/licenses/by-nc-sa/4.0/deed.en_US.)

Background

Mrs. Verners' 30 fourth graders have been learning about place value during the first few weeks of the school year. They are approaching the end of their place value unit. Students have been engaged in lessons and math routines focused on their grade level standards for Number and Operations in Base Ten that are focused on place value. The upcoming task is one of their first experiences within a larger task focused on the same concepts. Students will work independently and collaboratively with their table groups during the task.

The student population is predominately Latinx students, and over half of the students are identified as ELs. Ms. Verners' roster includes ELs at each of the Emerging, Expanding, and Bridging levels. Two students in the class have identified learning disabilities. The fourth-grade team of teachers at this school meets weekly to discuss and plan their math lessons, discussing instructional strategies and resources that they are using to ensure all students feel supported accessing and understanding the content.

Lesson Context

Students have spent their time in the unit exploring place value through daily math lessons and routines. They have developed the ability to identify the place value of given digits, and write numbers in standard, word, and expanded form. Students compare numbers using their understanding of place value and inequality symbols. They have had some experiences describing these comparisons orally and in writing. Mrs. Verners is working to develop student understanding of how the places within the place value system are related through multiplying and dividing by ten. Students have analyzed the relationship between the value of a digit in two locations within a number. For instance, they understand that in the number 5,500, the 5 in the thousands place is ten times greater than the 5 in the hundreds place. In this task, they will explore the relationship between values of a common digit as they compare several different numbers.

Mrs. Verners' lesson is designed to provide students the opportunity to apply what they have learned about the relationships within the base ten place value system and comparing numbers within the context of a real-world situation. Students are asked to engage independently at first, then to work collaboratively in small groups to deepen their understanding of the relationship between the value of a digit located in different places within numbers. The previous lessons helped students establish a foundation through focused attention on place value concepts. Mrs. Verners and her grade-level team had opportunities to better develop background knowledge regarding the places described within the task before beginning the math portion. The teachers decided integrate a map and introductory activity during social studies to start a discussion and identify the location within the task on the map. The learning target and clusters of CA CCSSM and California English Language Development (CA ELD) Standards in focus for today's lesson are the following:

Learning Targets

The students will organize fourth grade population data for different locations across the United States in order to compare and describe the relationships between the values of digits within the number.

CCSS for Mathematics:

4.NBT.1 - Recognize that in a multi-digit whole number, a digit in one place represents ten time what it represents in the place to its right. For example, recognize that 700/70 = 10 by applying concepts of place value and division; 4.NBT.2 - Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons; 4.OA.1 - Interpret a multiplication equations as a comparison, e.g., interpret 35=5x7 as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations; 4.OA.2 - Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison; SMP 1 - Make sense of problems and persevere in solving them; SMP 7 - Look f and make use of structure.

CA ELD Standards (Expanding): ELD.PI.4.1 - Exchanging information and ideas with others through oral collaborative discussions on a range of social and academic topics ELD.PI.4.10 - Writing literary and informational texts to present, describe, and explain ideas and information.

Task

There are almost 40 thousand fourth graders in Mississippi and almost 400 thousand fourth graders in Texas. There are almost 4 million fourth graders in the United States.

We write 4 million as 4,000,000. There are about 4 thousand fourth graders in Washington, D.C. Use the approximate populations given to solve.

a. How many times more fourth graders are there in Texas than in Mississippi?
b. How many times more fourth graders are there in the United States than in Texas?
c. How many times more fourth graders are there in the United States than in Washington, DC?

(Source: Adapted from "Thousands and Millions of Fourth Graders," *Illustrative Mathematics*, https://www.illustrativemathematics.org/content-standards/tasks/1808)

Lesson Excerpts

Day 1

During social studies, Mrs. Verners introduces the math task to her students, saying that tomorrow they will be exploring populations in different locations in the

United State. She gives students the task handout with a map of the United States on the top. She begins the conversation with her class by asking students what state they live in. She refers to a copy of the map under the document camera to serve as a visual. Students discuss with their small groups and share their ideas with the whole class. She asks students to shade California yellow. Next, she asks them to discuss where they live and where they think it is located in California. Mrs. Verners models how to place a dot to represent their city in its approximate location in California. The teacher points to the section lab “key” on their handout. Mrs. Verners states that key is a multiple meaning word and asks students if they know of another way this word is used. Students respond that keys are used to unlock things. Mrs. Verners makes a connection between a key, like a house key, and the key on their map, which is used to help you understand the symbols and colors used on the map. The conversation continues and she helps students to identify the United States, Texas, Mississippi, and Washington D.C. on the map and represent them on the key. Mrs. Verners tells her students that they will use this map tomorrow during math as they explore populations of fourth graders in the different locations they identified.

Day 2

The next day, Mrs. Verners launches the math lesson through a three-read activity. She first asks students to make sense of the context with one another, revisiting the map and telling students that they will be talking about approximate populations of fourth graders in these different locations. She asks the students to use personal whiteboard to write synonyms for “estimate” or “approximate.” After a quick formative check where students show their whiteboards, Mrs. Verners asks for students to share with their partner their words, highlighting some of the examples she hears on the whiteboard at the front of the classroom. Mrs. Verners says that these words (pointing to her list on the whiteboard) are synonyms that mean about or close to. She explains that when we use numbers are not exact, we sometimes use the words almost or about to say that these numbers are estimates or approximations. She says that the English word “approximate” is “aproximado” in Spanish, and asks,”Quien sabe otras palabras matematicas que se oyen igual o similar en ingles?” (Who knows other math words that sound similar in English?) Possible student answers: “Estimado” (estimate), “Angulo” (angle), and “Linea” (line).

Next, she asks students to reason with each other about relevant quantities. Mrs. Verners asks the students to estimate the number of fourth grade students at their school. Students make individual estimates and records them on their individual whiteboards. Students share their estimates with a partner and justify how they decided on their particular estimate. She lists seven estimates on the whiteboard and asks students to discuss the estimates with their small groups to determine if all the estimates are reasonable (make sense) or not and why. Mrs. Verners asks two groups to share their thinking with the class. The two groups share similar explanations stating that 300 is an unreasonable estimate because they have three classes of fourth graders and each class about 30 students, not 100 students to make 300. She tells the class that they just estimated the population of fourth graders at their school and that today they will be using the approximate populations of fourth graders of the locations they marked on their map the previous day.

She asks the class to discuss with their partners what they think population means. Mrs. Verners reminded the class to use (if needed) their sentence starters. She circulates to listen to student conversations and then asks several students to share.

Mrs. Verners: As I listened to you talk with your partners, I heard different ideas about what a population is. Who would like to share what you and your partner discussed? Alex.

Alex: I think population is like the amount of people in a state.

Sara: I think it could be a city too.

Mrs. Verners: Would anyone like to add on to what Alex or Sara said? Yes, Maria.

Maria: So, the population is the amount of people in a city or state.

Mrs. Verners: Yes, for this task we are going to think about the population as the number of people in a given location such as a city, state, or country.

Mrs. Verners then asks students to turn to one another and reason about what mathematical questions they might ask about populations. Once they have shared ideas, Mrs. Verners tells the students that they will be looking at the population of fourth grade students in the different locations, the places they identified on their maps. She tells the students that she going to read the task aloud and wants the students to listen carefully and point to each location on the map when she reads it in the task. Students are asked to reread the task silently, underlining or circling important ideas in the task to help them make sense of what they are reading. Students take turns sharing something that they underlined or circled with their small group. Translations of the task are provided.

Next, students are asked to individually complete the data table by writing the fourth-grade population of each location using digits in standard form in order to organize the population data that they were given in the task. Mrs. Verners explains that table is a multiple meaning word. She explains that there are different types of tables. In math, tables are used to record information and organize data. She shows students the t-table on their task handout and says that this is an example of a table that use in math. After asking her students to begin working independently, Mrs. Verners asks for several of her students to meet her at her small group table. Here, she works with her Emerging ELs to collaboratively complete the t-table. She facilitates the conversation using the following types of questions:

- Where can you find the population of each location in the text? How is the population written?
- How can we rewrite the populations from word form to standard form?
- What are the digits in this number? What digits do we use in our base ten number system?
- What do you notice about the location of the digit 4 in the numbers in your table? What does the location of the digit 4 tell you about its value?

After working together to discuss and create their data tables, the teacher excuses her small group to return to their seats. Mrs. Verners brings the class back together and describes how they will work with their small group during the next portion of the task to answer several questions comparing the population of fourth graders in the different locations and explaining these comparisons in writing. Mrs. Verners poses the question, “How many times greater is [blank] than [blank]? She orchestrates discussion about the difference between additive comparisons and multiplicative comparisons. She then shows the class two sentence frames that she has written on the board and reads them to the class, and tells them that they may use these frames as they are writing or they may create sentences on their own. Her sentence frames are:

- The number of fourth grades in [blank] is [blank] times as many as the fourth graders in [blank].
- There are [blank] times as many fourth graders in [blank] than [blank].

Students are asked to complete a and b collaboratively with their group, saving c to complete on their own so that Mrs. Verners can use this information to check the level of student understanding:

a. How many times more fourth graders are there in Texas than in Mississippi?
b. How many times more fourth graders are there in the United States than in Texas?
c. How many times more fourth graders are there in the United States than in Washington, DC?

The teacher circulates as students are working in small groups and ask questions to support and extend student thinking. She poses the following types of questions:

- What do you notice about the numbers/populations listed in your table?
- What relationship do you notice between these numbers?
- Do you notice a pattern in the place value of the digit 4?
- What tools might help you as you’re trying to represent the place value of the 4 in each of these numbers? (base ten blocks, place value chart, etc.)

- How would you describe the relationship between the digit 4 in these numbers?
- You noticed that each place value is x 10 from the place before it. How might find the relationship between 4,000 and 4,000,000?

Mrs. Verners selects 3 groups to share their explanation from question a. Within each group, she selects one student to represent the group and present to the whole class. She considers students that have recently presented and intentionally selects students who have not had an opportunity to present their thinking to the whole class recently, tipping them off beforehand so they can prepare to share. While circulating around the room, she also continues efforts to support their class norm that all students have good math ideas and selects students that represent a range of strategies. Mrs. Verners asks the student who have been selected to practice what they will say to their table groups before presenting in front the whole class. After the students share their group's explanation, Mrs. Verners asks questions deepen student understanding and make connections between the different explanations that were presented. Next, she asks all students to reread their explanations in part a and provides them time to add on to their explanation to make it stronger or to revise their thinking.

Mrs. Verners asks the students to think about the explanations they have heard and practice with their partner. She asks them to use what they have learned from their work on parts a and b the task to complete part c independently. She tells the students that she is interested in looking their work and reading their writing in part c so that she can learn about what students understand about comparing numbers. Students write their explanations independently.

Teacher Reflection and Next Steps

Mrs. Verners collects the student work and reviews their independent work and explanation from part c. As she reads, she analyzes whether or not students were able to generalize their place value understanding to describe the relationship between the digit 4 in the population of fourth graders in Washington D.C. and the United States. Students have had experience describing the relationship between a digit in a given place value and the place to its right or left; however, this question asks them to describe the relationship of a digit three places to the left. As Mrs. Verners analyzes the student work, she discovers that while the majority of her students understand and are able to describe these place value relationships, a small group of students are struggling to express their thoughts in writing. This small group contains two Emerging ELs, one Expanding EL, one student with a learning disability, and two students that she has noticed are struggling with place value concepts. She decides that she will work with these students in small groups the following day to determine if they are having trouble with the concept or if they are having difficulty using writing to explain their thinking. Mrs. Verners sees that students were able to deepen their understanding of place value relationships through the use of this task and decides that she would like to give the students the opportunity to engage in another task to further develop these concepts before the end of the place value unit.

The phrase “all students” in California schools is inclusive all groups, including students from a range of diverse linguistic and cultural backgrounds and learning needs.

Emergent multilingual students who are learning English face a dual challenge in English-only settings as they endeavor to acquire mathematics content and the language of instruction simultaneously. Teachers can support their progress in a variety of ways, such as drawing on students’ existing linguistic and communicative repertoire or providing language resources during small group work. Teachers can also highlight specific vocabulary as it arises in context or **revoice** students’ mathematical contribution in more formal terms, describing how the precise mathematical meaning might differ from the common use of the same word, e.g., yard, difference, area. All students, including students with special needs, will benefit from these and similar attentions during whole class, small group/partner, or independent work periods. Additional discussion of these shifts in the teacher’s role is found in Chapter 2: Teaching for Equity and Engagement.

Mathematics: Investigating and Connecting, Grades TK–2

Young learners come to school with a rich set of mathematical knowledge and experiences. Starting from infancy and into the toddler years, children develop a knowledge base about mathematics. Infancy research shows that babies demonstrate an understanding about number essentially from birth (National Research Council, 2001). Some infants and most young children show that they can understand and perform simple addition and subtraction by at least three years of age, often using objects (National Research Council, 2001). These studies suggest that children enter the world prepared to notice and engage in it quantitatively.

In the early grades, students spend much of their time exploring, representing, and comparing whole numbers with a range of different kinds of manipulatives. For a student who is interested in dinosaurs, the opportunity to sort pictures or toy versions of the dinosaurs into herbivores and carnivores (or other interesting attributes) and then counting the number of dinosaurs in each category may be a highly engaging activity. Some students enjoy the challenge of recreating structures with building blocks that connect or snap together, or erecting with magnetic builders. Students might create a structure that other students duplicate, describe, and analyze.

Finally, nurturing of students' mathematical explorations may create a classroom atmosphere where students believe they can solve problems and learn engaging new concepts. Discovering repeating digits in a hundreds chart can be eye-opening for a young student and lead to new curiosities about number that can be investigated. Students might also be astonished to realize that one added to any whole number equals the next number in the counting sequence. Activities like these nurture students' interest and encourage future mathematical investigations.

Mathematics in the early elementary grades is rooted in exploration and discovery that build on and develop this early knowledge base. The CA CCSSM offer guidelines[1] for both what mathematics topics are considered essential to learn and how young mathematicians are to engage in the discipline through the practices (SMPs). The SMPs are central to the mathematics classroom and teachers should be intentional about teaching mathematical content through the SMPs. From the earliest grades, mathematics involves making sense of and working through problems. In kindergarten, first, and second grades, students begin to build the understanding that doing

[1] Unlike kindergarten and beyond, transitional kindergarten does not have grade-specific content standards. Therefore, the guidelines in this chapter draw from the California Preschool Learning Foundations (for children at age 60 months).

mathematics involves solving problems, as well as discussing how they solved them through a range of approaches. Young students also reason abstractly and quantitatively. They begin to recognize that a number represents a specific quantity and connect the quantity to written symbols. For example, a student may write the numeral 11 to represent an amount of objects counted, select the correct number card 17 to follow 16 on a calendar, or build two piles of counters to compare the numbers 5 and 8. In addition, young students begin to draw pictures, manipulate objects, or use diagrams or charts to express quantitative ideas.

Modeling and representing is central to students' early experiences with mathematizing their world. In early grades, students begin to represent problem situations in multiple ways—by using numbers, objects, words, or mathematical language; acting out the situation; making a chart or list; drawing pictures; or creating equations, and so forth. While students should be able to use any of these representations as needed, they need opportunities to connect the different representations and explain the connections. For example, a student may use cubes or tiles to show the different number pairs for 5, or place three objects on a 10-frame and then determine how many more are needed to "make a 10." Students rely on manipulatives and other visual and concrete representations while solving tasks and record an answer with a drawing or equation. Students need to be encouraged to answer questions such as, "How do you know?" which reinforces their reasoning and understanding and helps student develop mathematical language.

Students in the early grades must have frequent opportunities for mathematical discourse, including opportunities to construct mathematical arguments and attend to, make sense of, and critique the reasoning of others. Young students begin to develop their mathematical communication skills as they participate in mathematical discussions involving questions such as, "How did you get that?" and, "Why is that true?" They explain their thinking to others and respond to others' thinking. Students can learn to adopt and use these types of questions. For example, sentence frames or charts that teacher can refer to on the wall—especially if they reflect work generated by the class—would be helpful in building activities that support long-term engagement with mathematics. In activities like Compare and Connect (https://ell.stanford.edu/sites/default/files/u6232/ULSCALE_ToA_Principles_MLRs__Final_v2.0_030217.pdf), students compare and contrast two mathematical representations (e.g., place value blocks, number line, numeral, words, fraction blocks) or two methods (e.g., counting up by fives, going up to 30 and then coming back down three more) together. In this activity, teachers might ask the following:

- Why did these two different-looking strategies lead to the same results?
- How do these two different-looking visuals represent the same idea?
- Why did these two similar-looking strategies lead to different results?
- How do these two similar-looking visuals represent different ideas?

In another activity, Critique, Correct, Clarify (https://ell.stanford.edu/sites/default/files/u6232/ULSCALE_ToA_Principles_MLRs__Final_v2.0_030217.pdf), students are provided with ambiguous or incomplete mathematical arguments (e.g., "two hundreds is more than 25 tens because hundreds are bigger than tens") and students are asked to practice respectfully making sense of, critiquing, and suggesting revisions together.

As students engage in mathematical discourse, they begin to develop the ability to reason and analyze situations as they consider questions such as, "Do you think that would happen all the time?" and, "I wonder why…?" These questions drive mathematical investigations. Students construct arguments not only with words, but also using concrete referents, such as objects, pictures, drawings, and actions. They listen to one another's arguments, decide if the explanations make sense, and ask appropriate questions. For example, to solve 74 – 18, students might use a variety of strategies to discuss and critique each other's reasoning and strategies. This exercise is explored further in Chapter 4: Exploring, Discovering, and Reasoning With and About Mathematics (SMPs 3/7/8).

Through experiences with math centers, collaborative tasks, and other rich, open-ended activities, young learners understand ways to use appropriate tools strategically. Younger students begin to consider tools available to them when solving a mathematical problem and decide when certain tools might be helpful. For instance, a kindergartner may decide to use linking cubes to represent two quantities and then compare the two representations side by side—or, later, make math drawings of the quantities. In grade two, while measuring the length of the hallway, students are able to explain why a yardstick is more appropriate to use than a ruler. A student decides which tools may be helpful to use depending on the problem or task and explain why they use particular mathematical tools. Students use tools such as counters, place-value (base-ten) blocks, hundreds number boards, concrete geometric shapes (e.g., pattern blocks or three-dimensional solids), and virtual representations to support conceptual understanding and mathematical thinking. Students should be encouraged to reflect on and answer questions such as, “Why was it helpful to use?”

From early on, children look for and make use of mathematical structure. For instance, students recognize that $3 + 2 = 5$ and $2 + 3 = 5$. Students use counting strategies—such as counting on, counting all, or taking away—to build fluency with facts to 5. Students notice the written pattern in the “teen” numbers—that the numbers start with 1 (representing one 10) and end with the number of additional ones. While decomposing two-digit numbers, students realize that any two-digit number can be broken up into tens and ones (e.g., $35 = 30 + 5$, $76 = 70 + 6$). They use structure to understand subtraction as an unknown addend problem (e.g., 50 – 33 = [blank] can be written as 33 + [blank] = 50 and can be thought of as, “How much more do I need to add to 33 to get to 50?”). Children also thrive when they have opportunities to look for and express regularity in repeated reasoning. In the early grades, students notice repetitive actions in counting, computations, and mathematical tasks. For example, the next number in a counting sequence is one more when counting by ones and 10 more when counting by tens (or one more group of 10). Students should be encouraged to answer questions based on, “What would happen if …?” and “There are 8 crayons in the box. Some are red and some are blue. How many of each could there be?” Kindergarten students realize eight crayons could include four of each color ($8 = 4 + 4$), 5 of one color and 3 of another ($8 = 5 + 3$), and so on. Grade-one students might add three one-digit numbers by using strategies such as “make a ten” or doubles. Students recognize when and how to use strategies to solve similar problems. For example, when evaluating $8 + 7 + 2$, a student may say, “I know that 8 and 2 equals 10, then I add 7 to get to 17. It helps if I can make a ten out of two numbers when I start.” We explore these particular practices further in Chapter 4: Exploring, Discovering, and Reasoning With and About Mathematics (SMPs 3/7/8).

Standards-based instruction should be organized to support investigating big ideas in mathematics and connecting content and mathematical practices within and across grade levels. Big ideas in TK–2 mathematics content connect in the following four ways:

Content Connections

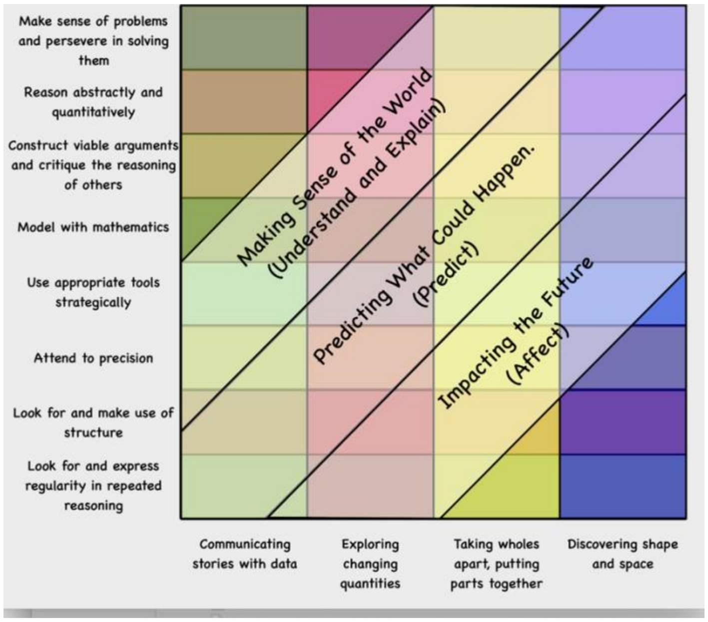
- (CC 1) Communicating Stories with Data
- (CC 2) Exploring Changing Quantities
- (CC 3) Taking Wholes Apart, Putting Parts Together
- (CC 4) Discovering Shape and Space

These content connections develop when students have opportunities to investigate mathematics. Mathematical investigations can fall into one or more of these Drivers of Investigation (DI):

- (DI 1) Making Sense of the World (Understand and Explain)
- (DI 2) Predicting What Could Happen (Predict)
- (DI 3) Impacting the Future (Affect)

Students might make sense of their world (D1) by working with data (CC1) or exploring the decomposition of number (CC2). Students might discuss solutions to a community problem (D1) by exploring changing quantities related to the problem topic (CC4) or examining the use of space within the problem context (CC3). Investigations should be situated in contexts that invite students to wonder in ways that motivate or require particular mathematical activity to drive the investigation. Any particular investigation can meaningfully include several CA CCSSM domains, such Measurement and Data, Number and Operations, and so on, through several of the Mathematical Practices (SMPs) as they conduct their investigations. Chapter 4 illustrates how Content Connections, Drivers of Investigations, and three SMPs come together across the grade bands.

Content Connections, Mathematical Practices and Drivers of Investigation



The figure illustrates the connections among the features of such an investigative, connected approach. These ideas are first illustrated for grades TK–2 and will be revisited later in this chapter for grades 3–5.

CC1: Communicating Stories with Data

With data all around us, even the youngest learners make sense of the world through data—including data about measurable attributes. In the early grades, students describe and compare measurable attributes, classify objects and count the number of objects in each category[2]. As they progress across the early grades, students represent and interpret data in increasingly sophisticated ways. Chapter 5 offers greater detail about how data can be explored across the grades through meaningful mathematical investigations. This Content Connection invites students to:

- Describe and compare measurable attributes (K.MD.A.1)
- Classify objects and count the number of objects in each category (K.MD.B.3)
- Measure lengths indirectly and by iterating length units (1.MDA.1, 1.MDA.2)
- Tell and write time (1.MD.B.3)
- Represent and interpret data (1.MD.C.4, 2.MD.C.8)

[2] Teachers should use their professional judgement in considering what attributes to measure, practicing particular sensitivity to any physical attributes.

- Measure and estimate lengths in standard units (2.MD.A.1, 2.MD.A.2, 2.MD.A.3, 2.MD.A.4)
- Relate addition and subtraction to length (2.MD.B.5, 2.MD.B.6)
- Work with time and money (2.MD.C.7)

Children are curious about the world around them, and might wonder about their classmates' favorite colors, kinds of pets, or number of siblings. Young learners can collect, represent, and interpret data about one another. Students can use graphs and charts to organize and represent data about things in their lives. This data supports the asking and answering of questions about the information in charts or graphs, and can allow them to make inferences about their community. Charts and graphs may be constructed by groups of students as well as by individual students.

Students learn that many **attributes**—such as lengths and heights—are measurable. Early learners develop a sense of measurement and its utility using **non-standard units of measurements**. Through explorations, students discover the utility of **standard measurements**.

This Content Connection can serve as the foundation for mathematical investigations around measurement and data. In an activity on comparing lengths, called Direct Comparisons, students place any three items in order, according to length:

- Pencils, crayons, or markers are ordered by length.
- Towers built with cubes are ordered from shortest to tallest.
- Three students draw line segments and then order the segments from shortest to longest.

In an activity on Indirect Comparisons, students model clay in the shape of snakes. With a tower of cubes, each student compares their snake to the tower. Then students make statements such as, "My snake is longer than the cube tower, and your snake is shorter than the cube tower. So, my snake is longer than your snake." (Adapted from ADE 2010.)

CC 2: Exploring Changing Quantities

Young learners' explorations of changing quantities support their development of meaning for operations, such as addition, subtraction, and early multiplication or division. This Content Connection can serve as the basis for mathematical investigations about operations. Students build on their understanding of addition as putting together and adding to and of subtraction as taking apart and taking from. Students use a variety of models—including discrete objects and length-based models (e.g., cubes connected to form lengths) to model add-to, take-from, put-together, and take-apart—and compare situations to develop meaning for the operations of addition and subtraction and develop strategies to solve arithmetic problems with these operations. Students understand connections between counting and addition and subtraction (e.g., adding two is the same as counting on two). They use properties of addition to add whole numbers and to create and use increasingly sophisticated strategies based on these properties (e.g., “making 10s”) to solve addition and subtraction problems within 20. By comparing a variety of solution strategies, children build their understanding of the relationship between addition and subtraction. By second grade, students use their understanding of addition to solve problems within 1,000 and they develop, discuss, and use efficient, accurate, and generalizable methods to compute sums and differences of whole numbers.

Investigating mathematics by exploring changing quantities invites students to:

- Know number names and the count sequence (K.CCA.1, K.CCA.2., K.CCA.3).
- Count to tell the number of objects (K.CC.B.4, K.CC.B.5).
- Compare numbers (K.CC.C.6, K.CC.C.7)
- Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from (K.OA.A1, K.OA.A2, K.OA.A3, K.OA.A4, K.OA.A5)
- Represent and solve problems involving addition and subtraction (1.OA.A.1, 1.OA.A.2, 2.OA.A.1)
- Understand and apply properties of operations and the relationship between addition and subtraction (1.OA.B.3, 1.OA.B.4)
- Add and subtract within 20 (1.OA.C.5, 1.OA.C.6, 2.OA.B.2)
- Work with addition and subtraction equations (1.OA.D.7, 1.OA.D.8)

- Work with equal groups of objects to gain foundations for multiplication (2.OA.C.3, 2.OA.C.4)
- Look for and make use of structure (SMP 7)
- Look for and express regularity in repeated reasoning (SMP 8)

Young learners benefit from ample opportunities to become familiar with number names, numerals, and the count sequence. While mathematical concepts and strategies can be explored and understood through reasoning, the names and symbols of numbers and the particular count sequence is a convention to which students become accustomed. Conceptually, students come to develop particular foundational ideas through experiences with early counting: **cardinality** and **one-to-one correspondence**.

In transitional kindergarten (TK), many opportunities arise for conversations about counting. Consider the exchange below:

Nora: "Sami isn't being fair. He has more trains than I do."

Teacher: "How do you know?"

Nora: "His pile looks bigger!"

Sami: "I don't have more!"

Teacher: "How can we figure out if one of you has more?"

Nora: "We could count them."

Teacher: "Okay, let's have both of you count your trains."

Sami: "One, two, three, four, five, six, seven."

Nora: "One, two, three, four, five, six, seven." (*Fails to tag and count one of her eight trains.*)

Sami: "She skipped one! That's not fair!"

Teacher: "You are right; she did skip one. We count again and be very careful to make sure not to skip—but can you think of another way that we can figure out if one of you has more?"

Sami: "We could line them up against each other and see who has a longer train."

Teacher: "Okay, show me how you do that. Sami, you line up your trains, and Nora, you line up your trains."

Opportunities to count and represent the count as a quantity, whether verbally or symbolically, allow students to recognize that, in counting, each item is counted exactly once and that each count corresponds to a particular number. Using manipulatives or other objects to count, students learn to organize their items to facilitate this one-to-one correspondence. Students also learn that the number at end of the count represents the full quantity of items counted and that each subsequent number represents an additional one added to the count. In Counting Collections (https://prek-math-te.stanford.edu/counting/counting-collections-overview), teachers ask young children to do the following:

- Count to figure out how many are in a collection of objects (a set of old keys, teddy bear counters, rocks collected from the yard, arts and crafts materials, etc.)
- Make a written representation of what they counted and how they counted it.

Source: http://www.research-and-play.com/2018/07/counting-collections-transform-your.html

There are many benefits when younger learners are provided opportunities to represent quantities with number words and numerals, as well as to represent number words and numerals as quantities. Activities related to this Content Connection can support teachers as they create opportunities for students to learn and grow.

To highlight representing quantities with number words in TK, teachers can add questions about numbers that arise during class reading activities. In a book about dogs, for instance, on the page showing a picture of two dogs, ask how many dogs there are, and then ask questions such as:

- How many legs does one dog have?
- How many legs do two dogs have?
- If one dog left the page, how many legs would be left?

Teachers can align instruction with proven English language development strategies, such as the use of gestures, facial expressions, and other non-verbal movements as communication strategies, sentence frames or **revoicing** student answers to support the participation of language learners.

To integrate representing number words as quantities, teachers can build steps for students to represent with their fingers the addends in a story problem. This can be particularly effective during small- or whole-group time. Individual students can explain to their classmates how they decided how many fingers to choose for each hand. For example, “One day, two baby dinosaurs hatched out of their eggs. The mama triceratops was so excited that she called to her auntie to come and see. Then four more baby dinosaurs hatched! How many dinosaurs hatched all together? Marisol, can you show me how many fingers you used?” Note that children across different communities of origin learn to show numbers on their fingers in different ways. Children may start with the thumb, the little finger, or the pointing finger. Support all of these ways of showing numbers with fingers.

In *Feet Under the Table* (Confer, 2005), a group of children sit at a table with counters, pencils, and paper. Without investigating or looking, students figure out how many feet are under the table. They can use mathematical tools, such as cubes or drawings, that will help them, and then represented their number on paper. Students then share how they represented the feet on their paper and how many feel they think there are altogether. When all the students are finished, they then peek under the table to check their answers.

Developmentally, children become more efficient counters through experiences over time and in ways that support early addition and subtraction. Young learners can build on what they know about counting to add on to an original count. For example, tasks from *Cognitively Guided Instruction* (Carpenter, et al, 2014) ask students to create a set of a particular amount, say five cubes and to then add three more cubes. Students can draw on what they know to first count out five cubes. Students might then use different strategies to add on three more. Some students might count out three more cubes separately, then start from one again and count out all eight cubes. Other students might count on from five, naming the numbers as they go along—six, seven, eight cubes. Students might also draw on other possible strategies. Teacher can notice student strategies as formative assessment, recognizing how their young learners become increasingly efficient counters. Young learners also draw on their counting strategies to develop early subtraction sense. Cognitively Guided Instruction tasks might prompt students to begin with, say, eight cookies, then note that three cookies were eaten. Students might count out eight cookies with manipulatives like counting cubes, and then employ a range of strategies to figure out how to "take away" three cookies. Students might remove three cubes from the original set and then count the remaining cubes to figure out how many remain. Other students might count backwards from the original set, landing of eight cookies.

Students will use different strategies to solve problems when teachers give the time and space to do so. Teachers should explore the various methods that arise as students work to understand general properties of operations. For example, in a number talk on the problem 8 + 7, students might come up with and share different computation strategies:

> Student 1: (Making 10 and decomposing a number) "I know that 8 plus 2 is 10, so I decomposed (broke up) the 7 into a 2 and a 5. First, I added 8 and 2 to get 10, and then I added the 5 to get 15."
>
> *This explanation could be represented as:* $8 + 7 = (8 + 2) + 5 = 10 + 5 = 15$.
>
> Student 2: (Creating an easier problem with known sums) "I know 8 is 7 + 1. I

also know that 7 and 7 equal 14. Then I added 1 more to get 15."

This explanation could be represented as: $8 + 7 = (7 + 7) + 1 = 15$.

The game "Pig,"[3] found on YouCubed, can be played to practice addition. The game involves students using dice (or an app to simulate a dice roll) and compete to be the first player to reach 100. Students take turns rolling the dice and determine the sum. Students can either stop and record that sum or continue rolling and add the new sums together as many times as they choose. When they decide to stop, they record the current total and add it to their previous score. Note that students should build understanding through activities that draw on concrete and representational approaches to operations before engaging in abstract fluency games. Other resources for addition activities include the National Council of Teachers of Mathematics' (NCTM) *Illuminations* and *Illustrative Mathematics*.

Classroom activities can also support students developing understanding of the equal sign as meaning that the quantity on one side of the equal sign must be the same quantity as on the other side of the equal sign. An activity from YouCubed, "Moving Colors," explores equality as students move around the room. Students are given red or yellow colored circles (or other shapes). Teachers ask, "How many students have red circles and how many have yellow circles?" With appropriate accommodations, students encouraged to get up and move around the room to work this out. Teachers ask, "How can we show that we have an equal number of each color or more of one color than the other color?"

[3] Pig is a folk jeopardy dice game described by John Scarne in 1945, and was an ancestor of the modern game Pass the Pigs® (originally called PigMania®); Scarne, John (1945). Scarne on Dice. Harrisburg, Pennsylvania: Military Service Publishing Co.

Table K-4. Methods Used for Solving Single-Digit Addition and Subtraction Problems

Level 1: Direct Modeling by Counting All or Taking Away Represent the situation or numerical problem with groups of objects, a drawing, or fingers. Model the situation by composing two addend groups or decomposing a total group. Count the resulting total or addend.
Level 2: Counting On Embed an addend within the total (the addend is perceived simultaneously as an addend and as part of the total). Count this total, but abbreviate the counting by omitting the count of this addend; instead, begin with the number word of this addend. The count is tracked and monitored in some way (e.g., with fingers, objects, mental images of objects, body motions, or other count words). For addition, the count is stopped when the amount of the remaining addend has been counted. The last number word is the total. For subtraction, the count is stopped when the total occurs in the count. The tracking method indicates the difference (seen as the unknown addend).
Level 3: Converting to an Easier Equivalent Problem Decompose an addend and compose a part with another addend.

Adapted from UA Progressions Documents 2011a.

CC 3: Taking Wholes Apart, Putting Parts Together

Children enter school with experience at taking wholes apart and putting parts together, a task that occurs in everyday activities such as slicing pizzas and cakes, building with Legos, clay, or other materials. Decomposing challenges and ideas into manageable pieces, and assembling understanding of smaller parts into understanding of a larger whole, are fundamental aspects of using mathematics. Often these processes are closely tied with SMP 7 (Look for and make use of structure). In the early grades, such investigations might include composing and decomposing the number 5 into parts such as 1 and 4 or 2 and 3, using manipulatives. This Content Connection spans and connects many typically-separate content clusters. For example, students might also decompose shapes, which connects to CC 4.

Understanding numbers, including the structure of our number system (place value or base 10) and relationships between numbers begins with counting and cardinality and extends to a beginning understanding of **place value**. Young learners use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set; counting out a given number of objects; comparing sets or numerals; and modeling simple joining and separating situations with sets of objects. As they progress across the early grades, students develop, discuss, and use strategies to compose and decompose numbers, noticing the numbers that exist inside numbers. Through activities that build number sense, they understand how numbers work and how they relate to one another.

PROBLEM-SOLVING SITUATIONS

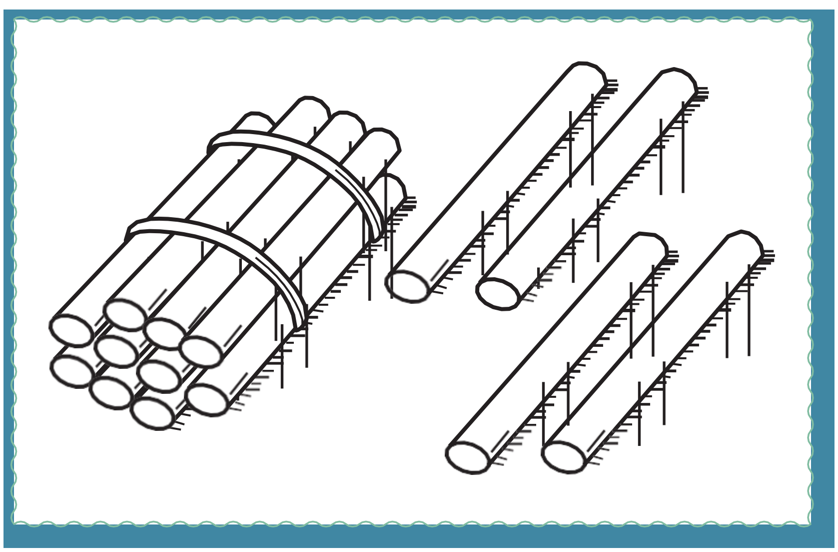
JOINING PROBLEMS		
Join: Result Unknown (JRU)	**Join: Change Unknown (JCU)**	**Join: Start Unknown (JSU)**
♦ Grandmother had 5 strawberries. Grandfather gave her 8 more strawberries. How many strawberries does Grandmother have now? 5 + 8 = □	♥ Grandmother had 5 strawberries. Grandfather gave her some more. Then Grandmother had 13 strawberries. How many strawberries did Grandfather give Grandmother? 5 + □ = 13	♠ Grandmother had some strawberries, Grandfather gave her 8 more. Then she had 13 strawberries. How many strawberries did Grandmother have before Grandfather gave her any? □ + 8 = 13
SEPARATING PROBLEMS		
Separate: Result Unknown (SRU)	**Separate: Change Unknown (SCU)**	**Separate: Start Unknown (SSU)**
♦ Grandfather had 13 strawberries. He gave 5 strawberries to Grandmother. How many strawberries does Grandfather have left? 13 - 5 = □	♥,Grandfather had 13 strawberries. He gave some to Grandmother. Now he has 5 strawberries left. How many strawberries did Grandfather give Grandmother? 13 - □ = 5	♠ Grandfather had some strawberries. He gave 5 to Grandmother. Now he has 8 strawberries left. How many strawberries did Grandfather have before he gave any to Grandmother □ - 5 = 8
PART -PART -WHOLE PROBLEMS		
Part-Part-Whole: Whole Unknown (PPW:WU)	**Part-Part-Whole: Part Unknow (PPW:PU)**	
♦ Grandmother has 5 big strawberries and 8 small strawberries. How many strawberries does Grandmother have altogether? 5 + 8 = □	♥,Grandmother has 13 strawberries. Five are big and the rest are small. How many small strawberries does Grandmother have? 13 - 5 = □ or 5 + □ = 13	
COMPARE PROBLEMS		
Comp. Difference Unknown	**Comp. Quantity Unknown**	**Comp. Referent Unknown**
♦ ♥,Grandfather has 8 strawberries. Grandmother has 5 strawberries. How many more berries does Grandfather have than Grandmother? 8 - 5 = □ or 5 + □ = 8	♠ Grandmother has 5 strawberries. Grandfather has 3 more strawberries than Grandmother. How many strawberries does Grandfather have? 5 + 3 = □	♠ Grandfather has 8 strawberries. He has 3 more strawberries than Grandmother. How many strawberries does Grandmother have? 8- 3 = □ or □ + 3 = 8
MULTIPLICATION & DIVISION PROBLEMS		
Multiplication	**Measurement Division**	**Partitive Division**
♦ Grandmother has 4 piles of strawberries. There are 3 strawberries in each pile. How many strawberries does Grandmother have? 4 x 3 = □	♦ Grandmother had 12 strawberries. She gave them to some children. She gave each child 3 strawberries. How many children were given strawberries? 12 ÷3 = □	♦ ♥,Grandfather has 12 strawberries. He wants to give them to 3 children. If he gives the same number of strawberries to each child, how many strawberries will each child get? 12 ÷ 3 = □

Problem chart based on Cognitively Guided Instruction Problem Types (Carpenter et al., 1996)

Investigating mathematics by taking wholes apart and putting parts together invite students to:

- Work with numbers 11–19 to gain foundations for place value (K.NBT.A.1).
- Extend the counting sequence (1.NBT.A.1).
- Understand place value (1.NBT.B.2, 1.NBT.B.3, 2.NBT.A.1, 2.NBT.A.2, 2.NBT.A.3, 2.NBT.A.4).
- Use place value understanding and properties of operations to add and subtract (1.NBT.C.4, 1.NBT.C.5, 1.NBT.C.6, 2.NBT.B.5, 2.NBT.B.6, 2.NBT.B.7, 2.NBT.B.8, 2.NBT.B. 9).
- Look for and make use of structure (SMP 7)

Understanding the concept of a *ten* is fundamental to young students' mathematical development. This is the foundation of the place-value system, which can be productively investigated through this Content Connection. Young children often see a group of 10 cubes as 10 individual cubes. Activities can support students in developing the understanding of 10 cubes as a bundle of 10 ones, or a *ten*. Students can demonstrate this concept by counting 10 objects and "bundling" them into one group of 10. Working with numbers between 11–19 are early ways to build the idea of numbers structured as a bundle of ten and remaining ones.



In The Pocket Game (Confer, 2005), children explore the smaller numbers inside larger numbers. Using number cards, students determine which of two numbers is larger, then place both numbers in a paper pocket labeled with the larger number. After playing the game, students are grouped to discuss what they notice about the numbers inside the different pockets, ultimately seeing that each pocket number contains all the smaller numbers within. After the discussion, teachers can prompt students to predict which numbers they will find in the paper pocket labeled “three” and rationalize their predictions, encouraging them to examine the paper pockets one by one and talk about what they notice (and see if their predictions were accurate). Conversation should focus on why those numbers were inside each pocket and why other numbers were not.

After the game is played over a number of weeks, teachers can facilitate a discussion about why the pockets look the way they do at the end of a game. For example, while viewing a pocket labeled “two,” students might be asked which numbers they think will be inside. With predictions recorded, teachers can facilitate an examination of the pocket and discuss why there are only ones and twos in the pocket. This continues as students question why some numbers are *not* in the pocket.

Later in the year, revisit the game again. When they finish the game they will figure out which paper pocket has the most cards. In the activity “Race for a Flat,” two teams of two players each roll number cubes in a place value game. The players find the sum of the numbers they roll and take Units cubes to show that number. Then they put their Units on a place value mat. When a team gets 10 Units or more, they trade 10 Units for one Rod. As soon as a team gets blocks worth 100 or more, they make a trade for one Flat. The first team to complete this wins the game.

Students in the early grades will be working with whole numbers, but TK–2 teachers may consider the effect of using number lines to represent whole numbers—not only to support students in noticing relationships between numbers (for example, that one and 11 are both equidistant from six), but also that numbers can be understood at points on a line. This understanding will support later exploration of fractions as points on a line between whole numbers. The Learning Mathematics through Representations project (https://sites.google.com/view/lmrberkeleyedu) offers activities for early and upper elementary grades that prepare students to make later connections to fractions. Fair sharing problems also support children’s developing understanding of fraction concepts through explorations with grouping (Empson, 1999; Empson & Levi, 2011).

CC4: Discovering Shape and Space

Young learners possess natural curiosities about the physical world. In the early grades, students learn to describe their world using geometric ideas (e.g., shape, orientation, spatial relations). They identify, name, and describe basic two-dimensional shapes, such as squares, triangles, circles, rectangles, and hexagons, presented in a variety of ways (e.g., with different sizes and orientations). They engage in this process with three-dimensional shapes as well, such as cubes, cones, cylinders, and spheres. They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes. As they progress through the early grades, students compose and decompose plane or solid figures (e.g., put two triangles together to make a quadrilateral) and begin understanding part-whole relationships as well as the properties of the original and composite shapes. As they combine shapes, they recognize them from different perspectives and orientations, describe their geometric attributes, and determine how they are alike and different, thus developing the background for measurement and for initial understandings of properties such as congruence and symmetry.

Investigating mathematics by discovering shape and space invite students to:

- Identify and describe shapes (K.G.A.1, K.G.A.2, K.G.A.3).
- Analyze, compare, create, and compose shapes (K.G.B.4, K.G.B.5, K.G.B.6)
- Reason with shapes and their attributes (1.G.A.1, 1.G.A.2, 1.G.A.3, 2.G.A.1, 2.G.A.2, 2.G.A.3).

Young learners can begin to explore the idea of classifying objects in relation to particular attributes—color, size, and shape. Students can build of these early experiences to identify geometric attributes at a fairly early age. In grades one and two, many teachers introduce terms like vertex, edge, and face. Students need ample time to explore these attributes and make sense of the ways they relate to one another and particular geometric shapes. Young learners often recognize shapes by appearance and need time to explore attributes and their relationship to shapes.

Teachers can provide opportunities for young learners to compose and decompose shapes around characteristics or properties and to explore typical examples of shapes, as well as variants, and both examples and non-examples of particular shapes. Classroom discussions can also surface and address common misconceptions students have about shapes, such as triangles always rest on a side and not on a vertex or that a square is not a rectangle.

In an activity on sorting shapes, students sort a pile of squares and rectangles into two groups. They discuss how the rectangles and squares are alike and how they are different. After students demonstrate an understanding of the differences between squares and rectangles, the teacher provides each student with one square or rectangle cutout. The teacher creates two groups—one side of the classroom includes students with the square cutouts, while the students with rectangle cutouts stand on the opposite side of the room. The differences in the rectangle and square cutouts (size and color) allow the students focus on the shape attributes as they compare in and across groups.

Another activity, based on the popular boardgame *Guess Who?*, offers students the opportunity to reason about the relationship between attributes and geometric shapes. In "Guess What?" the objective for students to guess an opponent's mystery shape before the opponent guesses theirs. Players take turns asking "yes" or "no" questions about character attributes (e.g., "Does your shape have angles?"). Shapes that no longer fit the description of the opponents' mystery shape are eliminated by flipping card holders over. The first player to correctly guess the other players' mystery shape wins.

Students can also use pattern blocks, plastic shapes, tangrams, or online manipulatives to compose new shapes. Teachers can provide students with cutouts of shapes and ask them to combine the cutouts to make a particular shape or to create shapes of their own. Peers can then work together to recreate or decompose one another's shapes.

Classroom discourse is an important aspect of such activities. It may be valuable to challenge students to test ideas about shapes using a variety of examples for a category, asking open-ended questions, such as:

- "What do you notice about your shape?"
- "What happens if you try to draw a shape with just one side?"

Such mathematics conversations are important even for the youngest learners. Teachers can provide access to sample questions as needed. TK teachers can take up students' own questions and curiosities as an opportunity to explore shapes. Consider the following exchange:

Andrew: Is this a triangle? (*Holds up a square.*)

Teacher: What do you think? (*Asks other students in the small group to contribute.*)

Students (in unison): No!

Teacher: Why not? Can you share how you can tell?

Zahra: Because a triangle doesn’t have four sides.

Teacher: I heard you say that a triangle doesn’t have four sides. How many sides does a triangle have?

Alexander: Three!

Teacher: So, Andrew, what do you think? Is your shape a triangle?

Andrew: No, it’s not a triangle.

Teacher: How can you tell?

Andrew: Because it has four sides and triangles have three sides.

Teacher: I heard you say that your shape is not a triangle because it has four sides and triangles have three sides. Is that right?

Andrew: Yes

Teacher: Class, do you agree with Andrew?

Students (in unison): Yes”

Teacher: Andrew, see if you can find a triangle and I’ll come back to check what you found.

Open-ended questions, such as, “What do we know about triangles?” or, “How did you figure that out?” elicit young learners’ thinking and encourage them to think and speak like mathematicians. Teachers can use responses to facilitate an organic conversation, as in the excerpt above, that allows students to collaborate, provide feedback, and built on one another’s reasoning.

Vignette: Alex Builds Numbers with a Partner (a two-day lesson)

In first grade, Alex's class is building understanding of making numbers. His teacher, Ms. Kim, launches the lesson with a whole class conversation with all students gathered on the carpet by the front of the room; half of the students hold a small rack of beads. Ms. Kim held a large rekenrek—an arithmetic rack with two rows of 10 beads each—and moved two beads from the top rack to one side and three beads from the second rack. She asked students, "How many beads do you see on this side of the rack? Turn and talk to your partner about how many beads you see altogether and how you can tell." Students turned to their peers excitedly and shared their ideas.

"Who wants to share? How many beads do you see?" she asks. Students raise their hands. Ms. Kim decided to ask Alex to share, who is usually so eager to respond that he sometimes overlooks key details. Alex says, "I see five beads." Ms. Kim presses, "You see five beads. And how do you see it?" Alex continues, "Because there are two on the top and three on the bottom and that makes one, two, three, four, five." Ms. Kim revoices, "I heard you say that you see five beads because there are two on the top and three on the bottom and two and three make five altogether. Is that right? Who agrees with Alex?" Several hands go up in the air.

"Are there other ways to make a five?" she wonders. "Work with your partner. If you are holding the rekenrek, you are Partner A. Raise your hand if you are Partner A. If you are not holding a rekenrek, you are Partner B. Raise your hand if you are Partner B. Ok, Partner A – How else can you make a five? Use your rekenrek to show another way to make a five. Then it will be Partner B's turn. Partner B—make five in a different way."

Students turn to their partners and begin to move beads. Some students move five beads over on the top row and none on the bottom. Others show four on the top row and one on the bottom. Several others are unsure, moving beads around playfully on the rekenrek.

Ms. Kim moves around the carpet area, squatting down to meet with particular partner groups and listen to their conversations about making the number five. After a few minutes, she reconvenes them for a discussion.

She opens with, "What were some other ways to make five?"

Students share ways to make five. Ms. Kim revoices their answers, checking with the class to see whether their different combinations of number count up to five and allowing students to revise their thinking when it does not.

Ms. Kim then introduces the activity they will be working for the rest of the lesson at their tables with their partners. Tables are provided number cards. Each partner will take turns turning over a number card and representing it on the rekenrek. The second partner is to ask, “How do you see it?” allowing the partner to explain. The roles are then changed. Partner B will represent the same number in a different way and Partner A will ask, “How do you see it?” Partners must agree that each combination does indeed count to the number on the card.

Alex moves to his table with his partner, who has been holding the rekenrek. As he does so, his teacher uses knowledge of Alex’s fidgetiness during partner work, reminds him to use his fidget spinner when it is his partner’s turn to hold the rekenrek. Alex relies on this as his partner, Partner A, quickly turns over a number card. “Eight!” she exclaims.

“So now you have to make an eight,” declares Alex. His partner moves the beads around playfully. She moves all 10 beads to the side and counts them one by one. When she reaches eight, she pauses and moves the remaining two beads away.

“Ok, I made eight. Now you say, ‘How did I see it?’” his partner states, chuckling.

“How do you see it?” asks Alex. His partner answers, “there are five on the top and one, two, three on the bottom. Your turn.”

Alex takes the rekenrek and move one bead away from the top row and adds one bead from the second row. “I see four and four.”

They continue to take turns with new numbers. Ms. Kim circulates around the room asking students to explain their representations and supporting partners’ interactions with one another. As she does so, she records their representations and explanations. She uses this time as a formative assessment opportunity as she plans for the next day’s discussion about patterns in representing numbers.

Mathematics: Investigating and Connecting, Grades 3–5

Mathematics in grades three, four, and five relies on students having acquired a solid understanding of the concepts developed in the earlier grades. New challenges in mathematics are exciting and meaningful for students when they are able to connect previous learning to make sense of current grade level concepts. The goal of the CA CCSSM at every grade is for students to understand mathematics. This means far more than expecting students to master procedures and memorize facts, and may call for adjustments to the ways mathematics instruction is structured in the classroom. To understand mathematics, students must be the doers of mathematics—the ones who do the thinking, do the explaining, and do the justifying. In this paradigm, teachers engage all students in authentic, relevant mathematics experiences; they support learning by recognizing, respecting, and nurturing their students' ability to develop deep mathematical understanding (Hansen and Mathern, 2008). Additional discussion of these shifts in the teacher's role can be found in Chapter 2: Teaching for Equity and Engagement.

What constitutes whether students are demonstrating understanding? A student can express an idea in their own words, build a concrete model, can illustrate their thinking pictorially, or can provide examples and possibly counterexamples. One might observe them making connections between ideas or applying a strategy appropriately in another related situation (Davis, Edward 2006). Many useful indicators of deeper understanding are embedded in the Standards for Mathematical Practice (SMPs). Teachers can note when students "analyze…the relationships in a problem so that they can understand the situation and identify possible ways to solve it," as described in SMP.1. Other examples of observable behaviors specified in the SMPs include students' abilities to:

- use mathematical reasoning to justify their ideas;
- draw diagrams of important features and relationships;
- select tools that are appropriate for solving the particular problem at hand; and
- accurately identify the symbols, units, and operations they use in solving problems (SMP.3, 5, 6).

To teach mathematics for understanding, it is essential to actively and intentionally cultivate students' use of the SMPs. The Introduction to the CA CCSSM is explicit on this point, “The MP standards must be taught as carefully and practiced as intentionally as the Standards for Mathematical Content. Neither should be isolated from the other; effective mathematics instruction occurs when the two halves of the CA CCSSM come together as a powerful whole” (CA CCSSM, p. 3).

SMPs are linguistically demanding, yet they provide opportunities to develop language; educators must remain aware of and provide support for students who may grasp a concept, yet struggle to express their understanding. Students who regularly incorporate the SMPs in their mathematical work develop mental habits that enable them to approach novel problems as well as routine procedural exercises, and to solve them with confidence, understanding, and accuracy. Specifically, recent research shows that an instructional approach focused on mathematical practices may be important in supporting student achievement on curricular standards and assessments, and that it also contributes to students’ positive affect and interest in mathematics (Sengupta-Irving and Enyedy, 2014). Regularly incorporating the SMPs gives students opportunities to make sense of the specific linguistic features of the genres of mathematics, and produce, reflect on, and revise their own mathematical communications. SMPs also offer teachers opportunities to engage in formative assessment, provide real-time feedback, and inform potential student language use issues that may arise as they develop their mathematical thinking.

The content standards were built on progressions of topics across grade levels, informed by both research on children’s cognitive development and by the logical structure of mathematics. TK–2 classes help students build a foundation for all their future mathematics as they explored numbers, operations, measurement and shapes. Students learned place value and used methods based on place value to add and subtract within 1,000. They developed efficient, reliable methods for addition and subtraction within 100. Students continue developing efficient methods throughout grade three, and learn the formal algorithms for addition and subtraction in grade 4 (4.NBT.B.4). In the earlier grades, students worked with equal groups and the array model, preparing the way for understanding multiplication. They used standard units to measure lengths and described attributes of geometric shapes. Mathematical investigations of core content—that is, the grade-level big ideas in mathematics—can be productively approached through the SMPs.

The mathematics content of grades three, four, and five is conceptually rich and multi-faceted. Students who engaged in meaningful mathematics in grades TK–2 are more likely to increase their mathematical understanding as they advance through subsequent grades. Across grades three, four, and five, they will expand this early mathematical foundation as they build understanding of the operations of multiplication and division, concepts and operations with fractions, and measurement of area and volume.

Preparing students to be the reflective problem solvers envisioned in the CA CCSSM requires educators to cultivate all students' abilities to persevere through challenges, explain the strategies they apply, and justify their conclusions. Research shows that students achieve at higher levels when they are actively engaged in the learning process (Boaler, 2016; CAST, 2020). Educators can increase student engagement by selecting challenging mathematics problems that invite *all* learners—including students who are EL and those with learning differences—to engage and succeed. Such problems are those that:

- involve multiple content areas;
- highlight contributions of diverse cultural groups;
- invite curiosity;
- allow for multiple approaches, collaboration, and representations; and
- carry the expectation that students will use mathematical reasoning.

Investigating and Connecting, Grades 3, 4, and 5

Chapters 6, 7, and 8 of the *Mathematics Framework* emphasize students' active engagement in the learning process. Instruction is organized and designed in the spirit of *investigating* the "big ideas" of mathematics and *connecting* content and mathematical practices within and across grade levels. A big idea becomes big when it includes connected mathematical content and a driver for investigation – it is the combination of content and investigation that makes content meaningful and important.

Instruction as described in this Framework intentionally draws conceptual connections within and across mathematical domains. The four Content Connections (CC) described in the framework organize content and provide mathematical coherence through the grades:

(CC 1) Communicating Stories with Data
(CC 2) Exploring Changing Quantities
(CC 3) Taking Wholes Apart, Putting Parts Together
(CC 4) Discovering Shape and Space

These content connections should be developed through investigation of questions in authentic contexts. Students actively engage in learning when they find purpose and meaning in the learning. Mathematical investigations will naturally fall into one or more of these Drivers of Investigation (DI):

- DI 1: Making Sense of the World (Understand and Explain)
- DI 2: Predicting What Could Happen (Predict)
- DI 3: Impacting the Future (Affect)

Big ideas that drive design of instructional activities will link one or more content connections with a driver of investigation, such as Communicating Stories with Data To Predict What Could Happen, or Exploring Changing Quantities To Impact the Future. Instruction should primarily involve tasks that invite students to make sense of these big ideas, elicit wondering in authentic contexts, and necessitate mathematics. Big ideas in math are central to the learning of mathematics, link numerous mathematical understandings into a coherent whole, and provide focal points for students' investigations. An authentic activity or problem is one in which students investigate or struggle with situations or questions about which they actually wonder. Lesson design should be built to elicit that wondering. An activity or task necessitates a mathematical idea or strategy if the attempt to understand the situation or task creates for students a need to learn or use the mathematical idea or strategy.

For example, a lesson may call for students to investigate for the purpose of predicting what could happen (DI 2), and to communicate the story with data (CC 1). The content involved in the course of a single investigation cuts across several CA CCSSM domains, perhaps Measurement and Data, Number and Operations in Base Ten (NBT), and Operations and Algebraic Thinking (OA). Simultaneously, students employ several of the Mathematical Practices as they conduct their investigations.

Specific Standards for Mathematical Practices, content standards, and activities are highlighted in the discussion of each content connection.

Content Connections, Grades 3, 4, and 5

CC 1: Communicating stories with data

In the upper elementary grades, students acquire important foundational concepts involving measurement, and increase the degree of precision to which they measure quantities as they engage in solving interesting, relevant problems. They measure various attributes including: time, length, weight, area, perimeter, and volume of liquids and solid figures (3. MD.1 – 4; 4. MD.1 – 4; 5. MD.1 – 5). Third-grade students develop an understanding of area, focusing on square units in rectangular configurations, and they build concepts of liquid volume and mass. As fourth-grade students solve problems in measurement, they discover and apply a formula to calculate areas of rectangles. They solve measurement problems involving time, money, distance, volume and mass. In fifth grade, students apply all of these skills as they focus on concepts of volume and use multiplicative thinking to calculate volumes of right rectangular prisms.

Measurement problem contexts are well-suited to connect with data science concepts. Students can gather and analyze measurement data to answer relevant questions. Chapter 5: Data Science, offers guidance as to how to integrate these content areas. Students apply reasoning and their growing understanding of multiplication and fractions to gather, represent, and interpret data in culturally meaningful contexts (SMP.1, 4, 7). While mathematical skills are necessarily in play when working with data, the emphasis is on representation and analysis; students need to be statistically literate in order to interpret the world (Van de Walle, 2014, p.378).

Students create and examine stories told by measurement and data as they

- solve problems involving measurement (3.MD.A.1, 2; 4.MD.A.1 – 3; 5.MD.1 – 5), and
- represent and interpret data (3.MD.B.3, 4; 4.MD.b.4; 5.MD.B.2) .

In their work with measurement and data, students use the SMPs to

- make sense of data and interpret results of investigations;
- construct arguments based on context as they reason about data; and
- select appropriate tools to model their mathematical thinking.

Key to creating lessons that promote student discourse, curiosity and active learning is the nature of the question being investigated. When the class determines what information to gather, they are likely to be fully engaged in the process. Students are naturally interested in themselves and their peers, and are curious about the world around them. Science, history–social science, and California’s Environmental Principles and Concepts are prime for integrating in mathematics, as they connect to local contexts that are relevant to students and their communities. These local contexts offer a wide array of opportunities for collection and analysis of real-world data and engage students in investigations about local environmental phenomena that can directly support math instruction and the objectives of the standards and frameworks for these other disciplines.

The internet provides access to almost unlimited sources of current data of interest to students. Some possible “about us” investigations might include the following:

- Minutes we spend traveling to school each day
- Our minutes of screen time in the past week
- Numbers of pets in our households

Other investigations may center on questions such as:

- What are typical temperatures in our area over the course of a year?
- What traffic patterns can we observe on nearby street(s)?
- What is the most common car color where we live? (youcubed.org, *Data Tells Us about Ourselves*)
- How far do players run during various professional sports games (soccer, basketball, baseball, etc.)?
- How far do people have to travel to the nearest hospital in different counties of the state?
- How long does it take for various seeds to germinate? (Van de Walle, 2014)

As students make decisions about how to gather the data, teacher guidance will likely be necessary. The question under investigation must be clearly defined and stated so that all data gatherers will be consistent as they collect and record responses. "Data Clusters and Distributions," a lesson for upper elementary grades (PBS Learning Media https://www.pbslearningmedia.org/), focuses on the importance of consistency in data collection. The video portion of the lesson demonstrates how inconsistent data gathering led to incorrect findings; the characters in the video then collaborate to remedy the problem and begin to analyze the data. The lesson poses additional questions highlighting the value of interpreting the results of a study in order to gain knowledge and make decisions or recommendations.

Investigations of data allow for integration and purposeful practice of the four operations and fractions concepts, both of which are major content areas in these grades. Third grade students use multiplication when they draw picture graphs in which each picture represents more than one object, or draw bar graphs in which the height of a given bar in tick marks must be multiplied by the scale factor to yield the number of objects in the given category. Fourth- and fifth-grade students convert measures within a given measurement system and use fractional values as they create and analyze line plots of data sets.

To understand the stories told by measurement and data, students are required to expand beyond collecting and presenting data; they must be actively engaged in analyzing and interpreting data as well.

Snapshot:

In this example (Lieberman and Brown), the teacher works with students to deepen their knowledge and skills of math, science, English language arts/literacy (ELA), and the California Environmental Principles and Concepts (EP&Cs) through an investigation of habitats on campus. They will be investigating how human activities can affect the number and diversity of organisms that live on campus.

The math-related focus of the learning will have students conduct an investigation that is local—ensuring it is relevant and meaningful to their lives. The teacher has decided to focus on content related to measurement and data, by having students: generate measurement data using rulers (3.MD.B.4); represent data by drawing a scaled picture graph and a scaled bar graph (3.MD.B.3); recognize area as an attribute of plane figures and understand the concept of area measurement (3.MD.C.5); and, solve real-world and mathematical problems involving perimeters of polygons (3.MD.D.8).

From a science perspective, students' investigations will focus on: gathering (CA NGSS SEP-3) and analyzing evidence (CA NGSS SEP-4); constructing an argument (CA NGSS SEP-7); and making a claim about the merit of a solution to a problem (CA NGSS 3-LS4-4).

In alignment with EP&C II, students will analyze the results of their investigation to examine how "the long-term functioning and health of terrestrial, freshwater, coastal and marine ecosystems are influenced by their relationships with human societies" (CA EP&C II); and, how "decisions affecting resources and natural systems are based on a wide range of considerations and decision-making processes (CA EP&C V).

Based on their investigations, mathematical analysis, and consideration of the environmental principles, students will choose to write either opinion pieces on topics or texts, supporting a point of view with reasons (ELA W.3.1), or informative/explanatory texts to examine a topic and convey ideas and information clearly (ELA W.3.2).

During an initial exploration of campus, students looked for places to observe plants and animals. They identified these areas on a simple map of the campus and recorded a few examples of what they observed.

Back in the classroom, students shared what they observed. The teacher introduced the concept of habitat and explained that a healthy habitat provides the resources and conditions necessary for a diversity of organisms (plants and animals) to survive. She also led a discussion about how human activity can affect the number and types of organisms that will survive in an area.

The teacher and students decided to work together to design an investigation to identify and gather data from both areas with different levels of human activity. They decided to compare areas: with more and fewer plants and animals; and, areas with more or less human activities. Prior to starting their outdoor investigation, the teacher introduced the relevant math standards and practices that they would use to analyze the data collected during the investigation. Students laid out and measured, using yardsticks, their rectangular study plots; recorded the numbers and types of plants and animals in the plots in a table; and, determined the types and levels of human activities taking place near each plot (by identifying the different types of activities and how many students and adults were involved in each type).

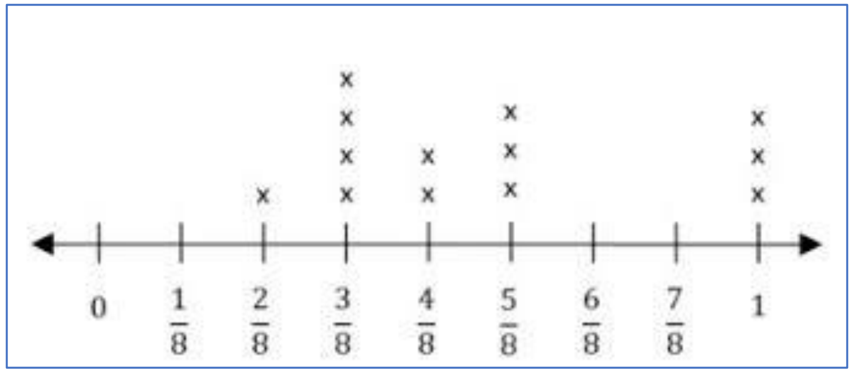
After collecting data, the class discussed the concept of area measurement. Students then recorded and analyzed their findings including the area of the different types of study plots and the nearby human activities. The students calculated the area of the rectangular study plots. They then used the data from their tables to create scaled bar graphs and/or scaled picture graphs of the number of animals and plants in the study plots. The teacher presented students with a real-world problem involving comparing the numbers of plants and animals in their study plots. The students used the graphs to make statements about the data (e.g., “There are x number of plants/animals in this study plot.” “There are more plants than animals in this plot.” “There are twice as many animals as plants in this plot.”). Students presented results using scaled bar and picture graphs.

The teacher posed the question, “How do human activities affect the number and diversity (types) of organisms that live on campus?” Students were asked to construct an argument based on the analysis of their data about the effects of human activities on habitats and the organisms that live there. Working in teams, they were asked to design a solution that might minimize the effects of human activities on organisms that live on campus. Using the results of their investigations, data collected and analyzed, and graphs, students wrote informative/explanatory texts that examined the topic of changes to habitats and conveyed their ideas about the problems and made claims about the merits of their solutions.

As they concluded their investigations, students began to wonder by whom and how the decisions had been made about the design and use of the campus. One student, a new arrival to the school, also mentioned that there were many more plants and animals at her previous school. This comment initiated another major question and discussion about why some schools have lots of green space, trees, and gardens, and others have few or none. This conversation created a direct connection to the teacher’s upcoming history-social science unit where she intended to focus on the distribution and use of resources and environmental justice.

The following week, the class began a unit on three important topics: the ways in which people have used the resources of the local region and modified the physical environment (HSS 3.1.2.); the importance of public virtue and the role of citizens, including how to participate in a classroom, in the community, and in civic life (HSS 3.4.2.); and, understanding that individual economic choices involve trade-offs and the evaluation of benefits and costs (HSS 3.5.3.).

One approach, called “turning the task around,” allows students to study a mystery graph that illustrates some unknown topic. For example, given the unlabeled line plot here, students can describe what they notice about the values shown, and make suggestions as to what this graph could reasonably represent.



Some possibilities might include:

- The lengths in inches of various insects
- The widths in inches of people’s fingers
- What fraction of a pizza different people ate
- What distance in miles students ran during physical education class
- Weights in grams of rocks in the class collection

In the PBS task “What’s Typical, Based on the Shape of Data Charts?” students analyze two sets of data (collected by two different students) showing the heights of all the members of the school band. Both students measured the heights of the same 21 band members, yet the numbers reported are not identical in the two data sets. (https://www.pbslearningmedia.org/). Preliminary tasks invite students to find the range of the data (4.MD.B.4) and the mode (a middle-school topic) for each set. Students then consider and offer explanations as to why the two data sets might differ, and make recommendations to the band director as to how many each of sizes small, medium, and large band uniforms they should order.

"Button Diameters," from *Illustrative Mathematics* (https://www.illustrativemathematics.org) emphasizes measurement skills: students measure buttons to the nearest fourth and eighth inch. After creating line plots of the data, students describe the differences between the two line plots they created, consider which line plot gives more information, and which is easier to read.

Chapter 3 is devoted specifically to data science; it describes the vital role data science plays in the modern world and enumerates important principles in the learning of data science in grades K–12.

CC 2: Exploring changing quantities

Upper elementary grade students extend their understanding of operations to include multiplication and division. They study several ways of thinking about these operations, represent their thinking with tools, pictures, and numbers, and make connections among the various representations. Full understanding of the meanings of multiplication and division is essential, as students will need to apply the same thinking strategies when they begin operations with fractions. The development of solid understanding of these operations also prepares students for mathematics in middle school and beyond.

In grades three through five, students advance their algebraic thinking as they

- understand properties of multiplication and the relationship between multiplication and division (3.OA.1, 2, 5, 6; 4.OA.2, 5, 6; 5.NF.3, 4, 7);
- use the four operations to solve problems with whole numbers (3.OA.7.8; 4.NBT.4, 5; 5.NBT.5, 6); and
- use letters to stand for unknowns in equations (3.OA.3, 8; 4.2, 3).

Simultaneously, they expand their use of all the SMPs. For example, they

- think quantitatively and abstractly using multiplication and division;
- model contextually based problems using a variety of representations;
- communicate thinking using precise vocabulary and terms; and
- use patterns they discover as they develop meaningful, reliable and efficient methods to multiply and divide (SMP.2, 4, 6, 8) numbers within 100.

Meanings of Multiplication and Division

In previous grades, students worked with the operations of addition and subtraction; now they develop understanding of the meanings of multiplication and division of whole numbers. They recognize how multiplication is related to addition (it can sometimes call for repeatedly adding equal-sized groups), how it is distinct from addition, and how it serves as a more efficient way of counting quantities.

Students engage initially in multiplication activities and problems involving **equal-sized groups, arrays,** and **area models** (NGA/CCSSO 2010c). Later (in grade four) they also solve **comparison** problems and use the terms **factor**, **multiple,** and **product.** Students who hear teachers consistently and intentionally using precise mathematics terms during instruction become accustomed to the vocabulary. Over time, as they gain experience and as their confidence increases, students begin to incorporate the language themselves.

The most common types of multiplication and division word problems for grades three, four, and five (from the 2013 Mathematics Framework, Glossary) are summarized in the table below:

Table GL-5. Common Multiplication and Division Situations*

	Unknown Product	Group Size Unknown	Number of Groups Unknown
	$\times 6 = \square$	$3 \times \square =$ and $\div 3 = \square$	$\square \times 6 =$ and $\div = \square$
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there altogether? **Measurement example** You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally and packed inside 3 bags, then how many plums will be in each bag? **Measurement example** You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed, with 6 plums to a bag, then how many bags are needed? **Measurement example** You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays,† Area‡	There are 3 rows of apples with 6 apples in each row. How many apples are there? **Area example** What is the area of a rectangle that measures 3 centimeters by 6 centimeters?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? **Area example** A rectangle has an area of 18 square centimeters. If one side is 3 centimeters long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? **Area example** A rectangle has an area of 18 square centimeters. If one side is 6 centimeters long, how long is a side next to it?
Compare	A blue hat costs $6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? **Measurement example** A rubber band is 6 centimeters long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs $18, and that is three times as much as a blue hat costs. How much does a blue hat cost? **Measurement example** A rubber band is stretched to be 18 centimeters long and that is three times as long as it was at first. How long was the rubber band at first?	A red hat costs $18 and a blue hat costs $6. How many times as much does the red hat cost as the blue hat? **Measurement example** A rubber band was 6 centimeters long at first. Now it is stretched to be 18 centimeters long. How many times as long is the rubber band now as it was at first?
General	$\times b = \square$	$\times \square =$ and $\div a = \square$	$\square \times b = p$ and $p \div b = \square$

*The first examples in each cell focus on discrete things. These examples are easier for students and should be given before the measurement examples.

† The language in the array examples shows the easiest form of array problems. A more difficult form of these problems uses the terms *rows* and *columns*, as in this example: *"The apples in the grocery window are in 3 rows and 6 columns. How many apples are there?"* Both forms are valuable.

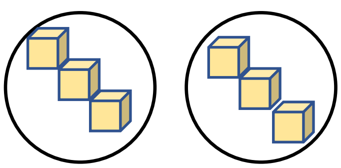
1044

‡ Area involves arrays of squares that have been pushed together so that there are no gaps or overlaps; thus array problems

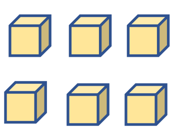
Views and Interpretations of the Operation of Multiplication

When students focus on the **equal-groups** interpretation of multiplication, they find the total number of objects in a particular number of equal-sized groups (3.OA.1). This references their understanding of addition, but it is important that instructional approaches include repeated addition as one of *several* distinct and necessary interpretations of multiplication. As they continue, students will use multiplication to solve contextually relevant problems involving **arrays, area**, and **comparison** using a variety of representations, including number lines, to show their thinking (SMP.4, 5, 6, 3.OA.3; 4.OA.2, 4.NBT.5).

Moving beyond the equal groups interpretation of multiplication can prove challenging for students. Arrays can serve as a likely next step, as they can be seen as the familiar equal-sized groups, but now the objects are arranged into orderly rows. This example shows, in each case, that when there are two groups of three cubes, there are six cubes, and 2 x 3 = 6.



Two Equal-sized Groups of three cubes

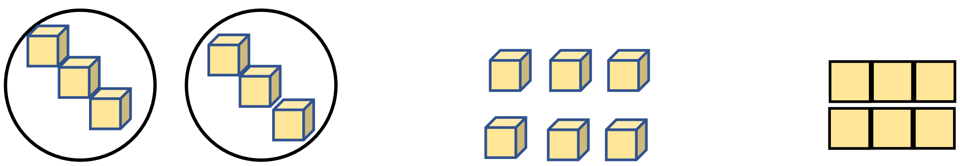


Array: Two rows (of equal size), with three cubes in each row

The instructional goal should be to move students beyond counting and re-counting items singly to determine the total; instead, they recognize the groups or **rows** as the quantities that comprise the total. In the example above, as students find the product, six, they should be counting by threes (three in each row) rather than counting single cubes.

To solve a problem such as, *“How many seats are in our multi-purpose room? There are 20 rows of seats and each row has 16 seats,”* students can think about and represent the problem with an array. Some students may use the distributive property to simplify the problem, perhaps realizing that 10 + 10 = 20, multiplying 10 x 16 = 160 and adding 160 + 160 = 320. Others might take the 16 apart, thinking 16 = 10 + 8. They can then apply the distributive property: 10 x 20 + 6 x 20 = 200 + 120 = 320.

Students begin to view multiplication as area by building rectangles using sets of square tiles, which allows them to connect the now familiar array models with the newer idea of the area of a rectangle. Once students learn various ways to solve contextual story problems through creating, representing, and interpreting arrays, introducing the area interpretation of multiplication makes sense.



Equal-sized Groups **Array** **Area**

In grade three, students develop an understanding of area and perimeter by using visual models. Fourth graders extend their work with area and use formulas to calculate area and perimeter of rectangles. Students in grade five will continue to apply the equal-sized groups and area models, and will begin to use the standard algorithm to multiply whole numbers (5.NBT.B.5). Fifth graders use their understanding of whole number multiplication, along with concrete materials and visual models, to multiply fractions (4.NBT.B.5; 5.NBT.B.6, 5.NF.B.6). The interpretation of multiplication as area connects the first category of investigation *Exploring Changing Quantities* with the third category, *Stories told by Measurement and Data*. Further discussion and illustration of these topics are found below, and in the vignette, “Alex Builds Rectangles to Find Area.”

Beginning in fourth grade, students solve **comparison** problems in multiplication and division (4.OA.A.1). Comparison multiplication requires students to engage in thinking about some number of “times as many.” This is particularly important in setting a foundation for scaling reasoning (5.NF.B.5) in grade five and demands careful introduction. The fifth-grade study of multiplication as scaling likewise sets the foundation for identifying scale factors and making scale copies in seventh grade and subsequent work with dilations and similarity (7.RP.1, 2, 3; 7.G.A.1) Presenting problems in familiar, culturally relevant contexts can help students to develop understanding and come to distinguish when **multiplicative** reasoning rather than **additive** reasoning is called for. They can compare quantities in the classroom (e.g., five times as many whiteboard pens as erasers, three times as many windows as doors, four times as much water as lemonade concentrate). Money can be a meaningful context, as seen in the following example, “Comparing Money Raised,” from *Illustrative Mathematics*, https://www.illustrativemathematics.org/.

> Luis raised $45 for the animal shelter, which was 3 times as much money as Anthony raised. How much money did Anthony raise?

In fifth grade, students prepare for middle school work with ratios and proportional reasoning by interpreting multiplication as **scaling**. They examine how numbers change as the numbers are multiplied by fractions. Students should have ample opportunities to examine the following cases:

a) When multiplying a number greater than one by a fraction greater than one, the number increases.

b) When multiplying a number greater than one by a fraction less than one, the number decreases. This is a new interpretation of multiplication that needs extensive exploration, discussion, and explanation by students.

Examples	**5.NF.5▲**
Student 1: “I know $\frac{3}{4}\times 7$ is less than 7, because I make 4 equal shares from 7, but I only take 3 of them ($\frac{3}{4}$ is a fractional part less than 1). If I’m taking a fractional part of 7 that is less than 1, the answer should be less than 7.”	
Student 2: “I know that $2\frac{3}{8}\times 8$ should be more than 16, because 2 groups of 8 are 16, and $2\frac{3}{8}>2$. Also, I know the answer should be less than 3 × 8 or 24, since $2\frac{3}{8}<3$.”	
Student 3: “I can show by equivalent fractions that $\frac{3}{4}=\frac{3\times 5}{4\times 5}$. I see that $\frac{5}{5}=1$, so the result should still be equal to $\frac{3}{4}$.”	
Adapted from ADE 2010 and KATM 2012, 5th Grade Flipbook.	

Story contexts matter greatly in supporting students' robust understanding of the operations. Multiplication and division situations move beyond whole numbers, as students develop understanding of fractions and measure lengths to the quarter inch in third grade (3.MD.B.4), and later calculate area of rectangles with fractional side lengths. As noted in Chapter 3: Number Sense, historically, the majority of story problems and tasks children experienced in the younger grades tended to rely on contexts in which things are counted rather than measured to determine quantities (how many apples, books, children, etc., versus how far did they travel, how much does it weigh). Increased use of measurement contexts in the primary grades will support a student's later work with fractions. A student who solves a measurement problem involving whole numbers will be able to apply the same reasoning to a problem involving fractions. Note that the Table of Common Multiplication and Division Situations (see p. XX) includes examples that call for measurement as well as for counting. The intent is to promote increased use of measurement contexts at all elementary grades.

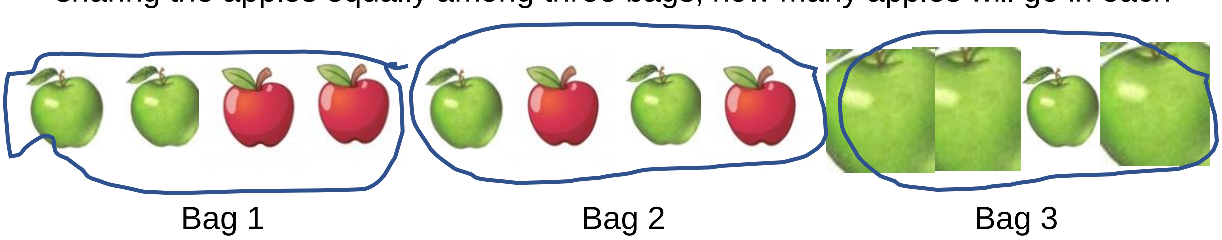
Views/Interpretations of the Operation of Division

Students work with division alongside multiplication, and develop the understanding that these are **inverse** operations. They come to recognize division in two different situations: **partitive** division (also referred to as fair-share division), which requires equal sharing (e.g., how many are in each group?); and **quotitive** division (repeated subtraction or measurement division), which requires determining how many groups (e.g., how many groups can you make?) (3.OA.A.2).

Partitive Division (also known as Fair-Share or Group Size Unknown Division)

In partitive division situations, the number of groups or shares to be made is known, but the number of objects in (or size of) each group or share is unknown.

Discrete (counting) Example: There are 12 apples on the counter. If you are sharing the apples equally among three bags, how many apples will go in each

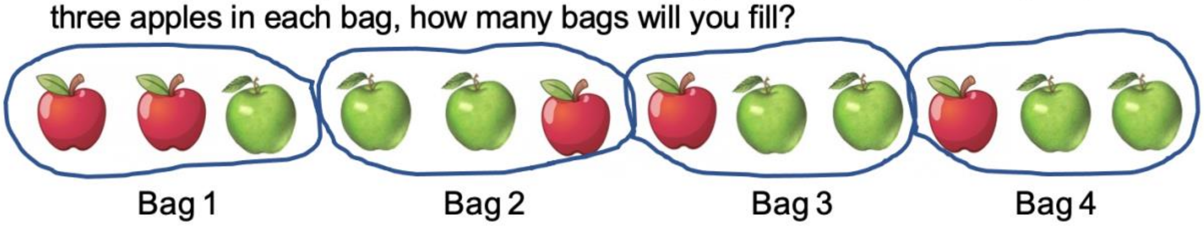


Measurement Example: There are 12 quarts of milk. If you are sharing the milk equally among three classes, how much milk will each class receive?

Quotitive Division (also known as Measurement Division, Repeated Subtraction Division or Number of Group Unknown Division)

In quotitive division situations, the number of objects in (or size of) each group or share is known, but the number of groups or shares is unknown.

Discrete (counting) Example: There are 12 apples on the counter. If you place three apples in each bag, how many bags will you fill?



There will be 4 bags of apples.

Measurement Example: There are three quarts of milk. If you give three quarts to each class, how many classes will get milk?

Both interpretations of division should be explored, as they both have important uses for whole number and for fraction situations. The action called for in the sample problems above illustrate that a quotitive situation typically differs from the action involved in a comparable problem posed in a partitive context. Representations of the actions will differ, and attention to how and why this occurs supports understanding of these two interpretations of division. In these grades, teachers use the language of **equal sharing**, **number of shares** (or groups), **repeated subtraction**, and the **size of each group**, with students rather than the more formal terms, partitive or quotitive.

Students use the inverse relationship between multiplication and division when they find the unknown number in a multiplication or division equation relating three whole numbers. Viewing division as the inverse of multiplication presents a natural opportunity for introducing the use of a letter to stand for an unknown quantity (SMP.4, 6; 3.OA.A.4; 4.OA.A.3). Students may be asked to determine the unknown number that makes the equation true in equations such as $8 \times n = 48$, $5 = n + 3$, $6 \times 6 = n$ (3.OA.A.4, 3.OA.D.8).

Acquiring understanding of variables is an ongoing process that begins in grade three and increases in complexity through high school mathematics.

Example: *There are four apples in each of the bags on the counter, and there are 12 apples altogether. How many bags must there be?* The student can write the equation *n* x 4 and solve for *n* by thinking "what times four makes 12?" a missing factor approach that utilizes the inverse relationship between multiplication and division.

In grade three, students learn and develop the concept of division, building on the understanding of the inverse relationship between multiplication and division (3.OA.B.5, 6, 3.OA.C.7). Grade-four students find whole number quotients, limited to single digit divisors and dividends of up to four digits (4.NBT.B.6). Students in grade five extend this understanding to include two-digit divisors and solve division problems (5.NBT.B.6). In grades four and five, students benefit from using methods based on properties, the relationship between multiplication and division, and place value to solve, illustrate, and explain division problems (Carpenter, et.al., 1997; Van de Walle et al, 2014). The acquisition of the standard algorithm for division of multi-digit numbers is reserved for grade six (6.NS.B.2).

CC 3: Taking Wholes Apart and Putting Parts Together – Whole Numbers

Elementary students come to understand the structure of the number system by building numbers and taking them apart; they make sense of the system as they explore and discover numbers inside numbers. A significant part of students' mathematical work in grades three, four, and five is the development of efficient methods—methods that they understand and can explain—for each operation with whole numbers. By engaging in meaningful activities and explorations, students gain fluency with multiplication and division with numbers up to 10. They discover ways to apply the commutative and associative properties to solve multiplication problems. They use place value understanding and the distributive property to simplify multiplication of larger numbers.

Students use place value, take wholes apart, put parts together, and find numbers inside numbers when they

- use the four operations with whole numbers to represent and solve problems (3.OA.A.3, 3.OA.C.7, 3.OA.D.8; 3.NBT.2; 4.OA.A.2, 3, 4.OA.B.4.NBT.B.4, 5, 6; 5.NBT.B.5, 6);

- use place value understanding and properties of operations to perform multi-digit arithmetic (3.OA.C.7, 3.OA.D.8; 4.NBT.B.4, 5; 5.NBT.B.5, 6);
- build fluency for products of one-digit numbers (3.OA.C.7);
- gain familiarity with factors and multiples (3.OA.B.6; 4.OA.B.4); and
- identify, generate, and analyze patterns and relationships (3.OA.D.9; 3.NBT.A.1; 4.OA.C.5, 4.NBT.A.1, 3).

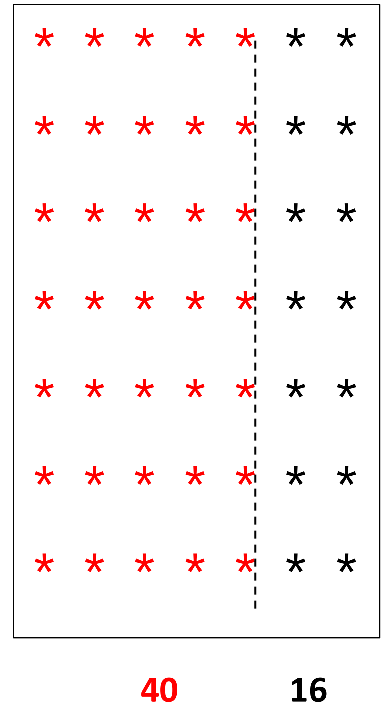
Development of students' use of the SMPs continues as they

- apply the mathematics they already know to solve multiplication and division problems;
- use pictures and/or concrete tools to model contextually based problems;
- communicate thinking using precise vocabulary and terms; and
- use patterns they discover as they develop meaningful, reliable and efficient methods to multiply and divide (SMP.2, 4, 6, 8) numbers within 100.

Strategies and Invented Methods for Multiplication and Division

Students need opportunities to develop, discuss, and use efficient, accurate, and generalizable methods to compute. The goal is for students to use general written methods for multiplication and division that they can explain and understand using visual models and/or place-value language. (SMP.2, 6, 8; 3.OA.1, 3.OA.C.7; 4.NBT.B.5). In grade five, students learn the standard algorithm for multiplying multi-digit numbers, connecting this abstract method to their understanding of the operation of multiplication (SMP.2, 8; 5.NBT.A.1). Research reminds us that students who use invented strategies *before* applying standard algorithms understand base-ten concepts more fully and are better able to apply their understanding in new situations than students who learn standard algorithms first (Carpenter, et.al., 1997). There is further merit in fostering students' use of informal methods before teaching algorithms. "The understanding students gain from working with invented strategies will make it easier for you to meaningfully teach the standard algorithms" (Van de Walle, et al, 2014).

Children often invent ways to take numbers apart to find an easier way to solve a problem. Students who knows some, but not all multiplication facts use invented strategies to calculate 7 x 8:



Student A: *I know that 5 x 8 = 40, and then there are two more eights, so that makes 16. And then I add 40 + 16 = 56, so 7 x 8 = 56.*

Student A used the distributive property. To help the class recognize the usefulness of the property, the teacher draws an array of stars: eight rows of stars with seven stars in each row. As shown at right, the teacher separates the columns to represent the student's thinking, showing eight rows with five (red) stars in each row and eight rows with two (black) stars in each row. The teacher invites Student A to show the class how this drawing represents their thinking.

Student A uses the pen to write "40" below the red part of the drawing, and 16 below the black part.

Student A explains: *The red part is 8 x 5, and then the black part is 8 x 2, so it's 40 + 16*.

Student B: *I knew that 7 x 7 = 49, and then there's one more seven, so I added 49 + 7 = 56.*

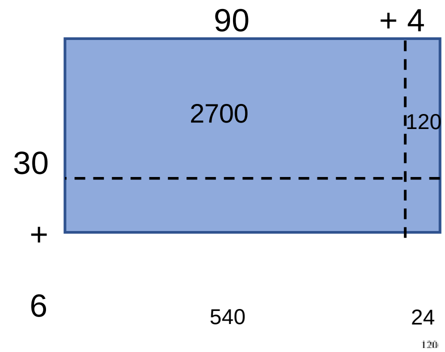
The teacher invites Student B to show the class the equations they used. Student B writes*: 7 x 7 = 49, and 49 + 7 = 56.*

The teacher checks with the class for understanding of what Student B did, and calls on two other students to re-explain this strategy.

The teacher asks the class to consider whether Student B used the distributive property, and how they could illustrate Student B's thinking.

As students begin to multiply two-digit numbers using strategies based on place value and properties of operations (SMP.2, 7, 8; 3.OA.B.5, 3.OA.C.7; 4.NBT.B.5, 6), they find and explain efficient methods. Grade-four students record their processes with pictures and manipulative materials as well as with numbers.

To multiply 36 x 94, three students use place value understanding and the distributive property, yet they use three different recording methods to show their thinking.



Student A

2700
540
120
+ 24
3384

Student A labels the partial products within each of the four rectangles in the picture: 2700, 540, 120, and 24, and calculates the final sum beside the sketch.

Student B calculates the four partial products and shows the thinking for each.

While it is essential that students understand and can explain the methods they use, variations in how they record their calculations are acceptable at this stage (Fuson and Beckmann, 2013). The recording method shown by Student C (below), for example, reflects the same thinking as that of Student D (below), but the locations where the students show the regroupings are different.

Student B

Showing the partial products

94	
X 36	Thinking:
24	6 x 4 = 24
540	6 x 90 = 540
120	30 x 4 = 120
+ 2700	30 x 90 =
3384	

Student C uses the standard algorithm with the regroupings shown above the partial products rather than above the "94" in the problem.

In this recording of the standard algorithm, the regroupings are shown above each of the partial products in their correct place value positions.

```
   94
X  36
  5 2
   44
 2 1
  720
 3384
```

Thinking:

6 x 4 = 24. The **4** is recorded in the ones place and the **2 tens** are recorded in the tens column.

6 x 90 = 540. The **40** is shown by the 4 in the tens place; the **5 hundreds** are recorded in the hundreds column.

30 x 4 = 120. The **20** is recorded in the tens and ones places; the **1 hundred** is recorded in the hundreds column.

30 x 90 = 2700. The 7 hundreds are recorded in the hundreds place; the **2 thousands** are recorded in the thousands place

Student D uses the standard algorithm with the regroupings shown above the factor "94."

Student D
Standard algorithm, traditional recording method:

1 This *1* represents the 100 in 30 x 4 = 120
2 The *2* represents two 10's in 6 x 4 = 24.

```
    94
X   36
   564
+ 2820
  3384
```

During thoughtfully guided class discussion, perhaps on several occasions, the connections among the pictorial representation (A), the partial products method (B), and the standard algorithm (C and D) become clear.

In order to use the standard algorithm successfully, and with understanding in grade five (5.NBT.B.5), students will need guidance in making connections between the increasingly abstract methods of multiplying two-digit numbers. Building understanding with concrete materials (e.g., base ten blocks) and visual representations (e.g., more generic rectangular sketches) allows students to build the necessary foundation for the formal algorithm. Students will rely on these skills and understandings for years to come as they continue to multiply and divide multi-digit whole numbers and to add, subtract, multiply, and divide rational numbers. The table below indicates the grade levels at which each of the standard algorithms is introduced. Note that the CA CCSSM call for no standard algorithms in grades TK, K, or 1. The progression of algorithm instruction begins with the standard algorithm for addition and subtraction in grade two, and concludes with the standard algorithm for whole number division in grade six.

Development of Standard Algorithms across Grades TK-6

Addition and Subtraction	Multiplication	Division	Operations with Decimals
Grade 2: 2.NBT.5 Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.	Grade 3: 3.NBT.3 Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9 × 80, 5 × 60) using strategies based on place value and properties of operations.	Grade 4: 4.NBT.6 Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.	Grade 5: 5.NBT.7 Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.
Grade 3: 3.NBT.2 Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.	Grade 4: 4.NBT.5 Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.	Grade 5: 5.NBT.6 Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.	
Grade 4: 4.NBT.4 Fluently add and subtract multi-digit whole numbers using the standard algorithm.	Grade 5: 5.NBT.5 Fluently multiply multi-digit whole numbers using the standard algorithm.	Grade 6: 6.NS.2 Fluently divide multi-digit numbers using the standard algorithm.	Grade 6: 6.NS.3 Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

Pattern investigation is a powerful means of building understanding, and can provide access for students with visual strengths and any students who lack confidence with numerical tasks. Investigating patterns helps students develop facility with multiplication, and supports them on their path to fluency. There are many patterns to be discovered by exploring the multiples of numbers. As they explore patterns visually, students find, describe, and color patterns on number charts. They engage in partner and/or class conversations in which they notice and wonder, explain their discoveries and listen to and critique others' discoveries. Examining and articulating these mathematical patterns is an important part of the work on multiplication and division (FW p. 162 plus).

Example: On a multiplication table, each student colors in the multiples of a designated factor (in this case, multiples of 4). The teacher poses questions, prompting students to notice and wonder why the pattern they see occurs, and what all these multiples of four have in common.

Students next circle on the 4s chart all the multiples of four that are also multiples of 5 (20, 40, 60, 80, 100) and analyze why only those 5 multiples coincide, where they are located on the table, what those numbers have in common.

x	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Fluency: The acquisition of fluency with multiplication facts begins in third grade and development continues in grades four and five. Together, this acquisition establishes the foundation for work with ratios and proportions in grades six and seven. To support this development, teachers must provide students with learning opportunities that are enjoyable, make sense, and connect to previous learning about the meanings of operations and the properties that apply. Research shows that when students are under time pressure to memorize facts devoid of meaning, working memory can become blocked. Such stressful experiences tend to defeat learning, and for many students can lead to persistent, generalized anxiety about their ability to succeed in mathematics (Boaler, Williams, 2015).

The following are some general strategies that can be used to help students know from memory all products of two one-digit numbers (3.OA.C.7).

Strategies for Learning Multiplication Facts (SMP.2, 4, 8; 3.OA.C.7):

- Multiplication by zeros and ones
- Doubles (twos facts), doubling twice (fours), doubling three times (eights)
- Tens facts (relating to place value, 5 x 10 is 5 tens, or 50)
- Fives facts (knowing that the fives facts are half of the tens facts)
- Know the squares of numbers (e.g., 6 x 6 = 36)
- Patterns (e.g., for nines, 6 x 9 = 6 x 10 – 6 x 1 = 60 – 6 = 54)

Fluency

Fluency is an important component of mathematics; it contributes to a student’s success through the school years and will remain useful in daily life as an adult. What do we mean by fluency in elementary grade mathematics? Content standard 3.OA.C.7, for example, calls for third graders to “Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division … or properties of operations.” Fluency means that students use strategies that are ***flexible**, **efficient**,* and *accurate* to solve problems in mathematics. Students who are comfortable with numbers and who have learned to compose and **decompose** numbers strategically develop fluency along with conceptual understanding. They can use known facts to determine unknown facts. They understand, for example, that the product of 4 x 6 will be twice the product of 2 x 6, so that if they know 2 x 6 = 12, then 4 x 6 = 2 x 12, or 24. In the past, fluency has sometimes been equated with speed, which may account for the common, but counterproductive, use of timed tests for practicing facts. But in fact, research has found that, “Timed tests offer little insight about how flexible students are in their use of strategies or even which strategies a student selects. And evidence suggests that efficiency and accuracy may actually be negatively influenced by timed testing.” (Kling, G and Bay-Williams J.M. 2014, p.489).

Fluency is more than the memorization of facts or procedures, and more than understanding and having the ability to use one procedure for a given situation. Fluency builds on a foundation of conceptual understanding, strategic reasoning, and problem solving (NGA Center & CCSSO, 2010; NCTM, 2000, 2014). To develop fluency, students need to connect their conceptual understanding with strategies (including standard algorithms) in ways that make sense to them.

Reaching fluency with multiplication and division within 100 represents a major portion of upper elementary grade students' work. Practice should be organized to maximize student success. Some additional suggestions to support fluency and increase efficiency in learning multiplication and division facts include:

- Focus most heavily on products and unknown factors students understand but in which they are not yet fluent.
- Continue meaningful practice—and extra support as necessary—for those students who need it to attain fluency.
- Encourage students to use, work with and explore numbers.

When practice is varied, playful, and tailored to student needs, students enjoy and learn more mathematics readily (Boaler, 2015; Kling, Bay-Williams, 2014, 2015). Interesting, worthwhile facts practice can be accomplished by engaging students in number talks/strings and games. Familiar card games such as *Concentration* or *War* are easily adapted to provide fact practice (Kling, Bay-Williams, 2014, p.493).

For example: Pairs of students can use a deck of playing cards (with the face cards removed) to practice multiplication facts: The cards are shuffled and four cards are turned face up between the players. The remaining cards are placed face down in a stack. Player A selects two of the face-up cards, calculates the product and explains the strategy they used. Player B confirms or challenges the product—they may ask for further explanation of the strategy—and if the product is correct, Player A claims those two cards. Player B turns over two cards from the stack to replace those taken by Player A, and then takes their own turn. For further discussion of fluency and additional resources, see Chapter 3: Number Sense.

Investigating and Applying Properties of Multiplication

As students develop strategies for solving multiplication problems, they naturally use properties of operations to simplify the tasks. Students are not explicitly required to call the properties they use by their formal names, but they are expected to apply them strategically throughout these grades as they calculate quantities (SMP.5, 7; 3.OA.B.5, 3.OA.C.7; 4.NBT.B.4, 6; 5.OA.A.1, 2; 5.NBT.A.4, 5.NBT.B.5). Teachers support this by providing frequent opportunities for students to explore and discuss various multiplication strategies and properties (SMP.3, 4, 5, 8, ELD.P9), and by highlighting the efficacy of the strategies as they arise in practice (Kling, Bay-Williams, 2015).

Students provide several methods as they explain their thought processes for solving 7 x 24. The teacher records students' methods on the board, as shown below, using symbolic notation.

- Jax: I skip counted by two seven times, and 7 x 2 = 14, so that means 7 x 20 = 140 because 20 is ten times as much as two. Then I had to multiply 7 x 4, and that was 28. I know 2 x 7 is 14, so I added 14 + 14. Then I added 140 + 28 and got 168.
- Lucca: I used 25 instead of 24. I did 7 x 25 and that equals 175, because that's like 7 quarters. But it's not really 25, it is *24*, so I had to take away an extra seven. So I took away five (of the seven) to get 170, and then took away two more to get to 168.
- Pippin: My way is kind of like Jax's. I know 7 x 10 = 70, and there are two tens in 24, so I did 7 x 10 again. 70 + 70 = 140. And 7 x 4 = 28, so 140 + 28 = 168.

Jax:

2, 4, 6, 8, 10,12,14 so

7 x 2 = 14

7 x 20 = 140

7 x 2 = 14, and 14 + 14 = 28, so 7 x 4 = 28

140 + 28 = 168

Lucca:

7 x 25 = 175

175 – 5 = 170

170 – 2 – 168

Pippin:

7 x 10 = 70

7 x 10 = 70

70 + 70 = 140

7 x 4 = 28

The teacher asks the class to consider what is the same and what is different about the three methods. Students point out that all three methods produce the same result, and that they all took the number 24 apart, but that they did that differently. A few students say that Lucca's method is tricky and they don't know why she did that. The teacher replies that they will talk about Jax and Pippin’s methods first and then ask Lucca to explain the thinking behind that method.

The teacher asks Jax and Pippin to describe more about how their methods are alike.

- Jax: *We both broke the 24 apart and we both multiplied 7 x 4.*
- Pippin: *And we both got the same product*.
- Teacher: *So, you both knew that you could multiply 7 x 24 by taking the 24 apart, finding parts of the product and then putting all the parts together?*
- Jax and Pippin: *Yes!*
- Teacher: *Aha! So, you used the distributive property! We will have to try some more problems and see if your method always works.*
- Teacher: *Now let’s figure out whether Lucca used the distributive property, too.*

The class focuses attention on Lucca’s method, and at the end of the discussion the teacher tells the students that they will have more opportunities to try out these methods on other problems and to see when they are useful and how they can help solve problems more easily.

Commutative Property: As they work with equally-sized groups, arrays, and area, students encounter many opportunities to employ the commutative property of multiplication. They may notice that they also use commutativity to solve addition problems. In story contexts, there is a difference between “two groups of three objects each” (e.g. pencils, ants, pounds, quarts) and “three groups with two objects each.” Students discover the commutative property by noticing that the result in both cases is a total of six objects. This also supports their ability to become fluent with multiplication within 100: if a student knows 4 x 6 = 24, then they know that 6 x 4 also is equal to 24.

Associative Property: Experiences in which students must multiply three factors, such as 3 x 5 x 2, provide opportunities to apply the associative property. A student can first calculate 3 x 5 = 15, then multiply 15 x 2 to find the product 30. Another student may find 5 x 2 = 10 first, then multiply 3 x 10 to find the same product, 30. Again, students can observe that the associative property applies to both addition and multiplication.

Distributive Property: Students frequently use the distributive property to discover products of whole numbers (such as 6 x 8) based on products they can find more easily.

A student who knows that 3 x 8 = 24 can use that to recognize that since 6 = 3 + 3, then 6 x 8 = (3 + 3) x 8 = 3 x 8 + 3 x 8, and that 3 x 8 + 3 x 8 = 24 + 24 = 48.

Another student may use knowledge that 6 x 8 = 6 x (4 + 4) to solve: 6 x 8 = 6 x (4 + 4) = 6 x 4 + 6 x 4 = 24 + 24 = 48.

The distributive property may also involve subtraction. A student may solve 6 x 8 by beginning with the familiar 6 x 10: 6 x 8 = 6 x (10 – 2) = 6 x 10 – 6 x 2 = 60 – 12 = 48.

Taking Wholes Apart and Putting Parts Together – Fractions

NOTE: all fractions need to be written in vertical format, such as $\frac{1}{4}$. For speed and convenience in typing this draft, some fractions here are shown in the slanted format, ¼.

In grades one and two, students partitioned circles and rectangles into two, three, and four equal shares and used fraction language (e.g., halves, thirds, half of, a third of). Their experiences with fractions were concrete and related to geometric shapes. Starting in grade three, important foundations in fraction understanding are established, and the topic calls for careful development at each level.

There are several ways to think about fractions, which increases the complexity and significance of this body of learning. Children begin formal work with fractions in third grade, with a focus on **unit fractions** and **benchmark fractions**. Fourth and fifth grade students move on to fraction equivalence and operations with fractions. Fifth grade mathematics includes the development of the meaning of division of fractions, a sophisticated idea which needs careful attention and preparation in prior grades. Students often struggle with key fraction concepts, such as “Understand a fraction as a number on the number line…” (3.NF.2) and “Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions” (5.NF.B.7). It is imperative to present fractions in meaningful contexts and to allow ample time for the careful development of fraction concepts at each stage.

Proficiency with fractions is essential for success in more advanced mathematics such as percentages, ratios and proportions, and algebra.

To develop fraction concepts, upper-elementary students should

- develop understanding of fractions as numbers (3.NF.1, 2);

- understand decimal notation for fractions, and compare decimal fractions (4.NF.B.5, 6, 7);
- extend understanding of fraction equivalence and ordering (3.NF.3; 4.NF.A.1, 2); and
- apply and extend previous understandings of operations to add, subtract, multiply and divide fractions (4.NF.B.3, 4; 5.NF.1–7).

As students work with fractions, they use the SMPs. For example:

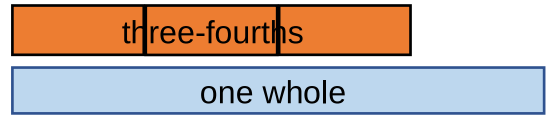
- Think quantitatively and abstractly, connecting visual and concrete models to more abstract and symbolic representations of fractions.
- Model contextually based problems mathematically, and using a variety of representations.
- Select and use tools such as number lines, fraction squares, or illustrations appropriately to communicate mathematical thinking precisely.
- Make use of structure to develop benchmark fraction understanding.

Understanding fractions as numbers; equivalence, and ordering fractions

Grade three students begin with **unit fractions**, building on the idea of partitioning wholes into equal parts and become familiar with **benchmark** fractions such as one-half. In fourth grade, the emphases are on equivalence, ordering, and beginning operations with fractions and decimal fractions. Fifth-grade students apply their previous understandings of the operations to add, subtract, multiply and divide fractions (in limited situations).

An important goal is for students to see unit fractions as the basic building blocks of all fractions, in the same sense that the number one is the basic building block of whole numbers. Students make the connection that, just as every whole number is obtained by combining a sufficient number of ones, every fraction is obtained by combining a sufficient number of unit fractions (adapted from UA Progressions Documents 2013a). While the idea of "$\frac{3}{4}$," as a number may be difficult for students to grasp initially, "putting together three one-fourths" is more readily accessible. To develop this concept, students can use concrete materials to build a number, and then see the connections between the concrete model and the representational, and abstract approaches.

Students might use concrete materials such as fraction bars (in this case, one orange rectangle is identified as one-fourth of the whole) to physically put together three one-fourth pieces. They can illustrate this rectangular representation on paper, and record it symbolically as ¼ + ¼ + ¼ = ¾. Teachers support students in making these connections by asking that they record their thinking in several ways, giving opportunities for discussion and comparison of various representations, and being explicit about how the representations express the same idea.



The teacher poses a question to the class: “What fraction of this square is the blue triangle?”

Jamie and Parker study the square arrangement of four tangram pieces. Jamie says, “The blue triangle is $\frac{1}{4}$, because there are four pieces.” Parker says, “I don’t think that’s $\frac{1}{4}$, but I’m not sure what it is.” As they worked with their tangram pieces, Parker put two of the small triangles together, forming a square. Jamie commented, “The two little triangles make a square just like the purple square. What if we build our own square like this one?” They used tangram pieces to build their own four-piece square. Once they completed building the square, Parker picked up the large triangle, and flipped it over to cover the three smaller pieces (two triangles and square). Jamie exclaimed, “I get it! The big triangle is half of the square, not $\frac{1}{4}$!”

At the beginning stages of fraction work, students need considerable experience exploring various concrete and visual materials in order to build understanding of fractions as equal parts of a whole (3.NF.1,3; ELD I7). It is natural for students, using their understanding of whole numbers, to think that if a whole is split into 4 parts, regardless of whether those parts are of equal size, then each part must be $\frac{1}{4}$ of the whole. Similarly, if students rely on their whole number thinking, they often expect that a unit fraction with a smaller denominator will be less than a unit fraction with a larger denominator, e.g., $\frac{1}{4}$ must be less than $\frac{1}{6}$ (Van de Walle, 2014).

Third- through fifth-grade students explore fractions with concrete tools and develop the more abstract understanding of fractions on the number line (SMP.2, 4, 5; 3.NF.2, 4.NF.2, 3, 4; 5.NF.3, 4, 6). Round fraction pieces are commonly available, and serve well for establishing such ideas *as* $\frac{1}{4}$ is *half of one half*, and $\frac{1}{6}$ is a smaller size fraction piece than $\frac{1}{2}$, and that 3 sixths pieces together make a half-circle equal to $\frac{1}{2}$. Using multiple models for fractions can help to enlarge and solidify concepts. As with other tools used for building mathematical concepts, each fraction manipulative has advantages as well as limitations. While fraction circles are helpful for establishing relative sizes of unit fractions, a number line or fraction bars might be a better choice for finding the sum of $\frac{1}{2}$ and $\frac{1}{3}$.

Other useful manipulatives for fractions include:

- Fraction bars
- Fraction squares or rectangles
- Tangrams
- Pattern block pieces
- Cuisenaire rods
- Paper strips, used for folding halves, fourths, thirds, etc.
- Rulers/meter sticks
- Number lines
- Geoboards

The process of preparing some of their own fraction tools is valuable for young students (Burns, 2001). It increases their understanding of fractions as parts of a whole and supports recognition of the relative sizes of fractional parts. For example, they can create fraction strips from construction paper. As they cut halves, fourths, and eighths of the whole, students discover that $\frac{1}{4}$ is half of $\frac{1}{2}$, and $\frac{1}{8}$ is half of $\frac{1}{4}$, leading to the generalization that when a whole is partitioned into more equal shares, the parts become progressively smaller.

Alternatively, students can fold paper strips to create fractional parts.

Examples **3.NF.1▲**

Teacher: Show fourths by folding the piece of paper into equal parts.

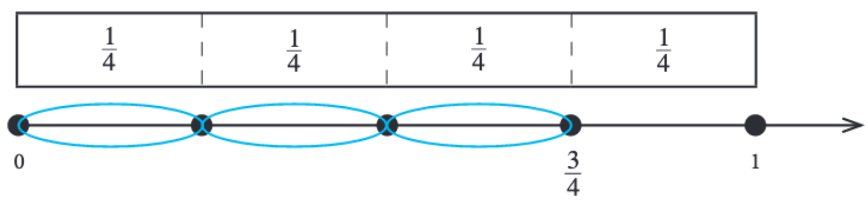
Student: I know that when the number on the bottom is *4*, I need to make four equal parts. By folding the paper in half once and then again, I get four parts, and each part is equal. Each part is worth $\frac{1}{4}$.

Teacher: Shade $\frac{3}{4}$ using the fraction bar you created.

Student: My fraction bar shows fourths. The *3* tells me I need three of them, so I'll shade them. I could have shaded any three of them and I would still have $\frac{3}{4}$.

Teacher: Explain how you know your mark is in the right place. **3.NF.2b▲**

Student (Solution): When I use my fraction strip as a measuring tool, it shows me how to divide the unit interval into four equal parts (since the denominator is *4*). Then I start from the mark that has *0* and measure off three pieces of $\frac{1}{4}$ each. I circled the pieces to show that I marked three of them. This is how I know I have marked $\frac{3}{4}$.



Ordering fractions from least to greatest provides opportunity for students to reason about relative sizes of fractions. Students can determine how to put fractions such as $\frac{5}{3}$, $\frac{2}{5}$, $\frac{5}{4}$., in order from least to greatest, using reasoning along with concrete materials or drawings. They can explain verbally how they know that $\frac{5}{3}$ is greater than $\frac{5}{4}$. "There are five thirds and five fourths, but thirds are bigger pieces than fourths, so $\frac{5}{3}$ is bigger than $\frac{5}{4}$." Benchmark reasoning is also useful here. "I know that $\frac{2}{5}$ is less than one and it's even less than $\frac{1}{2}$. And $\frac{5}{3}$ and $\frac{5}{4}$ are both more than 1. So, $\frac{2}{5}$ Is the smallest."

Comparing and ordering fractions can be challenging for upper elementary students. They need repeated experiences reasoning about fractions and justifying their conclusions using a variety of visual fraction models to develop benchmark reasoning (SMP.1, 2, 4, 5, 7; ELD I6, P9). Students in these grades who rely on their understanding of whole numbers may have particular difficulty recognizing the relationship between the numerator and denominator of a fraction. Frequent, sustained discussion of ideas in both small groups and whole class settings will be necessary. Three students were discussing how to put 1/3, 3/5, and 1/2 in order from least to greatest. Alana is an Emergent EL student with strong problem-solving skills. She is reluctant to share her ideas with the whole class, but is more confident in small group settings. The teacher has paired her with Miriam, who helps Alana practice expressing her ideas in English, and Gus, who often uses visual representations to make sense of mathematics situations.

- Miriam: 1/3 and 3/5 are equal because you just add 2 to 1 (the numerator of 1/3) to get 3 (the denominator of 1/3) and you add 2 to 3 (the numerator of 3/5) to get 5 (the denominator of 3/5). So, they're the same.
- Alana: But wait! That doesn't make sense! 1/3 is less, isn't it? Because 3/5 is more than half and 1/3 is not as big as ½.
- Gus: Let's do it with our fraction pieces.

The children build 1/3, 3/5, and ½ with their fraction pieces. They compare and find that 1/3 is less than ½ and ½ is less than 3/5. The conversation continues.

- Miriam: Why didn't my way work?
- Alana: I think because the thirds pieces are not the same size as the fifths pieces.
- Gus: But we only had 1 third, and there are three 1/5ths, so when you put them together to make 3/5, that's bigger than just one third.
- Alana: Isn't ½ a benchmark fraction? I can tell that 1/3 is less than ½ because when a fraction is the same as ½, the denominator is always two times as big as the numerator. Like, ½, 2/4, 3/6, 4/8 and 5/10.

- Miriam: Oh yeah—I remember we talked about how ½ can have lots of names. But would you tell me again how you know that 3/5 is bigger than 1/3?

Alana explains again, pointing to the fraction pieces. The teacher, observing the conversation, is pleased to note Alana's involvement, and notes that she used the word "benchmark". In several groups, some confusion remains; the teacher decides to conduct a whole-class discussion to develop this idea further.

The grade-four task, *Doubling Numerators and Denominators*, from *Illustrative Mathematics,* https://www.illustrativemathematics.org/, provides opportunity for such reasoning and class discussion of fraction concepts.

The task is based on the following:

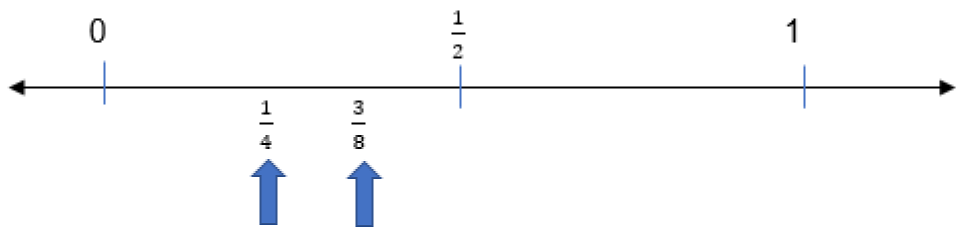
1. How does the value of a fraction change if you double its numerator? Explain your answer.
2. How does the value of a fraction change if you double its denominator? Explain your answer.

As students are developing fraction concepts and beginning to use fractional notation, they need to recognize $\frac{a}{b}$ as a quantity that can be placed on a number line, and that it may be located between two whole numbers, or may be equivalent to a whole number (where $a = b$). Students develop an understanding of order in terms of position on a number line, following the mathematical convention that the fraction to the left is said to be smaller and the fraction to the right is said to be larger.

The use of precise mathematical terms is essential in order to support all students' understanding. $\frac{3}{4}$ is read as "three fourths." Casual language such as "three over four" or "three out of four" (except when discussing ratios or probability situations) undermines fragile understanding of fractions, interferes with academic language acquisition, and may lead to misapplication of whole number reasoning in fraction situations.

The number line reinforces the analogy between fractions and whole numbers. Just as 5 is the point on the number line reached by marking off 5 times the length of the unit interval from 0, so is $\frac{5}{3}$ the point obtained by marking off 5 times the length of a different interval as the basic unit of length, namely the interval from 0 to $\frac{1}{3}$.

Locating fractions on the number line calls for reasoning about relative sizes of fractions and whole numbers (SMP.2, 5, 7). In this context, familiarity and comfort with the use of benchmark fractions is of great value. Where, for example, does $\frac{3}{8}$ belong on the number line pictured here? A student who uses benchmark reasoning can begin by locating $\frac{1}{4}$ midway between 0 and $\frac{1}{2}$, and then place $\frac{3}{8}$ midway between $\frac{1}{4}$ and $\frac{1}{2}$.



In the process of labelling locations on the number line in relation to benchmark numbers such as $\frac{1}{2}$, students expand understanding of equivalence. For example, they find that the location marked $\frac{1}{2}$ coincides with $\frac{2}{4}$. Such observations can lead to powerful insights; students need time to think and talk about fraction ideas. Consider this conversation between two third graders and their teacher:

Desmond: We found one-fourth on the number line, and then is this two-fourths (pointing to $\frac{1}{2}$)?

Teacher: Can that place on the number line be both $\frac{2}{4}$ and $\frac{1}{2}$? Would that make sense?

Ellie: Yes, because $\frac{1}{4}$ is half of $\frac{1}{2}$, like with our fraction pieces! See? It takes 2 of these (pointing to the distance from 0 to $\frac{1}{4}$ on the number line) to get to $\frac{1}{2}$. So, that's $\frac{1}{4}$, then $\frac{2}{4}$, and then that will be $\frac{3}{4}$.

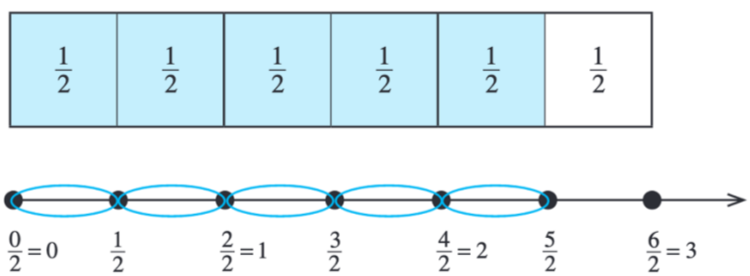
Teacher: What about this place, then? (pointing to 1). How does that fit in here?

Desmond: Four fourths. So, 1 can be 1 whole or it can be four fourths!

Teacher: I wonder what other names can you find for one-half? What if we ask the class to investigate that question?

The CA CCSSM have updated the language describing fractions in which the numerator is greater than the denominator: fractions can be described as *less than one, equal to one, or greater than one.* The term “improper fraction” carries with it the implication that the fraction must be rewritten in another format, such as a mixed number. Fractions greater than one, such as $\frac{5}{2}$, are simply numbers in themselves and are constructed in the same way as other fractions. Further, depending on the context of a problem, re-naming a fraction greater than one as a mixed number may cause a problem to be less readily understood and/or solved.

For example, to construct $\frac{5}{2}$, we might use a fraction strip as a measuring tool to mark off lengths of $\frac{1}{2}$, Then we count five of those halves to get $\frac{5}{2}$.



Important Concepts Related to Understanding Fractions

- Fractional parts must be the same size.
- The number of equal parts tells how many make a whole.
- As the number of equal pieces in the whole increases, the size of the fractional pieces decreases.
- The size of the fractional part is relative to the whole.
- When a shape is divided into equal parts, the denominator represents the number of equal parts in the whole and the numerator of a fraction is the count of the demarcated congruent or equal parts in a whole (e.g., $\frac{3}{4}$ means that there are 3 one-fourths or 3 out of 4 equal parts).
- Common benchmark numbers such as 0, $\frac{1}{2}$, $\frac{3}{4}$, and 1 can be used to determine if an unknown fraction is greater or smaller than a benchmark fraction.

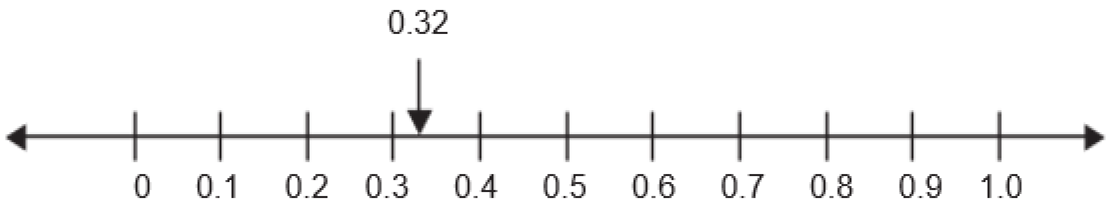
Adapted from ADE 2010 and KATM 2012, 3rd Grade Flipbook.

Understanding decimal notation for fractions, and comparing decimal fractions

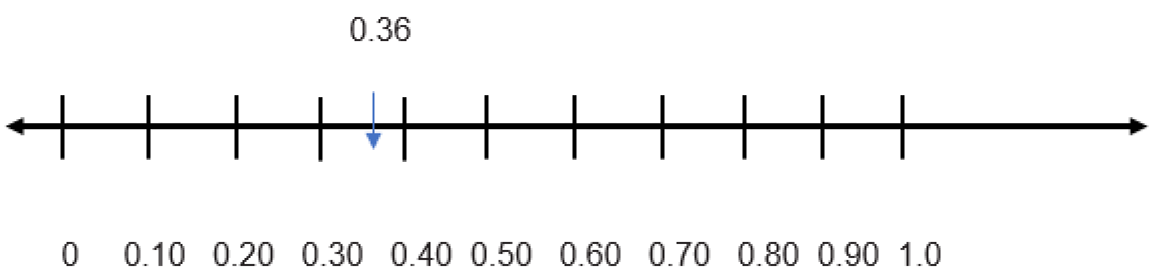
In grade four, students use decimal notation for fractions with denominators 10 or 100 (4.NF.C.6), understanding that the number of digits to the right of the decimal point indicates the number of zeros in the denominator. This lays the foundation for performing operations with decimal numbers in grade five. Students learn to add decimal fractions by converting them to fractions with the same denominator (SMP.2; 4.NF.C.5). For example, students express 3/10 as 30/100 before they add 30/100 + 4/100 = 34/100. Students can use graph paper, base-ten blocks, and other place-value models to explore the relationship between fractions with denominators of 10 and 100 (adapted from UA Progressions Documents 2013a).

Students make connections between fractions with denominators of 10 and 100 and place value. They read and write decimal fractions, and it is important that teachers encourage students to read decimals in ways that support developing understanding (Van de Walle, 2014). When decimals are read using precise language, students learn to write decimals flexibly, e.g., by writing thirty-two hundredths as both 0.32 and 32/100. Conversely, imprecise reading of decimals, such as “0 point 32” rather than as “thirty-two hundredths” undermines sense-making and obscures the connection between fraction and decimal values.

Students represent values such as 0.32 or 32/100 on a number line. They reason that 32/100 is a little more than 30/100 (or 3/10) and less than 40/100 (or 4/10). It is closer to 30/100, so it would be placed on the number line near that value (SMP.2, 4, 5, 7).



Students compare two decimals to hundredths by reasoning about their size (SMP.3, 7; 4.NF.7). They relate their understanding of the place-value system for whole numbers to fractional parts represented as decimals. Students compare decimals using the meaning of a decimal as a fraction, making sure to compare fractions with the same denominator and ensuring that the wholes are the same. For example, if the number 0.36 is located as indicated by the blue arrow, where are the numbers 0.67 and 0.92 located?



In grade three, students begin to develop understanding of benchmark fractions. Fourth grade students extend this understanding to connect familiar benchmark fractions with corresponding decimals.

- The teacher asks the students to write the number “five tenths.” Some write it as a decimal, and others use the fraction form. To help students recognize that 0.5 is equivalent to ½, the teacher calls for students to name the benchmark fraction equal to 5/10, and highlights this connection.
- On a 10 x 10 square grid, students color in 25 small squares to illustrate the decimal 0.25. On a comparable grid, students color ¼ of the whole grid, and

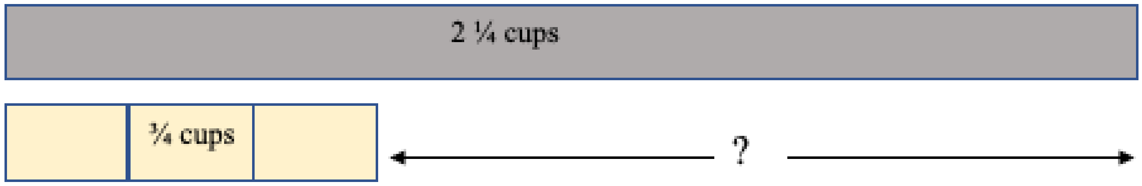
discover that ¼ of the grid is the same number of small squares, 25. They can use this visual model to see that ¼ = 0.25 (Van de Walle, 2014). This exercise can be done with other familiar fractions such as ½, 3/5, or 75/100.

Applying and extending previous understanding of operations to add, subtract, multiply and divide fractions

Students are expected to “apply and extend previous understandings” to operate with fractions. To do so, they must deeply understand the meanings of the four operations and be supported in their efforts to make connections between operations with whole numbers and operations with fractions (SMP.2, 4, 7; 4.NF.B.3, 4; 5.NF.1 – 7). In grades four and five, students begin operating with fractions; the algorithms for operations with decimals are addressed in grade six (6.NS.B.3). In an active learning environment, where students explore, challenge ideas, and make connections among various topics, they experience mathematics as a coherent, understandable body of knowledge and come to expect that previous learning will support their acquisition of new concepts.

A solid understanding of the relationship between addition and subtraction helps a fourth grader solve a problem such as: *The recipe calls for 2¼ cups of rice. Avery already has ¾ cup of rice. How much more rice does Avery need?* While the story problem can be solved using subtraction, the context does not suggest a **take-away** situation. This problem is more logically interpreted as **comparison** subtraction (2¼ – ¾), to find the difference between the quantities or as **missing addend** addition (¾ + ___ = 2¼), with the intention of finding how much more is needed. Students can represent the situation with visual fraction models as they have done in whole number problem situations. The problem can be modeled quite literally, using measuring cups filled with rice (or a substitute for rice, such as sand), or with fraction tools (fraction bars, for example), a number line, or a bar diagram, as shown here. Class conversation paired with written recordings of the various actions, representations, and equations support students in making the necessary connections between the concrete, representational, and abstract expressions of the problem.

The recipe calls for 2 1/4 cups of rice. Avery already has 3/4 cup of rice. How much more rice does Avery need?



The longer bar, labeled 2 1/4 cups, is compared to a shorter bar, representing ¾ cup. The unknown in the problem is represented by the gap between the two lengths.

Intentional, guided class discussion of how these subtraction strategies and illustrations work equally well to solve whole number problems can help students to make necessary connections (SMP.2, 7; 4.NF.B.4, 5.NF.B.6, 7; ELD. Connecting ideas 6).

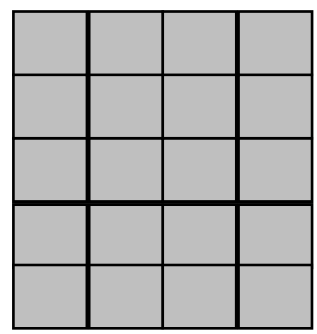
> Teacher: What if the problem involved whole numbers rather than fractions? What if the problem asked instead: The recipe calls for five cups of rice. Avery already has two cup of rice. How much more rice does Avery need? How would you solve it and illustrate it?
>
> Students describe to their partners how the two problems are alike.
>
> Teacher: Would the same approach and a similar diagram work to solve the whole number problem? Show us!
>
> Students respond, sharing the thinking and diagrams they used in each case, and make connections between the two.

More examples can be found among the fourth grade tasks at Illustrative Mathematics, https://www.illustrativemathematics.org/. *Peaches, Plastic Building Blocks*, and *Cynthia's Perfect Punch* are contextual problems that call for students to apply strategies of fraction addition and/or subtraction.

Multiplication of a fraction by a whole number can be seen as parallel to multiplication of whole numbers. This is an opportunity for reflection on whole number strategies and active investigation and discussion of how these strategies apply with fractions. If 5 x 4 is understood as "five groups of four," "a rectangle with dimensions of five meters by four meters," or "five copies of the quantity four," then 5 x ¼ can be understood as "5 groups of ¼," "a rectangle with dimensions of 5 x ¼ meters," or "five copies of the quantity ¼." The strategies and representations used with whole number multiplication—repeated addition, jumps on the number line, or area—can be used with fractions. Tasks and problems presented in contexts that make sense to students make learning accessible, even without direct instruction on “how to multiply fractions.”

Whether the student illustrates with fraction manipulatives (five one-fourth pieces), or perhaps 5 jumps of distance ¼ on a number line, the reasoning is the same as would be used with whole number multiplication (4.NF.B.4).

- The recipe says to bake the pan of cookies for ¼ of an hour. How long will it take to bake five pans of cookies, one pan at a time?
- Dean and Jean ran the ¼ mile track five times. How far did they run?
- At our party, we will give each friend ¼ pound of candy. There will be five friends at the party. How much candy do we need?
- We are painting a line of the playground to mark the start for the runners. The line will be five feet long, and ¼ foot wide. If the paint we have will cover four square feet, will that be enough?



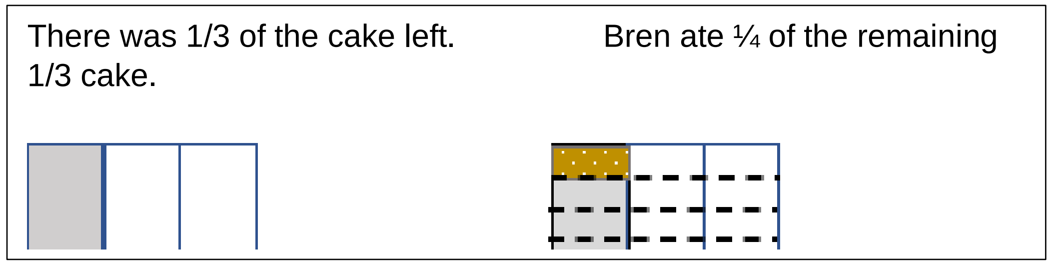
To solve the whole number multiplication 5 x 4, one could use an area interpretation, illustrating the problem with a rectangle of dimensions five units by four units. In the rectangle at right, there are five rows of squares, with four squares in each row, for a total of 20 square units.

Using the same reasoning and a comparable illustration, one can use an area interpretation to solve 5 x ¼. In this example, the rectangle will have a height of five units and a width of ¼ unit. The area of this figure can then be seen as five ¼ unit pieces, or ¼ + ¼ + ¼ + ¼ + ¼, = 5/4.

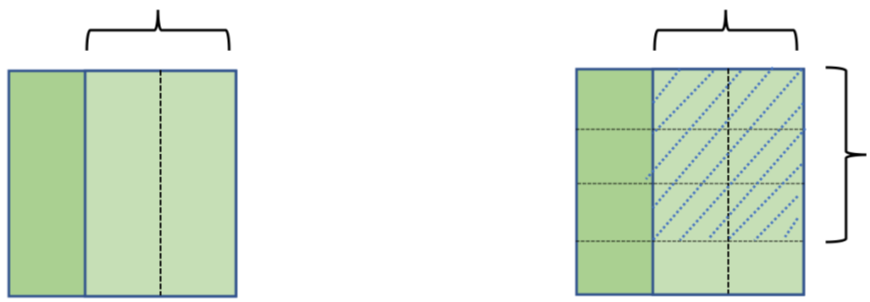
When both factors are fractions less than one, students may expect that multiplication will result in a product that is greater than either factor, as is often the case with whole number multiplication. It can be helpful to remind students that with whole numbers, the product is not always greater than the factors. Multiplying any number (*n)* by one results in a product equal to that number, e.g., 1 x 14 = 14. Students can then reason about how the product of two fractions that are less than one can be less than either of the factors, e.g., ¼ x 2/5 = 2/20 (SMP.1, 6, 7).

Students sometimes lose sight of what is the whole as they multiply fractions. The understanding that we are finding a part of a part of a whole underlies fraction multiplication and requires emphasis and thoughtful discussion. Illustrations can often mitigate the difficulty of making sense of these situations. Again, the illustrations correspond to the ways used for representing whole number multiplication.

- *After the party, there was 1/3 of the cake left. Bren ate ¼ of the remaining 1/3 cake. How much of the whole cake did Bren eat?*

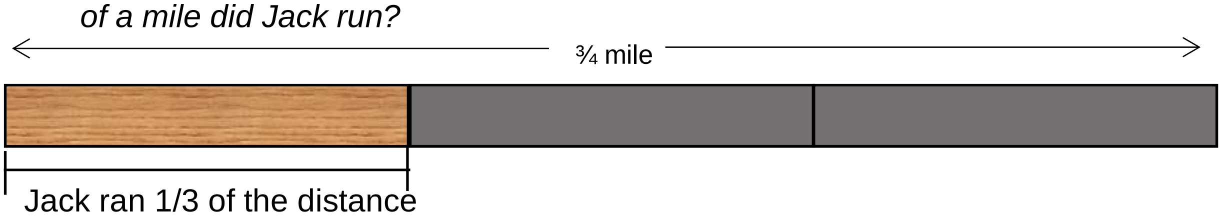


- *Zack had 2/3 of the lawn left to cut. After lunch, he cut ¾ of the grass he had left. How much of the whole lawn did Zack cut after lunch? (Van de Walle, 2014, p. 243)*



- *The milk carton is labelled ½ gallon. If Idalia drank 3/8 of the full carton, what fraction of a gallon did Idalia drink?*

- *Jack ran 1/3 of the distance along the ¾ mile track. What fraction of a mile did Jack run?*



Solidly establishing the meaning of multiplication with fractions is essential in order for students to develop the concept of division with fractions in fifth grade. Identifying how fraction division relates to previous work with whole number division supports students in making sense of the concept of fraction division. The goal in fifth grade is for students to understand what it means to divide with fractions, with applications limited to instances involving a unit fraction and a whole number (5.NF.B.3, 7). This conceptual understanding deserves thoughtful attention to prepare students to continue with proportional relationships in later grades. As with whole-number operations, students who develop and discuss methods that make sense to them as they begin to calculate with fractions will be more capable of applying reasoning in new situations than if they are prematurely taught an algorithm for solving division of fractions problems. The development of algorithms for fraction calculation, such as the common denominator method, is reserved for middle school grades.

Dividing a unit fraction by a whole number, such as 1/3 ÷ 4, can be related to a comparable problem with whole numbers, such as 3 ÷ 4. *If there are three cups of soup to share equally among four people, how much soup will each person have*? A fraction question that calls for the same reasoning: *If there is 1/3 gallon of juice to share equally among four people, how much juice can each person have?*

Fifth graders also divide a whole number by a unit fraction, such as 4 ÷ 1/3. Again, understanding of division with whole numbers and a meaningful context support students in making sense of this problem: *If there are 4 cups of soup, and each serving is 1/3 cup, how many servings of soup are there?*

When a fraction problem is presented in a familiar context, students can illustrate the problem in ways that make sense to them, and solve it using logic and invented strategies. It may not be obvious to the student which operation is involved, and yet the solution is accessible.

Snapshot:

The fifth-grade teacher has selected the *Illustrative Mathematics* grade-five task, “Dividing by One-Half” (illustrativemathematics.org/), as a means for students to grapple with the idea of dividing a whole number by a fraction. Student partners will solve these four fraction problems using their own illustrations and strategies. Then the class will work together to determine which of the four problems can be solved by calculating 3 ÷ ½, and explain how they know.

1. Shauna buys a three-foot-long sandwich for a party. She then cuts the sandwich into pieces, with each piece being 1/2 foot long. How many pieces does she get?
2. Phil makes three quarts of soup for dinner. His family eats half of the soup for dinner. How many quarts of soup does Phil's family eat for dinner?
3. A pirate finds three pounds of gold. In order to protect the riches, they hide the gold in two treasure chests, with an equal amount of gold in each chest. How many pounds of gold are in each chest?
4. Leo used half of a bag of flour to make bread. If Leo used three cups of flour, how many cups were in the bag to start?

Once the students have found solutions, they will discuss with their partners which operation is involved, and write the equation that could be used to calculate the answer. During the class discussion, students will focus on reaching consensus on which of the four problems calls for the division calculation 3 ÷ ½ = 6 and justifying their conclusions.

- Number 1 is easily solved based on an illustration of a three-foot long sandwich. The corresponding calculation is 3 ÷ ½, and the question being asked in this case is," how many ½ foot long pieces of sandwich are there in a 3-foot long sandwich?" This is an example of measurement, or quotitive division.
- Number 2 is a multiplication situation, in which the question calls for finding part of a whole. It can be solved by the calculation ½ x 3 = 1½.
- Number 3 calls for the calculation 3 ÷ 2 = 1½. It is a division problem, but is not solved by dividing 3 by ½.
- Number 4 is another division situation and can be calculated using the equation 3 ÷ ½ or the equation 3 = ½ x [blank]? This can be thought of as partitive division or as a missing factor situation which asks the question, "three cups of flour is half of what amount of flour?"

The teacher will facilitate a whole-class discussion during which students justify their conclusions and find consensus. The expectations include the following:

- Most (if not all) student pairs will solve at least three of the four problems correctly.
- Justifying which operation is used in each case will be challenging.
- Students will disagree about which operation was used in some cases.
- Careful analysis of the meaning of the operations, particularly of division by a fraction, will be necessary; the teacher’s questioning and prompts will play a vital role.

CC 4: Discovering Shape and Space

Second-grade students work in one-dimensional space, using rulers to measure length.

The development of two- and three-dimensional space takes place in grades 3–5. Younger grade students learned to identify common geometric figures and to count the numbers of sides and corners. In grades three through five, students deepen their understanding of the properties of shapes and apply their understanding to organize shapes into categories and analyze hierarchical relationships.

Students explore shape and space in the upper elementary grades as they develop the following:

- Strategies for solving problems involving measurement and conversion of measurements from larger to smaller units (4.MD.A.1; 5.MD.A.1)
- Understanding of concepts of area, perimeter, and volume of solid figures (3.MD.C.6; 4.MD.B.3; 5.MD.C.3, 4, 5)
- Understanding of concepts and measurement of angles; draw and identify lines and angles (4.MD.C.5, 6, 7, 4.G.1, 2).
- Ability to reason with shapes and their attributes; categorize shapes by their properties and recognize the hierarchical relationships among two-dimensional shapes (3.G.1, 2; 4.G.2; 5.G.B.3, 4)

In their work with shapes and space concepts, students use the SMPs to

- think quantitatively and abstractly, connecting visual and concrete models to more abstract and symbolic representations;

- select appropriate tools to model their mathematical thinking;
- communicate their ideas clearly, specifying units of measure accurately; and
- discern patterns and structural commonalities among geometric figures.

Students begin exploration of area concepts by covering rectangles with square tiles and learning that these can be described as **square units**. Two-dimensional measure is a significant advance beyond students' previous experience with linear measure, and deserves reflection and careful instruction. Initially, students count the number of square units used to find the area.

Students can use one-inch square tiles to cover the surface of a book's cover or the surface of their desks. As students work, the teacher looks for organization in their arrangements of the tiles, wondering, "Are they creating rows? Do they start by forming a frame around the edge of the surface?" Based on observation of various approaches, the teacher asks students to share strategies that enabled them to cover the whole surface without leaving any gaps. By posing questions and inviting comparison of results, the teacher can guide students' development of accurate and efficient methods of measuring area*. I see that this group has 6 rows of tiles. How many tiles are in each row? What do we notice about the number of tiles in each row? How can that help us to figure out the area of this rectangle?*

Explorations of area need not be limited to one-inch tiles as the unit of measure. Large squares cut from cardboard or other sturdy materials can be used to measure area of larger areas such as rectangular regions on the playground.

With further tiling experience, students discover that they can multiply the side lengths (the number of rows of tiles x how many tiles are in each row) to find the area more efficiently, and no longer need to count square units singly. They make sense of this by connecting to their prior work with the array model of multiplication. In third grade, students measure only areas of rectangles with whole number length sides as they develop these understandings. They will apply this thinking in grades four and five, when rectangles involve fractional side lengths (SMP.2, 5, 6, 7; 3.OA.A.3; 3.MD.C.5, 6, 7; 4.MD.A.3). Students should understand and be able to explain why multiplying the side lengths of a rectangle yields the same measurement of area as counting the number of tiles (with the same unit length) that fill the rectangle's interior, and to explain that one length tells the number of unit squares in a row and the other length tells how many rows there are (3.MD.C.7; 4.MD.A.3).

Along with developing area concepts, upper elementary students now recognize perimeter as an attribute of plane figures. The concept of perimeter is introduced in grade three, but confusion between the terms area and perimeter is common throughout grades 3–5, a reminder that the distinction between linear and area measurement needs to be explored and emphasized at this stage of learning.

> As students find the perimeter of a 4 x 6 rectangle, one student offers: *“I added 4 + 6 + 4 + 6 (pointing to each of the four sides of the rectangle in turn), and that was 10 + 10, so 20 cm.”* Another student reports, *“I added the sides like this: 4 + 4 = 8 and 6 + 6 = 12, so 8 + 12 = 20 cm.”* A third student explains, “I added 4 + 6 and that was 10, so it’s 2 x 10 = 20 cm.” The teacher displays these examples and asks the class to describe how the methods are alike and how they differ, and whether they will all work for finding the area of other rectangles. In the discussion that follows, the class observes that the methods all use addition to find the perimeter, and one method uses addition and multiplication. The students agree the methods all work because the opposite sides of a rectangle have the same lengths. The teacher draws attention to this idea to highlight the linear nature of perimeter, and invites a student to outline with a colorful pen the perimeter of the rectangle under discussion.

Questions about how we can measure the length of the perimeter (add the four side lengths) versus how we can find the area of the interior of the rectangle (multiply the number of rows by the number of tiles in a row) give students a chance to deepen their understanding of how and why area and perimeter are measured differently, and are identified by different types of units.

The vignette in this chapter, “Alex Builds Rectangles to Find Area,” presents a multi-day lesson incorporating many of the space and measurement concepts developed in grades three through five.

In “Garden Design,” a grade three performance assessment found at Inside Mathematics (https://www.insidemathematics.org/), students find and compare areas of rectilinear figures. The task explores the idea that figures can have different dimensions, yet contain the same area.

Fifth-grade students expand on their understanding of two-dimensional area measurement to develop concepts of volume of solid figures, with a particular focus on the volume of rectangular prisms (5.MD.C.3, 4, 5). Students need concrete experiences building with three-dimensional cubes to reach understanding of the concept and eventually to derive a formula for calculating volume (SMP.2, 4, 6, 7). When students build rectangular prisms from cubes, they find they will make layers of cubes and can recognize how each layer represents the area of the corresponding two-dimensional rectangle.

Fifth-grade students explore the ideas of volume and scaling with a focus on rectangular solids (5.MD.C.3, 4, 5). They might investigate what happens when, for example, we double the length, width, and height of a rectangular solid. They find that the volume increases not by two or by four, but by a factor of eight, since $2 \times 2 \times 2 = 8$. This discovery is often quite surprising to students. Before they get to the point of generalizing this phenomenon, they should think about the effects of scaling the different dimensions by different factors.

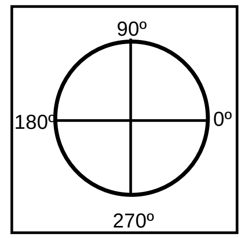
The task “Box of Clay,” at *Illustrative Mathematics* ([http://tasks.illustrativemathematics.org/)](http://tasks.illustrativemathematics.org/), challenges students’ understanding of volume and scaling, as well as whether they recognize how length x width x height can be used to calculate volume (5.MD.C.3, 4, 5).

> *A box 2 centimeters high, 3 centimeters wide, and 5 centimeters long can hold 40 grams of clay. A second box has twice the height, three times the width, and the same length as the first box. How many grams of clay can it hold?*

Tasks such as this help students understand what happens when we scale the dimensions of a right rectangular solid (SMP.2, 5, 7; 5.MD.C.3, 4, 5). In this case, the volume is increased by a factor of 6: the height is doubled, the width is tripled, and the length remains the same (2 x 3 x 1), so the volume of the larger box is 240 grams of clay.

Exploring angles, the space between two rays that have a common endpoint, begins in grade four (4.MD.C.5, 6, 7). Students have had previous experience identifying and counting the corners of plane figures, and often assume that an angle is that point where two line segments join. It is important that students come to understand an angle as some portion of a 360º rotation around the point where two rays meet. Fourth-grade students are expected to sketch and measure angles using a protractor. Students can make their own protractors as a means of deepening understanding of an angle as a measure of rotation around the center of a circle (4.MD.C.6,7; SMP.1, 3, 5, 7).

Snapshot:

> Mr. Flores provides each student or pair of students with a set of fraction circles, a square of cardstock (larger than the diameter of the whole fraction circle), and a straightedge ruler. He directs students to outline the whole fraction circle on the cardstock to create a protractor.
>
> 
>
>
> The students work to align their ½ fraction piece within the circle, and eventually draw a line across to create a diameter. With further observation, Mr. Flores helps them label one end of this diameter as 0º, and the opposite end as 180º. The students then place the right angle of the ¼ fraction piece at the origin, which allows them to find and mark 90º angle. They place a second ¼ fraction piece adjacent to the first (180º is already marked), and a third ¼ fraction piece, which allows the marking of 270º. When they place the final ¼ fraction piece, the full circle is complete, and the marking 360º coincides with the 0º spot.
>
> Students explore with other fraction pieces (1/8, 1/3, 1/12, etc.), figuring and marking as many degree measures as the fraction pieces permit. Once completed, Mr. Flores engages students in an academic conversation to compare their results. He provides them with vocabulary words from the lesson to support the discussion, including how they found any measures that others may not have discovered. Students discuss the use of the protractors as a tool, and demonstrate how they measure angles on various polygons or other available objects, then justify the measurements they identify.

The growth of students' reasoning about geometric shapes across grades three to five is considerable. Along with growth of reasoning in this content area, students also encounter significant new vocabulary. Mathematics instruction should seek to support all students' language facility—including content and language development of students learning English. Graphic displays of terms and properties, choral responses, partner talk, and the use of gestures can be helpful. Manipulative tools such as two- or three-dimensional geometric figures, straws or other straight objects that can be used to construct and compare geometric figures, and technological tools that allow students to illustrate figures with specified properties can all support students as they make sense of the vocabulary involved.

The Understanding Language/Stanford Center for Assessment, Learning, and Equity (SCALE) project at Stanford University (Zweirs, et al., 2017) describes eight specific *Math Language Routines* designed to support and develop students' academic language. These include student-centered routines that are readily implemented in the classroom; one example is "Convince Yourself, a Friend, a Skeptic." This routine calls for students to justify their mathematical argument as a way to

1. satisfy themselves;
2. convince a friend (who asks questions and encourages further verbal or written explanation, or perhaps an illustration); or
3. convince a student skeptic, who will challenge and offer counter-arguments to help refine the argument.

By presenting multiple examples of **regular** and **irregular** figures in various sizes and orientations, we can help students recognize the similarities and differences among properties of geometric figures. Note that "regular" is a word that has one meaning in everyday usage and a distinct, specific meaning as it applies to geometric figures. Multi-meaning terms often present a challenge to EL students and others with learning differences; teachers can provide additional supports and/or time. Thoughtful attention to student partners/groups, non-verbal cues, or verbal prompts (e.g., "You can tell this shape is regular because …") can help a student develop the concept as well as the pertinent academic language.

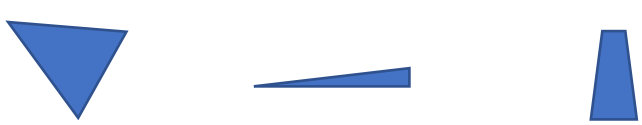
- Third-grade students categorize shapes by **attributes** and recognize that different shapes may share certain attributes. Vocabulary includes: rhombus, rectangle, square, and quadrilateral.

- Fourth-grade students gain familiarity with additional attributes and shape names, including **symmetry, parallel** and **perpendicular** lines, **parallelograms**, and **trapezoids**. They identify angles and specific types of triangles: **acute, obtuse, right, isosceles, equilateral** and **scalene**.
- In fifth grade, a greater degree of analysis is demanded as students describe and diagram the hierarchical relationships of properties among two-dimensional figures. For example, they verify that, based on properties, squares are a **sub-category** of rectangles.

Research on the development of geometric thought describes a progression in the elementary grades from **simple recognition** of how a shape looks through **analysis**, and **informal deduction**. Progress is sequential; a child must work through each level to move to the next higher stage, and experiences rather than age determine when a child is ready to advance (Van de Walle, 2014, p. 246 – 361; Breyfogle and Lynch, 2010). Consequently, instruction at any grade must account for students who are progressing at various rates. Activities that have multiple entry points, call for hands-on, active learning, and invite student discourse enable all students to contribute and to advance their thinking. When justification of conclusions is an expectation in a classroom, students have opportunity to evaluate results and to recognize and to challenge claims that are not sufficiently supported by mathematical reasoning (SMP.3).

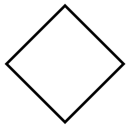
Overgeneralization of geometric ideas often occurs in these grades, as students attempt to integrate the new concepts with previous knowledge. For example, students may come to believe that all rectangles have two longer and two shorter pairs of parallel sides, and thus that squares are not rectangles. Or, that a triangle that is “tilted” is not a triangle (e.g., triangle a, below). Instruction must include examples of geometric figures in many orientations and with unusual dimensions (e.g. triangle b, trapezoid c, below).

(a)



Students need repeated opportunities to examine and discuss examples and non-examples to strengthen a concept.

Possible tasks:

-  *My friend said that this was not a square: Is she right? Why/why not?*
- *Draw an example of a quadrilateral that is a parallelogram and another quadrilateral that is not a parallelogram. Explain why the second one is not a parallelogram.*
- *Cut two paper squares diagonally to create four congruent right triangles. Then, using the 4 triangles, how many different shapes can you make? We will use the rule that touching sides must be the same length. Draw each shape you made, and be ready to share and explain your thinking.*
- *On a page, using a straight edge, draw five lines, no two of which may be parallel. Convince your partner that your drawing matches the requirements* (Sullivan and Lilburn, 2002).
- *I drew a shape with four sides but none of the four sides were the same length. Draw what my shape might have looked like* (Sullivan and Lilburn, 2002, p. 81). *After drawing, plan to compare your shape with your partner's.*
- *A shape is made of two smaller shapes that are the same shape and the same size and that are not rectangles. What might the larger shape look like* (Sullivan and Lilburn, 2002, p. 83)? *Convince your group members that your shape fits the requirements. How many different shapes did your group find? How can we know if others are possible?*

When fifth grade students organize two-dimensional shapes in a hierarchical structure, they are demonstrating the informal deduction stage of growth. At higher grade levels, students move to formal deduction and **rigor**.

Vignette: Alex Builds Rectangles to Find Area

Alex's third-grade class is building understanding of the operations of multiplication and division and concepts of perimeter and area. His teacher plans a two- to three-day lesson, knowing that these are pivotal concepts and that integrating multiple concepts in a meaningful context is more effective than addressing a single concept in isolation. Alex, like many of his classmates, responds with excitement, is actively engaged, and retains learning well when their classroom tasks involve using math tools and allow students to approach problems in a variety of ways. For Alex, working with an instructional aide is an additional tool to support his full participation in these activities.

The teacher has chosen a task that addresses third grade measurement and area content while simultaneously calling on skills of multiplication and division. To conclude the lesson, each student will compose a paragraph explaining their reasoning.

Alex and his instructional aide, listen as his teacher, Ms. B, describes what the class will be doing.

> "Our challenge is to find all the ways to make a rectangle with a loop of string that is 36 inches long. Then we will make some decisions about what these rectangles could be used for, and which would be the best choices."

Ms. B asks the students to think about what that would look like, and what part of the rectangles the string would represent. She draws a rectangle on the board, and tells students to think about the line she draws as if it were the string. After a few seconds, Ms. B asks children to talk to partners about what part of the rectangle the string represents.

As the students discuss with their partners, Alex and his instructional aide discuss a few ideas in preparation for the whole-class discussion: *it's the outside of the rectangle; it's the edge; it's like a fence or maybe a wall.* The aide nudges Alex to record his thinking and rehearse his contribution to the upcoming discussion.

Ms. B opens the floor to the whole class and listens as children talk, and records their ideas, including Alex's. The list includes *edge, side, outside, fence, area, perimeter, line*. In a short discussion, in which Ms. B reminds the students of their previous lesson about what they called the "outside" of a polygon, the class agrees that "perimeter" is the word that fits best, and that the class will be making rectangles with a perimeter of 36 inches (SMP.3, 6; 3; 3.MD.D.8). Ms. B notes that the word "area" appeared in the list, and asks students to recall what they have previously learned about area. Ms. B reminds the class that they may find it useful to refer to the math wall (a large space on the wall where the class has posted definitions, drawings, and counter-examples of the shapes they have studied so far this year) in the classroom. During the lesson, Alex's aide supports his shifts of attention to the word "area," to the math wall, and so on.

After a brief discussion, Ms. B tells the students that after they explore, finding rectangles with a perimeter of 36 inches, they will talk more about area.

Ms. B continues, posting directions:

1. Arrange the string to form rectangles along the grid lines on your paper.
2. Draw each rectangle on the grid paper, recording length and width in inches along the sides (SMP.2, 5, 6; 3.MD.B.4).
3. Talk with your group about how you know you have found all the possible rectangles (SMP.3, 6; 3.G.1).
4. Bring your ideas to the class when we gather to share.

Ms. B supplies each group with a large sheet of one-inch grid paper, rulers, and a string loop. Children gather paper, pencils, and markers they will use to record the rectangles they make and move to their work spaces.

Alex wonders whether it is possible to make many different rectangles (how many?) with the same string, and whether they will all have the same area. When he joins his partners, he immediately picks up the string and tries to make a rectangle on the grid paper. Alex's aide joins the group and supports their interactions by asking peers to repeat what others have said, and making sure that Alex both listens and is heard. When Alex tries to form the corners, he cannot hold the string still, so he asks a teammate for help. The group decides on a plan: each person will make one rectangle with a helper, and then they will pass the string to the next person so each person gets to build some of the rectangles. Another team member will draw the rectangle and record its dimensions on the grid paper.

Alex tries again to form a rectangle that is 4 inches wide. His partner helps by holding the string still at two corners while Alex stretches the string to find that it makes a length of 14 inches. The team works together to draw this first rectangle, and they write down the dimensions.

Work proceeds until the group is satisfied they have found all the possible rectangles.

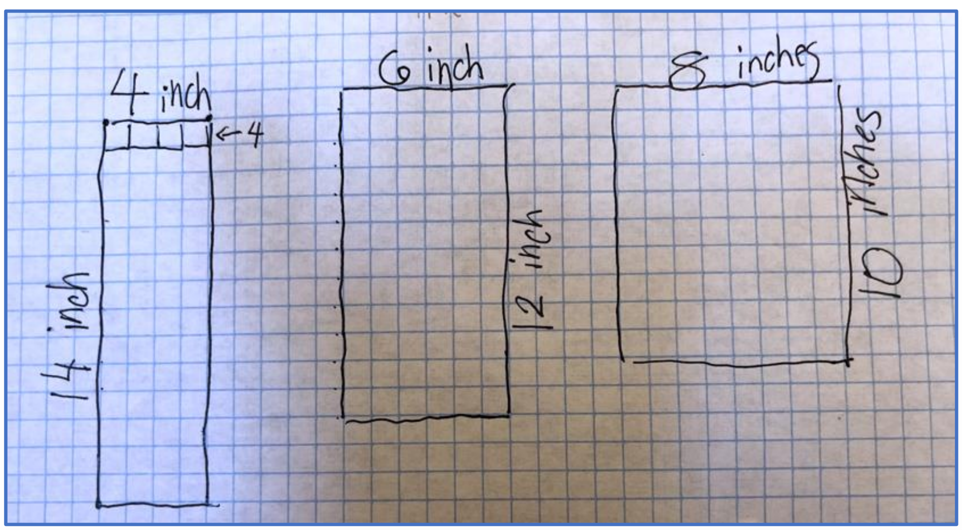
After the students have worked to find all the rectangles, Ms. B calls for attention. She tells the class they get to continue the investigation, and directs them to:

- work with your group to find the *area* of each rectangle you found; record the area for each rectangle on your drawing (SMP.2,6; 3.MD.C.5, 6)

- talk with your group about what each rectangle could represent in the world and be ready to share with the class (SMP.2,3; ELD P 10,11,12).

Ms. B circulates as groups find the areas of the rectangles. She notes strategies students use: some count single unit squares, others count how many rows there are in the figure (e.g., four square inches in each row), and count by fours to find the total number of square inches. A few students make multiplication connections, such as "Well, there are four in each row and there are 14 rows, so isn't that like a multiplication problem?" She hears a student say the area is like an **array**. Some students have a discussion about whether they should count the 9 x 9 square they have drawn; they are debating whether a square is also a rectangle. Several students express surprise that there were so many rectangles possible and they all have the same perimeter, but not the same area.

Ms. B reminds students to think and talk to each other about what each shape of rectangle might represent in the real world, and to get ready to share their discoveries and ideas. As she circulates, Ms. B encourages partners to practice out loud with each other what they will say to the class. She is particularly attentive language development, pausing a few minutes to support all students, including ELs, in their efforts to express their thinking. During this final group work period, she identifies a few groups' posters that represent different approaches and/or organizational methods; she will invite students to present these samples to initiate the class discussion.



Alex is excited that Ms. B asked his group to share their poster and how they found the areas of their rectangles. He and his team explain how they found each rectangle and report the areas, which they found by counting by 1s, 2s, 3s, up to 9s (the lengths of the rows they made).

Another team shares their thinking; they figured out they could find areas by multiplying. A rectangle of width 1 inch had a length of 17, and there were 17 square inches in that area. They noticed that 1 x 17 = 17, and that meant they could multiply to find the area.

A lively discussion develops regarding whether the 9 x 9 inch square should be included in the list of rectangles. Ms. B asks teams to review their knowledge of what makes a rectangle, a topic they had discussed previously. The points included the following:

- Rectangles have four sides.
- Rectangles include square corners.
- Rectangles have two sides across from each other that are the same lengths.

Casey agrees, but says to include that rectangles have to have two long sides and two short sides. Sumira challenges: “Why do there have to be long sides and short sides? I thought when we talked before we said all the sides could be the same, like in a square.” Alex walks to the math wall, and reviews the pictures and descriptions of “rectangle” and “square” posted. He comes back, and excitedly tells Sumira he agrees with her. With a few more minutes of discussion, the class comes to agreement and includes the 9 x 9 inch square rectangle in the list of nine possible rectangles with whole number length sides, and a perimeter of 36.

Ms. B focuses attention on the questions of which rectangle has the greatest area, and which of the rectangles would be most useful at school, at home, or in the community, and why.

Students talk a few moments about whether a “long, skinny” or a “shorter, wider” rectangle is better. When the class discussion resumes, Alex comments that the 1 x 17 rectangle is so long and skinny it would not be useful for many things, and wider ones are probably better for most things. Another student says that some of the rectangles look like they are the shape of a book or a door. Others describe how various rectangles could be the shape of a playground, a pool, a garden, or a sandbox. A number of students claim the rectangles that have the largest areas (the 8 x 10 rectangle and the 9 x 9 square rectangle), would be the “best” for most things.

Ms. B introduces the plan for students to write in their journals: they will explain why there are so many different rectangles that have the same perimeter, describe how they could use one of the rectangles to represent something real (dog run, pool, garden, etc.), and explain why they made that choice. Having already decided that a pool would be the perfect way to use a rectangle; Alex explains the choice and illustrates a sunny day, blue sky, and a “long, medium-skinny” pool in the journal.

Transition from Grades TK–5 to Grades 6–8

Similarly to this chapter, Chapter 7: Mathematics for Understanding, Grades 6–8 is organized around the same four Content Connections:

1. (CC 1) Communicating Stories with Data
2. (CC 2) Exploring Changing Quantities
3. (CC 3) Taking Wholes Apart, Putting Parts Together
4. (CC 4) Discovering Shape and Space

The preparation in younger grades is essential for students’ continued development in mathematics in every area of instruction in grades 6–8.

How does learning in grades TK–5 lead to success in grades 6–8 when students communicate stories told by data?

In the TK–5 years, students gather, represent, and interpret data. Engagement and understanding are enhanced when the question under investigation is of interest and relevance to the students. The ability to analyze data developed in the elementary years is essential to students in grades 6–8 as they focus on the importance of data as the source of most mathematical situations that students will encounter in their lives.

How does learning in grades TK–5 lead to success in grades 6–8 when students are exploring changing quantities?

Students in grades 6–8 extend their understanding of number types to the set of rational numbers, which includes whole numbers, integers, fractions and decimals. They make connections among ratios, rates, and percentages, and use proportional reasoning to solve authentic problems. Whole number foundations are established in the primary grades, and fraction and decimal ideas are key elements of grades 3–5. In grades 6–8, students deepen their understanding of fractions, especially division of fractions. When this concept is introduced with meaning in grade five, it enables students to succeed in later work.

Students in grades 6–8 work extensively with expressions and equations, and solve multi-step problems. This new content relies heavily on foundations developed in the youngest grades. Understanding of equality is evident when a kindergartener compares quantities of objects; a first or second grade student expresses a statement of equality with objects, verbally, or symbolically; a third, fourth, or fifth grade student finds and recognizes equivalent fractions or explains equivalence between a decimal and fractional value.

How does learning in grades TK–5 lead to success in grades 6–8 when students are taking numbers apart, putting parts together, representing thinking, and using strategies?

Throughout grades TK–5, emphasis is placed on students using objects and drawings to illustrate their ways of solving problems, describing their strategies verbally, and developing written methods that make sense within the context of a particular problem. Connections among various representations are an important feature of mathematical discourse, whether this occurs in a small group or a whole class setting.

In grades six through eight, students build their ability and inclination to see connections between representations, and to base strategies on different representations in order to gain insight into problem situations. Their efforts to make connections in younger grades will support students as they build representations for, understanding of, and facility in working with ratios, proportions, and percents, and for the new category of rational number.

How does learning in grades TK–5 lead to success in grades 6–8 when students are discovering shape and space?

Developing mathematical tools to explore and understand the physical world should continue to motivate explorations in shape and space. As in other areas, maintaining connection to concrete situations and authentic questions is crucial.

In grades TK–5, students use basic shapes and spatial reasoning to model objects in their environment to establish many foundational notions of two- and three-dimensional geometry. They develop concepts of area perimeter, angle measure, and volume. Shape and space work in grades 6–8 is largely about connecting these notions to each other, to students' lives, and to other areas of mathematics.

Developing mathematics for true understanding in grades TK–5 is pivotal. Students who experience meaningful mathematics that makes sense to them during the elementary grades will be well-prepared to increase their mathematical understanding as they advance through middle school and high school.

Conclusion

This chapter envisions investigating and connecting the big ideas of mathematics in grades TK–5 as a vibrant, interactive, student-centered endeavor. In an environment rich with opportunities for discourse and meaningful mathematics activities, curiosity and reasoning skills are nourished, and students see themselves as thinkers and doers of mathematics. Careful discussions of mathematical ideas supports all learners, particularly students who are English learners, as they acquire the language of mathematics. Children experience enormous growth in maturity, reasoning, and conceptual understanding in the span of years from transitional kindergarten through fifth grade. Students who enter grade six viewing themselves as mathematically capable and who have gained an understanding of elementary mathematics are positioned for success in the middle school years. They will be empowered to make choices that will affect all their future mathematics, throughout their school years and beyond.

Critical Areas for Instruction and Overview for Grades TK–5

Kindergarten Introduction

In kindergarten, instructional time should focus on two critical areas: (1) representing, relating, and operating on whole numbers, initially with sets of objects; and (2) describing shapes and space. More learning time in kindergarten should be devoted to number than to other topics.

(1) Students use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set; counting out a given number of objects; comparing sets or numerals; and modeling simple joining and separating situations with sets of objects, or eventually with equations such as 5 + 2 = 7

and 7 – 2 = 5. (Kindergarten students should see addition and subtraction equations, and student writing of equations in kindergarten is encouraged, but it is not required.) Students choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the cardinalities of small sets of objects, counting and producing sets of given sizes, counting the number of objects in combined sets, or counting the number of objects that remain in a set after some are taken away.

(2) Students describe their physical world using geometric ideas (e.g., shape, orientation, spatial relations) and vocabulary. They identify, name, and describe basic two-dimensional shapes, such as squares, triangles, circles, rectangles, and hexagons, presented in a variety of ways (e.g., with different sizes and orientations), as well as three-dimensional shapes such as cubes, cones, cylinders, and spheres. They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes.

Kindergarten Overview

Counting and Cardinality • Know number names and the count sequence. • Count to tell the number of objects. • Compare numbers. **Operations and Algebraic Thinking** • Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from. **Number and Operations in Base Ten** • Work with numbers 11–19 to gain foundations for place value. **Measurement and Data** • Describe and compare measurable attributes. • Classify objects and count the	**Mathematical Practices** 1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 5. Use appropriate tools strategically. 6. Attend to precision. 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.

number of objects in categories. **Geometry** • Identify and describe shapes. • Analyze, compare, create, and compose shapes.	

Grade 1 Introduction

In grade 1, instructional time should focus on four critical areas: (1) developing understanding of addition, subtraction, and strategies for addition and subtraction within 20; (2) developing understanding of whole number relationships and place value, including grouping in tens and ones; (3) developing understanding of linear measurement and measuring lengths as iterating length units; and (4) reasoning about attributes of, and composing and decomposing geometric shapes.

(1) Students develop strategies for adding and subtracting whole numbers based on their prior work with small numbers. They use a variety of models, including discrete objects and length-based models (e.g., cubes connected to form lengths), to model add-to, take-from, put-together, take-apart, and compare situations to develop meaning for the operations of addition and subtraction, and to develop strategies to solve arithmetic problems with these operations. Students understand connections between counting and addition and subtraction (e.g., adding two is the same as counting on two). They use properties of addition to add whole numbers and to create and use increasingly sophisticated strategies based on these properties (e.g., “making tens”) to solve addition and subtraction problems within 20. By comparing a variety of solution strategies, children build their understanding of the relationship between addition and subtraction.

(2) Students develop, discuss, and use efficient, accurate, and generalizable methods to add within 100 and subtract multiples of 10. They compare whole numbers (at least to 100) to develop understanding of and solve problems involving their relative sizes. They think of whole numbers between 10 and 100 in terms of tens and ones (especially recognizing the numbers 11 to 19 as composed of a ten and some ones). Through activities that build number sense, they understand the order of the counting numbers and their relative magnitudes.

(3) Students develop an understanding of the meaning and processes of measurement, including underlying concepts such as iterating (the mental activity of building up the length of an object with equal-sized units) and the transitivity principle for indirect measurement.[4]

(4) Students compose and decompose plane or solid figures (e.g., put two triangles together to make a quadrilateral) and build understanding of part-whole relationships as well as the properties of the original and composite shapes. As they combine shapes, they recognize them from different perspectives and orientations, describe their geometric attributes, and determine how they are alike and different, to develop the background for measurement and for initial understandings of properties such as congruence and symmetry.

Grade 1 Overview

Operations and Algebraic Thinking • Represent and solve problems involving addition and subtraction • Understand and apply properties of operations and the relationship between addition and subtraction. • Add and subtract within 20. • Work with addition and subtraction equations. **Number and Operations in Base Ten** • Extend the counting sequence. • Understand place value. • Use place value understanding and properties of operations to add and subtract. **Measurement and Data** • Measure lengths indirectly and by	**Mathematical Practices** 1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 5. Use appropriate tools strategically. 6. Attend to precision. 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.

[4] Students should apply the principle of transitivity of measurement to make indirect comparisons, but they need not use this technical term.

iterating length units. • Tell and write time. • Represent and interpret data. **Geometry** • Reason with shapes and their attributes.	

Grade 2 Introduction

In grade 2, instructional time should focus on four critical areas: (1) extending understanding of base-ten notation; (2) building fluency with addition and subtraction; (3) using standard units of measure; and (4) describing and analyzing shapes.

(1) Students extend their understanding of the base-ten system. This includes ideas of counting in fives, tens, and multiples of hundreds, tens, and ones, as well as number relationships involving these units, including comparing. Students understand multi-digit numbers (up to 1000) written in base-ten notation, recognizing that the digits in each place represent amounts of thousands, hundreds, tens, or ones (e.g., 853 is 8 hundreds + 5 tens + 3 ones).

(2) Students use their understanding of addition to develop fluency with addition and subtraction within 100. They solve problems within 1000 by applying their understanding of models for addition and subtraction, and they develop, discuss, and use efficient, accurate, and generalizable methods to compute sums and differences of whole numbers in base-ten notation, using their understanding of place value and the properties of operations. They select and accurately apply methods that are appropriate for the context and the numbers involved to mentally calculate sums and differences for numbers with only tens or only hundreds.

(3) Students recognize the need for standard units of measure (centimeter and inch) and they use rulers and other measurement tools with the understanding that linear measure involves an iteration of units. They recognize that the smaller the unit, the more iterations they need to cover a given length.

(4) Students describe and analyze shapes by examining their sides and angles. Students investigate, describe, and reason about decomposing and combining shapes

to make other shapes. Through building, drawing, and analyzing two- and three-dimensional shapes, students develop a foundation for understanding area, volume, congruence, similarity, and symmetry in later grades.

Grade 2 Overview

Operations and Algebraic Thinking • Represent and solve problems involving addition and subtraction. • Add and subtract within 20. • Work with equal groups of objects to gain foundations for multiplication. **Number and Operations in Base Ten** • Understand place value. • Use place value understanding and properties of operations to add and subtract. **Measurement and Data** • Measure and estimate lengths in standard units. • Relate addition and subtraction to length. • Work with time and money. • Represent and interpret data. **Geometry** • Reason with shapes and their attributes.	**Mathematical Practices** 1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 5. Use appropriate tools strategically. 6. Attend to precision. 7. Look for and make use of structure 8. Look for and express regularity in repeated reasoning.

Grade 3 Introduction

In grade 3, instructional time should focus on four critical areas: (1) developing understanding of multiplication and division and strategies for multiplication and division within 100; (2) developing understanding of fractions, especially unit fractions (fractions with numerator 1); (3) developing understanding of the structure of rectangular arrays and of area; and (4) describing and analyzing two-dimensional shapes.

(1) Students develop an understanding of the meanings of multiplication and division of whole numbers through activities and problems involving equal-sized groups, arrays, and area models; multiplication is finding an unknown product, and division is finding an unknown factor in these situations. For equal-sized group situations, division can require finding the unknown number of groups or the unknown group size. Students use properties of operations to calculate products of whole numbers, using increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving single-digit factors. By comparing a variety of solution strategies, students learn the relationship between multiplication and division.

(2) Students develop an understanding of fractions, beginning with unit fractions. Students view fractions in general as being built out of unit fractions, and they use fractions along with visual fraction models to represent parts of a whole. Students understand that the size of a fractional part is relative to the size of the whole. For example, 1/2 of the paint in a small bucket could be less paint than 1/3 of the paint in a larger bucket, but 1/3 of a ribbon is longer than 1/5 of the same ribbon because when the ribbon is divided into 3 equal parts, the parts are longer than when the ribbon is divided into 5 equal parts. Students are able to use fractions to represent numbers equal to, less than, and greater than one. They solve problems that involve comparing fractions by using visual fraction models and strategies based on noticing equal numerators or denominators.

(3) Students recognize area as an attribute of two-dimensional regions. They measure the area of a shape by finding the total number of same-size units of area required to cover the shape without gaps or overlaps, a square with sides of unit length being the standard unit for measuring area. Students understand that rectangular arrays can be decomposed into identical rows or into identical columns. By decomposing rectangles into rectangular arrays of squares, students connect area to multiplication, and justify using multiplication to determine the area of a rectangle.

(4) Students describe, analyze, and compare properties of two-dimensional shapes. They compare and classify shapes by their sides and angles, and connect these with

definitions of shapes. Students also relate their fraction work to geometry by expressing the area of part of a shape as a unit fraction of the whole.

Grade 3 Overview

Operations and Algebraic Thinking	Mathematical Practices
• Represent and solve problems involving multiplication and division. • Understand properties of multiplication and the relationship between multiplication and division. • Multiply and divide within 100. • Solve problems involving the four operations, and identify and explain patterns in arithmetic. **Number and Operations in Base Ten** • Use place value understanding and properties of operations to perform multi-digit arithmetic. **Number and Operations—Fractions** • Develop understanding of fractions as numbers. **Measurement and Data** • Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects. • Represent and interpret data. • Geometric measurement: understand concepts of area and	1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 5. Use appropriate tools strategically. 6. Attend to precision. 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.

relate area to multiplication and to addition. • Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures. **Geometry** • Reason with shapes and their attributes.	

Grade 4 Introduction

In grade 4, instructional time should focus on three critical areas: (1) developing understanding and fluency with multi-digit multiplication, and developing understanding of dividing to find quotients involving multi-digit dividends; (2) developing an understanding of fraction equivalence, addition and subtraction of fractions with like denominators, and multiplication of fractions by whole numbers; (3) understanding that geometric figures can be analyzed and classified based on their properties, such as having parallel sides, perpendicular sides, particular angle measures, and symmetry.

(1) Students generalize their understanding of place value to 1,000,000, understanding the relative sizes of numbers in each place. They apply their understanding of models for multiplication (equal-sized groups, arrays, area models), place value, and properties of operations, in particular the distributive property, as they develop, discuss, and use efficient, accurate, and generalizable methods to compute products of multi-digit whole numbers. Depending on the numbers and the context, they select and accurately apply appropriate methods to estimate or mentally calculate products. They develop fluency with efficient procedures for multiplying whole numbers; understand and explain why the procedures work based on place value and properties of operations; and use them to solve problems. Students apply their understanding of models for division, place value, properties of operations, and the relationship of division to multiplication as they develop, discuss, and use efficient, accurate, and generalizable procedures to find quotients involving multi-digit dividends. They select and accurately apply appropriate

methods to estimate and mentally calculate quotients, and interpret remainders based upon the context.

(2) Students develop understanding of fraction equivalence and operations with fractions. They recognize that two different fractions can be equal (e.g., 15/9 = 5/3), and they develop methods for generating and recognizing equivalent fractions. Students extend previous understandings about how fractions are built from unit fractions, composing fractions from unit fractions, decomposing fractions into unit fractions, and using the meaning of fractions and the meaning of multiplication to multiply a fraction by a whole number.

(3) Students describe, analyze, compare, and classify two-dimensional shapes. Through building, drawing, and analyzing two-dimensional shapes, students deepen their understanding of properties of two-dimensional objects and the use of them to solve problems involving symmetry.

Grade 4 Overview

Operations and Algebraic Thinking	**Mathematical Practices**
• Use the four operations with whole numbers to solve problems. • Gain familiarity with factors and multiples. • Generate and analyze patterns. Number and Operations in Base Ten • Generalize place value understanding for multi-digit whole numbers. • Use place value understanding and properties of operations to perform multi-digit arithmetic. **Number and Operations—Fractions** • Extend understanding of fraction equivalence and ordering.	1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 5. Use appropriate tools strategically. 6. Attend to precision. 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning

• Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers. • Understand decimal notation for fractions, and compare decimal fractions. **Measurement and Data** • Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit. • Represent and interpret data. • Geometric measurement: understand concepts of angle and measure angles. **Geometry** • Draw and identify lines and angles, and classify shapes by properties of their lines and angles.	

Grade 5 Introduction

In grade 5, instructional time should focus on three critical areas: (1) developing fluency with addition and subtraction of fractions, and developing understanding of the multiplication of fractions and of division of fractions in limited cases (unit fractions divided by whole numbers and whole numbers divided by unit fractions); (2) extending division to two-digit divisors, integrating decimal fractions into the place value system and developing understanding of operations with decimals to hundredths, and developing fluency with whole number and decimal operations; and (3) developing understanding of volume.

(1) Students apply their understanding of fractions and fraction models to represent the addition and subtraction of fractions with unlike denominators as equivalent calculations

with like denominators. They develop fluency in calculating sums and differences of fractions, and make reasonable estimates of them. Students also use the meaning of fractions, of multiplication and division, and the relationship between multiplication and division to understand and explain why the procedures for multiplying and dividing fractions make sense. (Note: this is limited to the case of dividing unit fractions by whole numbers and whole numbers by unit fractions.)

(2) Students develop understanding of why division procedures work based on the meaning of base-ten numerals and properties of operations. They finalize fluency with multi-digit addition, subtraction, multiplication, and division. They apply their understandings of models for decimals, decimal notation, and properties of operations to add and subtract decimals to hundredths. They develop fluency in these computations, and make reasonable estimates of their results. Students use the relationship between decimals and fractions, as well as the relationship between finite decimals and whole numbers (i.e., a finite decimal multiplied by an appropriate power of 10 is a whole number), to understand and explain why the procedures for multiplying and dividing finite decimals make sense. They compute products and quotients of decimals to hundredths efficiently and accurately.

(3) Students recognize volume as an attribute of three-dimensional space. They understand that volume can be measured by finding the total number of same-size units of volume required to fill the space without gaps or overlaps. They understand that a 1-unit by 1-unit by 1-unit cube is the standard unit for measuring volume. They select appropriate units, strategies, and tools for solving problems that involve estimating and measuring volume. They decompose three-dimensional shapes and find volumes of right rectangular prisms by viewing them as decomposed into layers of arrays of cubes. They measure necessary attributes of shapes in order to determine volumes to solve real-world and mathematical problems.

Grade 5 Overview

Operations and Algebraic Thinking	**Mathematical Practices**
• Write and interpret numerical expressions. • Analyze patterns and relationships.	1. Make sense of problems and persevere in solving them.

Number and Operations in Base Ten • Understand the place value system. • Perform operations with multi-digit whole numbers and with decimals to hundredths. **Number and Operations—Fractions** • Use equivalent fractions as a strategy to add and subtract fractions. • Apply and extend previous understandings of multiplication and division to multiply and divide fractions. **Measurement and Data** • Convert like measurement units within a given measurement system. • Represent and interpret data. • Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition. **Geometry** • Graph points on the coordinate plane to solve real-world and mathematical problems. • Classify two-dimensional figures into categories based on their properties.	2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 5. Use appropriate tools strategically. 6. Attend to precision. 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.

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