Chapter 4: Exploring, Discovering, and Reasoning With and About Mathematics

Contents

Note to reader: The use of the non-binary, singular pronouns *they*, *them*, *their*, *theirs*, *themself*, and *themselves* in this framework is intentional.

Introduction: Mathematical Practices

California schools must prepare students to be powerful users of mathematics to understand and affect their worlds, in whatever life path they embark upon. This charge is built on the California Common Core State Standards for Mathematics (CA CCSSM), which contain two types of standards. The content standards might be more familiar to many educators; they describe for each grade the mathematical expertise, skills, and knowledge that students should develop. The criteria to teach and measure math practices, the Standards for Mathematical Practice (SMPs), describe the ways of interacting with mathematics individually and collaboratively that make up the practices of the discipline. Eight SMPs are included in the CA CCSSM.

Habits of Mind and Habits of Interaction

The past several decades in mathematics education has included a national push to focus on both the habits of mind and habits of interaction that students need in order to become powerful users of mathematics and better interpret and understand their world. Habits of mind include making or using mathematical representations, attending to mathematical structure, persevering in solving problems, and reasoning. Reasoning includes the following processes: inferencing, conjecturing, generalizing, exemplifying, proving, arguing, and convincing (Jeannotte & Kieran, 2017). Habits of interaction include such things as explaining one’s thinking, justifying a solution, making sense of the thinking of others, and raising worthy questions for discussion. Both kinds of habits are fundamentally tied to language development and linguistic processes. Supporting reasoning processes and kinds of interactions involve supporting the development of language as students engage in these disciplinary practices. By the time California's students graduate from high school, they should be comfortable engaging in many mathematical practices, including those that are central to the SMPs highlighted in this chapter: exploration, discovery, description, explanation, generalization, and justification (including proof).

The capacity to use mathematics to understand the world influences every aspect of life, from participating in our communities to personal finances to everyday tasks such as cooking and gardening. For example, an understanding of fractions, ratios, and percentages is crucial to questions of fairness and justice in areas as diverse as incarceration, environmental and racial justice, and housing policy.

Being able to reason with and about the mathematics behind situations such as the above (using ideas such as recursion, shape of curves, and rate of change) empowers Californians in making important and consequential decisions not only for their own lives, but also for their communities. Making sense of the mathematics behind data-based claims about the benefits or dangers of particular foods or other substances empowers everyday decision making. This practice of reasoning about the world using data, described in the Data Science chapter, is another important example.

The ability to reason is a foundational skill for understanding the impact of stereotypes. Humans are quick to generalize from a small number of examples, and to construct causal stories to explain observed phenomena. In many situations, this tendency serves us well: people learn from very few examples that a stove might be painfully hot, and a Copernican model of a sun-centered universe enabled astronomers to predict the movement in the sky of planets and stars with reasonable accuracy.

There are, however, many situations in which humans are poorly served by such generalizations, especially those that lead to the treatment of people based on characteristics that call forth internal stories about expected capacities, motivation, behavior, or background. Such emotional stories are often based on little evidence and are socially buttressed, and action based on these stories does great harm to the communities and the individual students that comprise the schools they represent. This tendency to assume, without adequate justification, that generalizations are valid is reinforced by many poorly-constructed math assessment questions, e.g., “What is the next term in this sequence: 1, 2, 4, 8, …?” instead of the more informative and reasoning-reinforcing “What rule or pattern might generate a sequence that begins 1, 2, 4, 8, ...? According to your rule, what is the next term?” Mathematics education must prepare students to use mathematics to comprehend and respond to their world, deepening their understanding of mathematics and of the issues that impact their lives. The goal is that students learn to “use mathematics to examine… various phenomena both in one’s immediate life and in the broader social world and to identify relationships and make connections between them” (Gutstein, 2003, p. 45).

Deeper Practice or More Content Topics?

Mastering high-school level mathematics content to acquire the knowledge needed to understand the world can embolden students who will continue on to tertiary institutions where they will be expected to engage in career- and college-level mathematics. Despite this, there is a well-documented, persistent disconnect between high school mathematics teachers’ beliefs about what is important for their students to succeed in college, and what college instructors rate as most important for incoming students’ success. Even with the adopted the CA CCSSM, ongoing research into instructional practices, and annual results on statewide testing, this disconnect persists. The ACT’s National Curriculum Survey (widely administered every three to five years) reported in 2006 that “High school mathematics teachers gave more advanced topics greater importance than did their postsecondary counterparts. In contrast, postsecondary… mathematics instructors rated a rigorous understanding of fundamental underlying mathematics skills and processes as being more important than exposure to more advanced mathematics topics” (ACT, Inc., 2007, p. 5). Six years later, the same discrepancy was reflected in the fact that 19 of the 20 topics rated by college faculty as most important for incoming students are typically taught in ninth grade or earlier (ACT, Inc., 2013, p. 6).

Ranking of the 20 Content Topics Rated Most Important as Prerequisites by Instructors of Credit-Bearing First-Year College Mathematics Courses (ACT, Inc., 2013, p. 7)

Rank Topic

1. Evaluate algebraic expressions
2. Perform addition, subtraction, multiplication, and division on signed rational numbers
3. Solve linear equations in one variable
4. Solve multistep arithmetic problems
5. Locate points on the number line
6. Perform operations (add, subtract, multiply) on linear expressions
7. Find the slope of a line
8. Find equivalent fractions
9. Find and use multiples and factors
10. Perform operations (add, subtract, multiply) on polynomials
11. Locate points in the coordinate plane
12. Write expressions, equations, or inequalities to represent mathematical and real-world settings
13. Evaluate functions at a given value of x
14. Graph linear equations in two variables
15. Order rational numbers
16. Determine the absolute value of rational numbers
17. Manipulate equations and inequalities to highlight a specific unknown
18. Manipulate expressions containing rational exponents
19. Solve linear inequalities in one variable
20. Solve problems using ratios and proportions

This misunderstanding about the types of experiences that best prepare students for college mathematics success produces high-school graduates who enter college with a superficial grasp of superfluous procedures and little conceptual framework. The goal is to impart a deep, flexible procedural knowledge which helps students to understand important concepts, and deep conceptual knowledge which helps to make sense of and connect procedures and ideas. Clarified further, “procedural knowledge learning should be structured in a way that emphasizes the concepts underpinning the procedures in order for conceptual knowledge to improve concurrently” (Maciejewski & Star, 2016). In order to equip students for success in college level mathematics and in jobs that require an application of mathematical skills to novel situations, the SMPs describe habits and behaviors that develop and reflect a deep conceptual and procedural understanding.

Unlike the content standards, the SMPs are the same for all grades K–12 (with one addition in high school [SMP.3.1] below). As students progress through mathematical content, the opportunities they have to deepen their knowledge of and skills in the SMPs should increase.

- SMP.1: Make sense of problems and persevere in solving them
- SMP.2: Reason abstractly and quantitatively
- **SMP.3: Construct viable arguments and critique the reasoning of others**
- SMP.4: Model with mathematics
- SMP.5: Use appropriate tools strategically
- SMP.6: Attend to precision
- **SMP.7: Look for and make use of structure**
- **SMP.8: Look for and express regularity in repeated reasoning**

All of the SMPs are crucial, and most worthwhile classroom mathematics activities require students to engage in all of them to varying degrees throughout the year.

Exploring and Reasoning With and About Mathematics: SMP.3, 7, 8

Certain curricula more clearly represent the SMPs and, as a result, this chapter addresses the progression through the grades of a cluster of three of the SMPs, highlighted above: Construct Viable Arguments and Critique the Reasoning of Others

(SMP.3, including the California-specific high school SMP.3.1 regarding proof); Look for and Make Use of Structure (SMP.7); and Look for and Express Regularity in Repeated Reasoning (SMP.8). These practices do not develop without careful attention across all grade levels and in relation to mathematical content. In addition, these three SMPs all require a high degree of language proficiency in order to access content knowledge and reasoning. The California English Language Development (ELD) Standards describe structures to assist in the building of the English language proficiency for English learners (Els). The ELD Standards, along with the SMPs and content standards, would help illustrate how best to integrate language development in the lessons. For many students, having small groups in which students can do the investigations, critiques, and reasoning in their native or preferred language may support and strengthen their understanding. In designated ELD time, the language of critiquing, reasoning, generalizing, and arguing is a space to help prepare EL students for engagement in the SMPs and the mathematical content. The framework's approach integrates the three SMPs in the context of mathematical investigations to highlight ways that mathematical practices can come together through exploration and reasoning. The following four processes might be useful guideposts for designing mathematical investigations that integrate multiple content and practice standards at the lesson or unit level (see Chapters 6, 7, and 8 for more grade-level guidance on mathematical investigations):

1. Exploring authentic mathematical contexts
2. Discovering regularity in repeated reasoning and structure
3. Abstracting and generalizing from observed regularity and structure
4. Reasoning and communicating with and about mathematics in order to share and justify conclusions

A classroom where students are engaged in these processes might look different to a visitor (or to the teacher!) than math classes as portrayed in popular media. While these processes focus on communication as sharing and justifying mathematical ideas, mathematical investigations involve multiple communicative processes for connecting and interacting with others and mathematics. Evidence of SMPs 3, 7, and 8 (among others) might include the following:

- Students trying multiple examples and comparing (SMP.1, 7): Ex., “I tried 6; what did you do?”
- Students challenging each other (SMP.3): Ex., “I see why you think that from what you tried. I don’t think that always works because….”
- Predictions being shared (often these reflect early noticing of repeated reasoning and structure, SMP.7 and SMP.8): Ex., “I think that when we try with a hexagon, we’ll get….”
- Students justifying their predictions (SMP.3, 7, and 8): Ex., “No matter what number we use, it will always be true that….”

In short, a classroom with evidence of SMP.3, 7, and 8 will include students using their own understanding to reason about authentic mathematical contexts and to share that reasoning with others.

It is important to revisit these SMPs as they appear in the CA CCSSM.

- SMP.3: Construct viable arguments and **critique** the reasoning of others. Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing **arguments**. They make **conjectures** and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. CA 3.1 (for higher mathematics only): Students build proofs by **induction** and **proofs by contradiction**.

It is important to point out that neither “argument” nor “critique” has negative connotations in this context. In the sense used here, argument is “a reason or set of reasons given in support of an idea, action or theory,” and critique means “evaluate (a theory or practice) in a detailed and analytical way” (Oxford, 2019). Everyday notions of these terms can inadvertently invite students to interpret mathematics classroom discussions as competitions for status; expressing disagreement can feel like an insult rather than an invitation for reasoning (Langer-Osuna & Avalos, 2015).

Building a classroom culture in which students can become proficient at constructing and critiquing arguments requires rich contexts and problems in which multiple approaches and conclusions can arise, creating a need for generalization and justification (see figure X below). Teaching for the development of SMPs, especially SMP.3, includes developing classroom norms for discussions that focus on examining the “truthiness” (i.e., validity) of the mathematical ideas themselves, rather than evaluating the student offering ideas in what Boaler (2002, drawing on Pickering, 1995) referred to as the “dance of agency.” According to *Principles to Actions: Ensuring Mathematical Success for All*, “Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments” (NCTM, 2014, p.12).

Suggested Math Class Norms:

1. Everyone can learn math to the highest levels
2. Mistakes are valuable
3. Questions are really important
4. Math is about creativity and making sense
5. Math is about connections and communicating
6. Depth is much more important than speed.
7. Math class is about learning not about performing

8. Everyone has the right to share their thinking
9. We attend to and make sense of the thinking of others

It is possible to prompt this culture by valuing the role of skeptic through the use of purposeful and probing questions, removing or delaying teacher validation of reasoning in favor of class-negotiated acceptance, and explicitly reminding students frequently that mathematicians prove claims by reasoning (Boaler 2019). To do so, students must experience a classroom environment where teachers and all students have the right to share their thinking and will be supported in doing so. Further, classroom norms must set the expectation that students respectfully attend to and make sense of the thinking of others; this is especially important with respect to differences in mathematical ideas, cultural experiences, and linguistic expressions.

- SMP.7: Look for and make use of structure.
  Mathematically-proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 × 8 equals the well-remembered 7 × 5 + 7 × 3, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2 × 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.
- SMP.8: Look for and express regularity in repeated reasoning.
  Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention

> to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Patterns in SMP.7 might be numeric, geometric, algebraic, or a combination. Structure is "the arrangement of and relations between the parts or elements of something complex" (Oxford 2019). SMP.7 and SMP.8 are key to abstracting. Stepping back from concrete objects to consider, all at the same time, a class of objects in terms of some set of identical properties—and generalizing—extending a known result to a larger class. Reasoning abstractly and developing, testing, and refining generalizations are essential components of doing mathematics, including solving problems (National Governors Association Center for Best Practices [NGACBP], 2010).

Abstracting, Generalizing, Argumentation

Bringing all three SMPs together—abstracting, generalizing, and argumentation—points to the power of classroom discussions and other collaborative activities where students make sense of mathematics together. Teacher facilitation of high-quality mathematics discourse is the key to unlocking these practices for students and bringing them holistically into practice. Historically, proficiency in mathematics has been defined as an individual cognitive construct. However, the past three decades of mathematics classroom research has revealed the ways in which learning and doing mathematics is rooted in social activity (Lerman, 2000; National Academies of Sciences, Engineering, and Medicine, 2018). Still, merely asking students to talk to each other in math class is insufficient. The facilitation of high-quality discourse needs to be intentional, especially with attention to language development. Random assignments for student interactions could prevent high-quality math discourse. Intentional patterns of grouping, such as primary language grouping, to support effective interactions and communication is important. Another option is to consider assigning a student to serve as a bilingual broker for each small group of ELs and English-only students. This student is given extra training and support to provide the language support leading to understanding by each group member and an appreciation of everyone's thinking. In the following progressions through the grade bands, the framework illustrates ways that students might progress in the SMPs through such classroom discourse activity, based on thoughtful whole-class and small-group activities where students are offered plentiful opportunities to grapple with and discuss mathematical ideas and problems through engagement in the SMPs—especially SMP.3, 7, and 8.

Progressions in the Mathematical Practices

Young learners begin to engage with mathematical ideas through real-world contexts. As domains of mathematics become more accessible to students, they can increasingly explore purely mathematical contexts; for instance, even young learners who have become comfortable with the natural numbers—as a context in which reasoning can occur—can explore patterns in even and odd numbers and use shared definitions to reason about them. Yet even as students increasingly explore mathematical worlds, opportunities to mathematize the real world continue to be important from the early grades into adulthood (as illustrated in both the Number Sense and Data Science chapters of this framework).

While the practice standards remain the same across grade levels, the ways in which students engage in the practices progress and develop through experience and opportunity. In early grades, mathematical reasoning is primarily representation-based: When justifying a claim about even and odd numbers, students will typically refer to some representation like countable objects, a story, or a number line or other drawing. Representational and visual thinking remains important through high school and beyond.

As students become comfortable in additional mathematical contexts and develop more shared understanding in those contexts, reasoning may sometimes stay at that level and rely on mathematical definitions and prior results. However, teachers should recognize the importance of concrete ways of making conjectures and justifying them mathematically, to avoid unduly privileging more abstract reasoning. Moving too early to abstract reasoning, before all students have an adequate base of representations (physical, visual, contextual, or verbal) with which to reason, can have the effect that many students experience mathematical arguments as meaningless abstract manipulation. Ample mathematical reasoning and argumentation with concrete representations (such as appropriate manipulatives and visual representations) and with contextual examples helps to foster a classroom learning environment that provides access for and builds understanding for all students. (Note that concrete is used here not in the sense of tangible and physical, but in the sense of making sense; see Gravemeijer, 1997; Van Den Heuvel-Panhuizen, 2003.)

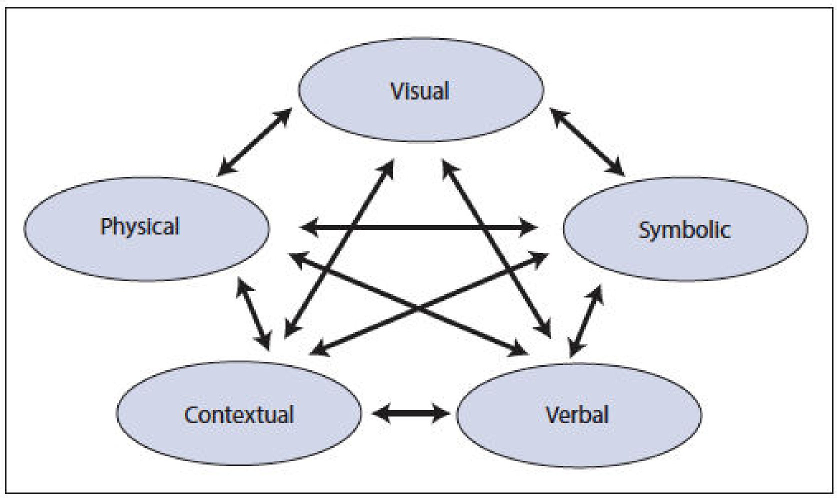
The principle of learning an abstract idea through access to concrete representations and examples is not just applicable at younger grades; it applies any time that a new concept is encountered. For example, students in grades five and six, working on their understanding of percentage, benefit from a bar representation that is used in increasingly abstract ways, finally simplifying to a double number line (Van Den Heuvel-Panhuizen, 2003). The use of representations and visuals provides scaffolding that English learners and others may use to connect the academic language to their conceptual understanding.

Consider a sixth-grade class that is using such a bar representation to explore percentages. Different students will see different uses of the representation, and use it to reason in different ways. Some may quickly generalize calculation patterns that they observe (SMP.7), and begin to calculate without reference to the bar representation: “If the price after a 25% discount is $96, then I just divide that by three and add it to $96 to get the original price of $128.”

This realization can be used productively, both to help these students to connect their method to the sense-making bar representation (SMP.8) and to help other students understand their classmates’ ideas. One useful routine for this is careful selecting, sequencing, and connecting of student work as described in *5 Practices for*

Orchestrating Productive Mathematics Discussions (Smith & Stein, 2018). However, it is easy—even when attempting to implement the 5 Practices routine—to hold up the work of students who have moved beyond the concrete representation as the preferred method (because it might appear to be quicker, or more generalized, or closer to a final understanding teachers hope all students will reach). This can create the false notion that reliance on sense-making representations is an indication of weakness. Therefore, it is important for teachers to support all students to make sense of each other's approaches by building connections between them.

Evidence from neuroscience suggests that some of the most effective understandings come about when connections are made between visual/physical and numerical or symbolic representations of ideas (see figure from NCTM, 2014). When students relate numbers to visual representations, they make connections between brain pathways that link ideas they hold in different parts of the brain. These connections are important to students at all ages and grade levels (Boaler, Chen, Williams & Cordero, 2016).



At all grades, students should have ample experience in all of the processes above (exploring authentic contexts, discovering regularity and structure, abstracting and generalizing, and reasoning and communicating). As with the **modeling cycle** (see Chapter 8: Mathematics: Investigating and Connecting, Grades Nine through Twelve), some of these processes are historically emphasized far more than others, contributing to many students' loss of a belief in mathematics as a sense-making activity. Classroom activities that are designed to engage students in these processes therefore must be sufficiently open ended, to allow students room to explore, must give access to the regularity and structure that is present, and must allow generalization to broader settings.

Teaching practices for SMP development

Principles to Action: Ensuring Mathematical Success for All (NCTM, 2014) lays out eight "Mathematics Teaching Practices:"

1. Establish mathematics goals to focus learning.
2. Implement tasks that promote reasoning and problem solving.
3. Use and connect mathematical representations.
4. Facilitate meaningful mathematical discourse.
5. Pose purposeful questions.
6. Build procedural fluency from conceptual understanding.
7. Support productive struggle in learning mathematics.
8. Elicit and use evidence of student thinking.

Detailed discussion of teaching is in Chapter 2: Teaching for Equity and Engagement, and (NCTM, 2014) contains detailed discussion of these teaching practices; here we limit the list to items that are specifically important for developing SMPs, especially SMP.3, 7, and 8.

First, mathematical goals (Teaching Practice 1) must include SMPs as a central driver of activity design at a more detailed level than simply "this is a rich task, and students will engage in all eight SMPs." Second, posing purposeful questions (Teaching Practice 5) is crucial in establishing students' inclination to engage in the SMPs as they encounter mathematical situations. Reprinted here is a framework for teacher question types (NCTM, 2014). All question types are important; type 1 (Gathering information) is traditionally over-represented while types 2, 3, and 4 help make clear that students are expected to engage in the SMPs.

Question type	**Description**	**Examples**

1	Gathering information	Students recall facts, definitions, or procedures.	When you write an equation, what does the equal sign tell you? What is the formula for finding the area of a rectangle? What does the interquartile range indicate for a set of data?
2	Probing thinking	Students explain, elaborate, or clarify their thinking, including articulating the steps in solution methods or the completion of a task.	As you drew that number line, what decisions did you make so that you could represent 7 fourths on it? Can you show and explain more about how you used a table to find the answer to the Smartphone Plans task? It is still not clear how you figured out that 20 was the scale factor, so can you explain it another way?
3	Making the mathematics visible	Students discuss mathematical structures and make connections among mathematical ideas and relationships.	What does your equation have to do with the band concert situation? How does that array relate to multiplication and division? In what ways might the normal distribution apply to this situation?
4	Encouraging reflection and justification	Students reveal deeper understanding of their reasoning and actions, including making an argument for the validity of their work.	How might you prove that 51 is the solution? How do you know that the sum of two odd numbers will always be even? Why does plan A in the Smartphone Plans task start out cheaper but become more expensive in the long run?

Finally, this table from (Barnes & Toncheff, 2016) helps to connect the mathematical teaching practices above (MTPs) with all of the SMPs.

<table>
<tr><th colspan="2">Standards for Mathematical Practice (SMP)</th><th rowspan="8">Teacher Action Connections

Mathematics lessons align to the essential learning standards and teachers clearly communicate them to students (MTP1). Lessons include complex tasks (MTP2), opportunities for visible thinking (MTP8 and MTP4), and intentional questioning (MTP5) to promote deeper mathematical thinking (MTP6). Teachers design lessons from the student’s perspective to provide multiple opportunities to make sense of the mathematics (MTP7).

To build SMP1, teachers focus on MTP7 and MTP2.
To build SMP2, teachers focus on MTP2 and MTP3.
To build SMP3, teachers focus on MTP4 and MTP5.
To build SMP4, teachers focus on MTP3 and MTP8.
To build SMP5, teachers focus on MTP2 and MTP3.</th><th colspan="2">Mathematics Teaching Practices (MTP)</th></tr>
<tr><td>SMP1</td><td>Make sense of problems and persevere in solving them.</td><td>MTP1</td><td>Establish mathematics goals to focus learning.</td></tr>
<tr><td>SMP2</td><td>Reason abstractly and quantitatively.</td><td>MTP2</td><td>Implement tasks that promote reasoning and problem solving.</td></tr>
<tr><td>SMP3</td><td>Construct viable arguments and critique the reasoning of others.</td><td>MTP3</td><td>Use and connect mathematical representations.</td></tr>
<tr><td>SMP4</td><td>Model with mathematics.</td><td>MTP4</td><td>Facilitate meaningful mathematical discourse.</td></tr>
<tr><td>SMP5</td><td>Use appropriate tools strategically.</td><td>MTP5</td><td>Pose purposeful questions.</td></tr>
<tr><td>SMP6</td><td>Attend to precision.</td><td>MTP6</td><td>Build procedural fluency from conceptual understanding.</td></tr>
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</table>

SMP7	Look for and make use of structure.	To build SMP6, teachers focus on MTP4 and MTP2. To build SMP7 and SMP8, teachers focus on tasks (MTP2).	**MTP7**	Support productive struggle in learning mathematics.
SMP8	Look for and express regularity in repeated reasoning.		**MTP8**	Elicit and use evidence of student thinking.

K–5 Progression of SMPs 3, 7, and 8

Imagine a teacher puts the number 36 on the board and asks students to determine all the ways they can make 36. In the context of an open problem such as this, young learners conjecture, notice patterns, use the structure of place value, notice and make use of properties of operations, and make sense of the reasoning of others. These practices often occur together as part of classroom discussions that focus on argumentation and reasoning through engaging mathematical contexts. The choice of number here makes a big difference; a grade-three teacher might choose 36 to build multiplication ideas; a kindergarten teacher might use 12 to both formatively assess and work to strengthen students' emerging operation understanding.

Consider, for example, the following first-grade snapshot of a number talk activity. Number talks are brief, daily activities that support number sense. Prior to the lesson, the teacher understands that presenting a question or problem to the whole class and asking for individual responses will be challenging for some English learners. In the designated ELD lessons prior to this whole-group instruction, the teacher practices the discourse needed to explain their thinking and problem solving while giving them the language they need to be able to participate.

First-Grade Snapshot: Number Talks for Reasoning

Big Idea: Flexibility in composing and decomposing numbers

The teacher introduces the number talk by placing the problem 7+3 on the board, waiting patiently as small silent thumbs pop up communicating they are ready to offer an answer and the strategy they used to figure it out. The teacher selects a first student, Iggy, to share.

Teacher: Iggy, how did you figure out 7+3?

Iggy: I knew 7+2 is 9 and 9+1 is 10.

Teacher records Iggy's thinking on the board and re-voices their response, then probes Iggy further: Iggy, where did the 2 and the 1 come from?

Iggy: That number.

Teacher: Which number? Who can add on to Iggy's strategy? How did they know to add 2 more and then 1 more? Sam?

Sam: 2 and 1 are both in 3. Iggy broke down 3.

Teacher: You noticed that 2 + 1 is 3. Iggy is that what you did? Did you think, let me break down 3 because I know 7+2 is 9 and 9 +1 is 10?

Iggy: Yes

Teacher: Who else wants to share how they figured out the answer? Alex?

Alex: Counting on? I did like, I started with 7 and then I counted, 8, 9, 10.

Teacher records Alex's thinking and re-voices their response, then adds: So that's a different strategy? (Alex nods.) Did anyone else count on like Alex?

The teacher selects other students who share their own strategies and make sense of their peers' reasoning, all based in a relatively straightforward computation problem. This approach supports mathematical sense-making and communication. While students certainly arrive at the answer "10," the focus of the activity is making sense of the addition problem, thinking flexibly and creatively about a range of ways to solve it, communicating one's thinking and making sense of the reasoning of others.

Exploring authentic mathematical contexts

Authentic (from Chapter 1: Introduction): An authentic problem, activity, or context is one in which students investigate or struggle with situations or questions about which they actually wonder. Some principles for authentic problems include 1) Problems have a real purpose; 2) Relevance to learners and their world; 3) Doing mathematics adds something; and 4) Problems foster discussion (Özgün-Koca, Chelst, Edwards, & Lewis, 2019).

Culturally Responsive-Sustaining Education: Education that recognizes and builds on multiple expressions of diversity (e.g., race, social class, gender, language, sexual orientation, religion, ability) as assets for teaching and learning. (NYSED, 2019)

SMP.3, 7, and 8 describe ways of exploring mathematical contexts such as numerical patterns, geometry, and place value structure. These activities might involve multiple visual representations, such as fractions represented in both area models like partitioned circles and linear models like number lines. Allowing students to explore the same mathematical ideas and operations using multiple representations and strategies is crucial for students to develop flexible ways of thinking about numbers and shapes (e.g., Rule of Four [http://www.sfusdmath.org/rule-of-four.html]). Students of all grade levels should engage in opportunities to create important brain connections through seeing mathematical ideas in different ways (also see Chapter 2: Teaching for Equity and Engagement).

At the elementary level, students work with numbers with which they are currently familiar. This may mean they generalize in ways that will be revisited and revised in the later grades, as new numbers and mathematical principles are introduced. For example, at the early elementary level, students may appropriately generalize about the behavior of positive whole numbers in ways that are revisited at the later elementary grades with the introduction of fractions (later called rational numbers), and then again later on at advanced grades with the introduction of imaginary numbers or irrational numbers. Students may also use everyday contexts and examples in order to make arguments. For example, a student might offer a story about two friends sharing cookies to demonstrate that an odd number, when divided by two, has a remainder of one. In the

Data Science chapter, we further illustrate ways that everyday contexts can become generative for learning and doing mathematics together. Importantly, contexts should be authentic to students (as defined above)—not the fake contexts used in many textbooks that require students to suspend their common sense in order to engage with the intended mathematics (see Boaler, 2009). It is important to make mathematical contexts culturally relevant to ensure that diverse student experiences are considered and possibly make connections with students' families. Chapter 2 offers examples of culturally relevant contexts for learning mathematics.

Discovering regularity in repeated reasoning and structure

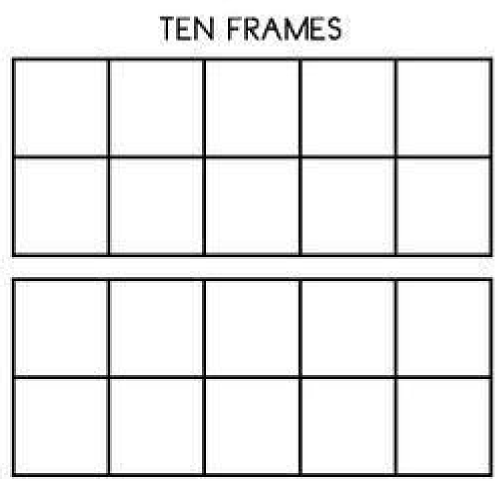
Students at the elementary level may notice and use structures such as place value, properties of operations, and attributes about shapes to make conjectures and solve problems. Additionally, students notice and make use of regularity in repeated reasoning. At the elementary level, students may notice, through repeatedly multiplying with the number 4, that it is always the same as doubling twice. Students might also notice a pattern in the change of a product when the factor is increased by 1. For example, that since 7 x 8 = 56, then 7 x 9 will be 7 more than 56. These regularities may lead to claims about general methods or the development of shortcuts based on conceptual reasoning.

A variety of reasoning activities support students in thinking flexibly about operations with numbers and relationships between numbers. In number talks and dot talks, students share and connect multiple strategies by explaining why the strategies work or comparing advantages and disadvantages. The Number Sense chapter offers a grade two number talk vignette where children work on doubles posed as addition problems. In the vignettes, students share strategies to solve 13 + 13. Many of the strategies made use of place value structure and counting strategies. As students in the snapshot offer ideas and take the ideas of their peers into consideration, some students revise their answers. In a "Collect and Display" activity (Zwiers, et al, 2017), teachers can scribe student responses (using students' exact words whenever possible and attributing authorship) on a graphic organizer on the board during the whole class discussion comparing two mathematical ideas, such as expressions and equations. In a

“Compare and Connect” activity (Zwiers, et al, 2017), students relate the expressions to the diagrams by asking specific questions about how two different-looking representations could possibly mean the same thing. For example, a teacher might ask, “Where is the 2w in this picture?” or “Which term shows this line on the rectangle?”

Abstracting or generalizing from observed structure and regularity

Young learners might explore place value structure through manipulatives like ten frames. In a number talk with ten-frame pictures, students offer various strategies used to figure out the quantity shown. Students also attend to and discern patterns and structure as they construct and critique arguments. A student might notice that four sets of six gives the same total as six sets of four, and that this applies to three sets of seven and seven sets of three, and so on, to conjecture about the commutative property



during a number talk.

Reasoning and communicating to share and justify

Part of constructing mathematical arguments includes understanding and using previously established mathematical assumptions, definitions, and results. For example, an elementary aged student might conjecture that two different shapes have equal area because, as the class has already recognized and agreed upon, the shapes are each half of the same rectangle. The student draws on prior knowledge already been demonstrated mathematically in order to make their argument.

Constructing and critiquing mathematical arguments includes exploring the truth of particular conjectures through cases and counterexamples. At the elementary level, a student may use, for example, a rhombus as a counterexample to the conjecture that all quadrilaterals with four equal sides are squares. Students may use multiplication with fractions, decimals, one, or zero to counter the conjecture that multiplying always leads to a larger number.

6–8 Progression of SMP.3, 7, and 8

Students in middle school build on early experiences to deepen their interactions with mathematics and with others as they do mathematics together. During the elementary grades, students typically draw on concrete manipulatives and representation in order to engage in mathematical reasoning and argumentation. At the middle-school level, students may rely more on symbolic representations, such as expressions and equations, in addition to concrete referents (such as algebra tiles and area models for algebraic expressions; physical or drawn examples of geometric objects; and computer-generated simulation models of data-generating contexts). Number talks (Parrish, 2010; Humphreys & Parker, 2015) and number strings (a series of related number talks or problems designed to build towards big mathematical ideas; see Fosnot & Dolk, 2002) are useful at the middle school level as well, and offer a range of opportunities for students to build on their elementary grades experiences to make sense of mathematical ideas with peers. For example, consider the following classroom snapshot:

Grade Seven Snapshot: Estimating using structure

Big Idea: Connecting multiple approaches leads to flexible, transferable understanding

Prior to the lesson, a seventh-grade teacher, in order to ensure that all students, including English learners, are supported, engages students in an activity to practice the discourse needed to explain their thinking and problem solving. This activity, they hope, will also increase participation. The activity transitions into the teacher introducing the number string activity and writes this problem (from http://www.mathtalks.net) on the board:

Are there more inches in a mile or seconds in a day?

After some wait time for individual thinking, the teacher asks students to show where they are in their thinking using their fingers, a routine the class knows well: closed fist for "still trying to find an approach to try;" one finger for "have an approach and haven't got an answer yet;" two fingers for "have an answer with an explanation, and not very confident;" three fingers for "have an answer and an explanation that I'm confident in;" and four fingers for "have tried two or more approaches and confirmed my answer." After a little more wait, she asks students to show again their status, and she chooses a student holding up two fingers:

Teacher: Can you describe your approach that might help us figure out which is bigger?

Courtney: I remember there are about 5,000 feet in a mile, so there are about 50,000 inches in a mile since there are about 10 inches in a foot. I rounded them both down. But then with seconds, I tried to figure out 24 × 60 and if I round those, it's only about 1,200 seconds but that seems too small. [*Teacher scribes both calculations, including units where the student included them*.]

Teacher: Is there anyone else who thinks they can go a little farther with this idea?

Tristán: I tried the same thing but I got 60,000 inches in a mile instead of 50,000.

Courtney: Did you round 12 inches in a foot down to 10?

Tristán: Oh yeah, I didn't.

Teacher: Courtney, can you explain again why you thought something wasn't right with your method?

Courtney: When I tried to figure out the number of seconds, the number seemed too small—it was a lot smaller than the 50,000 I got for inches in a mile.

Bethney: You did 24 × 60?

Courtney: Yeah.

Bethney: Where did you get the 60?

Courtney: Seconds in a minute. And the 24 is hours in a day. Wait… [*Teacher adds units to the 24 × 60 on the board from earlier*]

Bethney: I thought it was minutes in an hour [*Teacher adds alternate unit to 60*]. So, 24 × 60 is how many minutes in a day.

Courtney: Oh, so I have to times that by 60 again.

Teacher: So, Courtney, now it sounds like you think you could do 24 × 60 and then multiply by 60 again? [*scribes* (24 × 60) × 60 *on board*]. Can somebody else help me with units on these? What quantity is each of these numbers representing?

Cameron: The 24 is hours per day, and the first 60 is minutes per hour.

Michael: So, the thing in parentheses is minutes per day. And then the second 60 is seconds per minute.

The discussion continues, exploring several ways that students computed and estimated 24 hours/day × 60 minutes/hour × 60 seconds/minute and 5,280 feet/mile × 12 inches/foot. After several methods had been compared and connected, and students seemed to agree (with justification) that there are more seconds in a day than inches in a mile, the teacher added to the problem statement:

Teacher: What if I add this to the problem? [*scribes on board* “or breaths in a typical human lifetime?”]

After more wait time and a repeat of the finger routine, the teacher selects a student displaying three fingers, who hasn’t already participated:

Teacher: Ji-U, can you describe part of your approach?

Ji-U: I counted while I breathed, and decided that a breath takes about four seconds.

Teacher: Who else did something to decide how long a breath takes? [[most students raise hand] How long did you estimate? [*chorus of four seconds, five seconds, six seconds*]

The conversation continues with students adapting strategies from earlier, including:

- I searched and found to use 79 years for average lifespan
- Approximated number of seconds in a life, using earlier calculation of seconds/year, then divided by 5 seconds/breath
- Replaced 60 seconds/minute in earlier calculation with 15 breaths/minute to get number of breaths in a year since I thought each breath was 4 seconds
- Realized that $24 \times 60 \times 15 \times 79$ has to be much bigger than $24 \times 60 \times 60$ since 15×79 is more than 60
- So, there are more breaths in a 79-year human life!

The teacher concludes this final number talk in the string by asking students to think about and then share with a neighbor some descriptions of what they learned or noticed during the talk. Then a few students share something interesting their partner noticed, while the teacher highlights strategies that involve significant use of place value structure, others which make use of rounding with an explanation of the effect of the rounding, and others which compare products that share factors by comparing the other factors.

The number string offered students the opportunity to notice their own errors without the teacher's evaluation. As students made sense of the problems in multiple ways, they reflected on their own thinking, made connections, and revised their own thinking. Rather than positioning the student as lacking in mathematical competence, the number string positioned Student 1's error as an invitation for further sense-making, and as a normal part of doing mathematics. The teacher highlighted strategies which made significant use of structure of numbers and of operations.

Exploring authentic mathematical contexts

//callout box

Authentic: An authentic problem, activity, or context is one in which students investigate or struggle with situations or questions about which they actually wonder. (from Chapter 1: Introduction)

callout box//

Middle-school students become increasingly sophisticated observers of their everyday worlds as they develop new interests in understanding themselves and their communities. These budding interests can become engaging real-world contexts for mathematizing. The Data Science chapter offers examples of middle school students exploring data about the world around them.

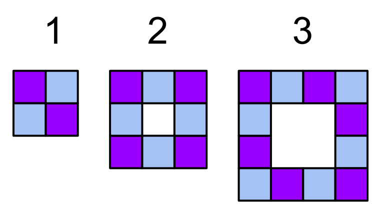
Mathematical contexts to explore, in addition to those carrying forward from earlier grades (number patterns and two-dimensional geometry), include the structure of operations, more sophisticated number patterns, proportional situations and other linear functions, and patterns in computation.

Discovering regularity in repeated reasoning and structure

Students at the middle level may build on their knowledge of place value structure and expand their use of structures, properties of operations, and attributes about shapes to make conjectures and solve problems. For example, middle-school students might draw on tables of equivalent ratios to conjecture about underlying multiplicative relationships.

Abstracting and generalizing from observed regularity and structure

Students might notice during a mathematical discussion that interior angle sums regularly increase in relation to the number of sides in a polygon and use this repeated reasoning to conjecture a rule for the sum of interior angles in any polygon. In a Compare and Connect activity (Zwiers, et al., 2017), students compare and contrast two mathematical representations (e.g., place value blocks, number line, numeral, words, fraction blocks) or two solution strategies (e.g., finding the eleventh tile pattern number recursively—"there were four more tiles each time, so I just added four to the four starting tiles, ten times"—compared to noticing a relationship between the figure number and the number of tiles—"I noticed that each side is always one more than the figure number, so I did 4 times the figure number plus 1. And then I had to take away 4 because I counted the corners twice.") together. As a whole class, students might address the following questions:



- Why did these two different-looking strategies lead to the same results?
- How do these two different-looking visuals represent the same idea?
- Why did these two similar-looking strategies lead to different results?
- How do these two similar-looking visuals represent different ideas?

The reference (Inside Mathematics, n.d.) includes a grade-eight illustration (with video) of SMP 7 (Look for and make use of structure) from the South San Francisco Unified School District. It illustrates students noticing mathematical structure in a concrete context—namely, water flowing in a closed system from one container into another. After observing the relationship between the two quantities (the water level in each container), they note constant rates of change and starting value. Students then apply the structure they discover, in order to recognize graphs corresponding to different systems—evidence of abstracting. Teacher moves that support their investigation include modeling of academic language, building on and connecting student ideas, restating student ideas, and more.

The Education Development Center (2016) has built student dialogue snapshots to illustrate the SMPs. The grade 6–7 example Consecutive Sums illustrates students working on the problem “in how many ways can a number be written as a sum of consecutive positive integers?” They work many examples, notice a pattern to their calculations, and connect that pattern to some structure of the numbers they are working with. They are then able to generalize that structure and develop a general strategy for writing integers as sums of consecutive integers.

Reasoning and communicating to share and justify

Part of constructing mathematical arguments includes understanding and using previously established mathematical assumptions, definitions, and results. Students might conjecture that the diagonals of a parallelogram bisect each other, after having experimented with a representative selection of possible parallelograms. Like in the elementary grades, where students may conjecture about shapes and area, students at the middle-school level continue this practice with mathematical content that builds on foundational ideas.

Constructing and critiquing mathematical arguments includes exploring the truth of particular conjectures through cases and counterexamples. An important use of counterexamples in middle school is the use of numerical counterexamples to identify common errors in algebraic manipulation, such as thinking that $5 - 2x$ is equivalent to $3x$.

In Boaler and colleagues’ Youcubed summer camp for middle-school students, which significantly increased achievement in a short period of time (Boaler 2019), students were taught that reasoning is a crucially important part of mathematics. They were told that scientists build evidence for theories by making predictions and then performing experiments to check their predictions; mathematicians, on the other hand, prove their claims by reasoning. Students were also told that it was important to reason well and to be convincing and there are three levels of being convincing: 1) It is easiest to convince yourself of something, 2) it is a little harder to convince a friend, and 3) the highest level of all is to convince a skeptic. Students were asked to be really convincing and also to be skeptics. An exchange between a convincer and a skeptic might include:

Jackie: I think that the difference between even and odd numbers is that when you divide them into two equal groups, even numbers have no left overs and odd numbers always have 1 leftover.

Soren: How do you know it's always one left over?

Jackie: Because, like, if you divide any odd number in half, like, look it—take the number 5, it would be two groups of two and then one left over. Or the number 7, it would be two groups of three and then one left over. There is always one left over.

Soren: Can you prove it? Maybe it just works for 5 and 7.

Jackie: Well, it's kind like, it will always be one left over because if it was two left over, they would just go in each of the groups, or if it was three left over, two would go in each of the groups. So, there's always only one left over.

In the summer camp, students loved being skeptics; and when others were presenting, they learned to ask questions of each other such as: "How do you know that works?" "Why did you use that method?" and "Can you prove it to us?" In essence, students were learning to "construct viable arguments and critique the reasoning of others." After only 18 lessons the students improved their achievement by the equivalent of 2.8 years of school. Students related their increased achievement to the classroom environment that encouraged discussion, convincing, and skepticism (see https://vimeo.com/245472639), as illustrated by this interview with two students, TJ and José:

Interviewer: So, what did it take in summer math camp to be successful?

TJ: Being able to communicate with your partner as you go.

José: And being able to show visuals, not just numbers.

TJ: Being able to explain things well.

José: And then someone says how, or why or...

TJ & José: Prove it! [laughing].

José: Uh, what, what is that called, a, um....

TJ: Skeptical question.

José: Yeah, skep-, yeah, skeptic.

Interviewer: And what does that mean and how does that feel?

TJ: It's fun to be.

José: [laughs]

Interviewer: Can you explain?

TJ: Because like it helps the other person that's not being skeptical...

José: Think about the problem.

TJ: Yes. For example, if Carlos said like, "This is a square," and I'm like, "Prove it."

José: Mmm, it has all, um, it, okay, it has all even sides and all, and all the corners are ninety degrees.

TJ: Why?

José: 'Cause it is.

TJ: Prove it!

José: It is! [laughs]

TJ: [laughs]

José: I just proved it.

There are many routines that help support students in being the skeptic, including tools to support English learners and others to develop the necessary language: In a "Critique, Correct, Clarify" activity (Zwiers et al., 2017), students are provided with teacher-made or curated ambiguous or incomplete mathematical arguments (e.g., "1/2 is the same as 3/6 because you do the same to the top and bottom" or "2 hundreds is more than 25 tens because hundreds are bigger than tens"). Students practice respectfully making sense of, critiquing, and suggesting revisions together. In a "Three Reads" activity (Zwiers et al., 2017), students make sense of word problems and other mathematical texts by discussing with each other: 1) the context of the situation, 2) relevant quantities (things that can be counted or measured), and 3) what mathematical questions we might ask about them before revealing what question the teacher has for them to answer.

High School Progression of SMP.3, 7, and 8

In high school, students build on their earlier experiences in developing their inclination and ability to explore, discover, generalize and abstract, and argue. It is important that high school teachers understand when designing student activities that the Standards for Mathematical Practice are as important as the content standards and must be developed together. The University of California, California State Universities, and California Community Colleges have a joint Statement on Competencies in Mathematics Expected of Entering College Students (ICAS, 2013) makes this clear, with expectations for students such as:

> "A view that mathematics makes sense—students should perceive mathematics as a way of understanding, not as a sequence of algorithms to be memorized and applied." (p. 3)
>
> "students should be able to find patterns, make conjectures, and test those conjectures; they should recognize that abstraction and generalization are important sources of the power of mathematics; they should understand that mathematical structures are useful as representations of phenomena in the physical world…." (p. 3)
>
> "Taken together the Standards of Mathematical Practice should be viewed as an integrated whole where each component should be visible in every unit of instruction." (p. 7)

Exploring authentic mathematical contexts

//callout box

Authentic: An authentic problem, activity, or context is one in which students investigate or struggle with situations or questions about which they actually wonder. (from Chapter 1: Introduction)

callout box//

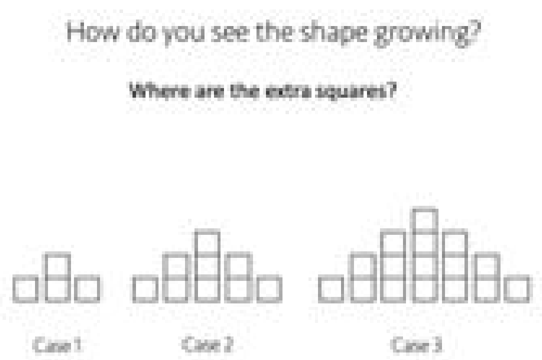
By high school, students have a wide array of contexts available for exploration. They continue to explore non-mathematical contexts—in the real world, in puzzles, etc. The Data Science chapter addresses one set of tools for exploring such contexts, and mathematical modeling represents another (overlapping) set. Often, data and modeling approaches yield mathematical contexts which then can be explored in the manner discussed here.

SMPs 7 and 8 afford opportunities to explore mathematical contexts and situations. Numerical patterns, geometry, and place value-based structure in the early grades, supplemented by structure and properties of operations in upper elementary and middle school, expand in high school to focus on algebraic, statistical, and geometric structure and repeated reasoning.

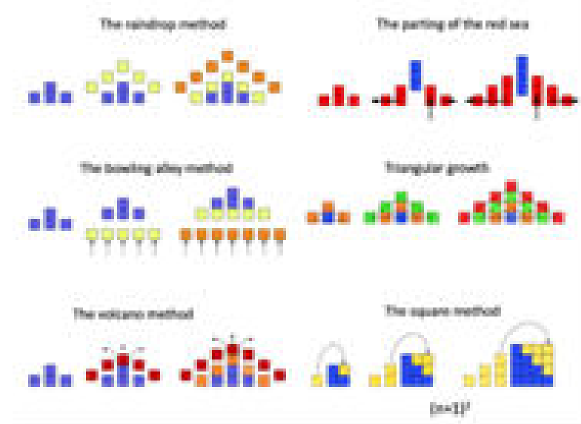
Important objects in algebraic settings include variables (letters or other symbols representing arbitrary elements of some specified set of numbers; distinct from unknowns and constants), graphs (often but not always graphs of functions), equations, expressions, and functions (often given by algebraic expressions—formulas—or implied by tables or graphs).

One very important skill in working with functions is to move fluently between contextual, graphical, symbolic, and numerical (e.g., table of values) representations of a function. Thus, activities that induce a need to switch representations are crucial. The exercise of moving from a formula (symbolic representation) to a graph is vastly overrepresented in most students' experience, often via sample values (numerical representation) and connecting dots. Examples of other pairings are described here.

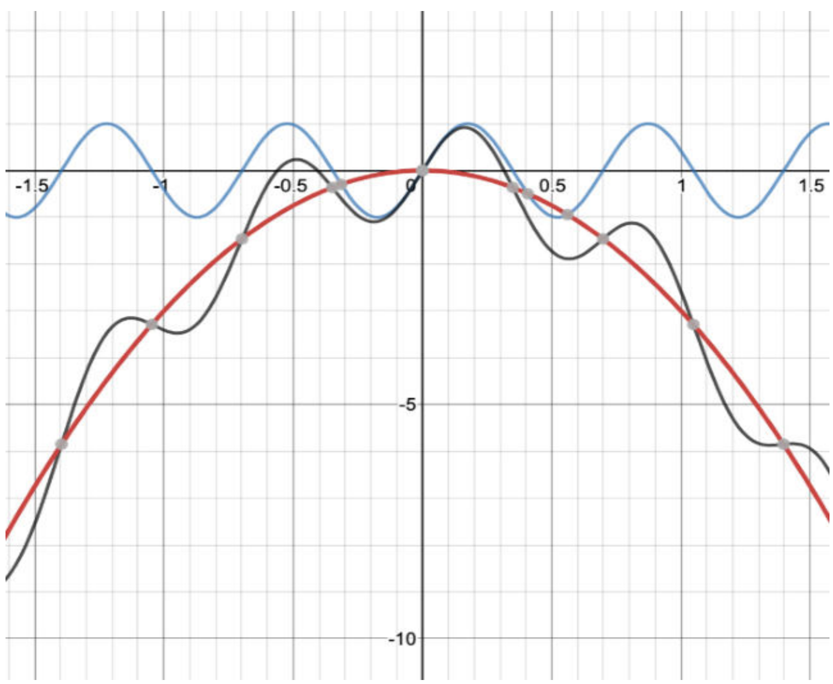
An engaging and important way to introduce patterns, expressions and functions, is through the context of visual or physical patterns (an easy-to-understand context). Students can first be asked to describe the growth of such a pattern with words, and then move to symbolic representations. In this way, students can learn that algebra is a useful tool for describing the patterns in the world and for communication. Note the examples below showing patterns for this type of work:



How do you see the shape growing?
Where are the extra squares?
(http://www.youcubed.org)



Further examples of this visual approach to algebra (with videos of lessons) can be seen at http://www.visualpatterns.org/ and https://www.youcubed.org/algebra/

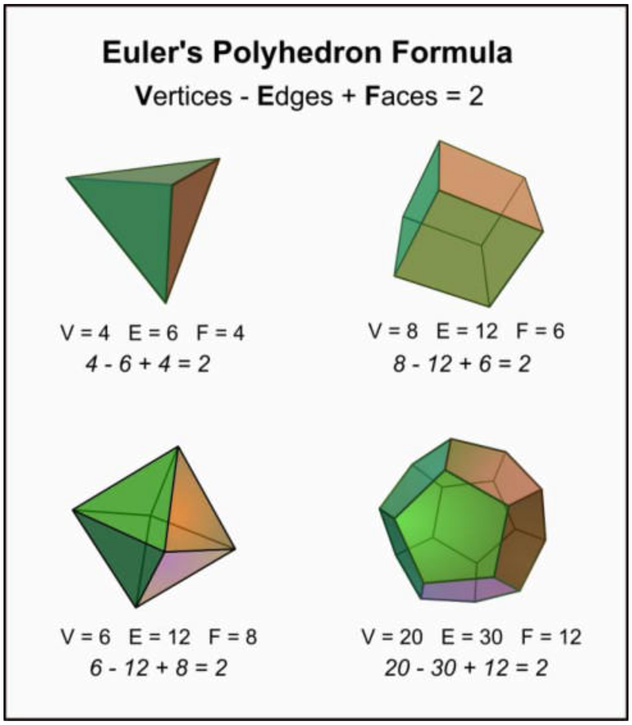


“Guess my rule” games (with student-generated sequences) require students to attempt to move from numerical representations to formulas. Students often can find a recursive formula first; “find the 100th term”-type questions force an attempt to move to a formula in terms of the sequence number. It is important that students have some experience with “guess my rule” games whose rule does not match the most obvious formula, as any finite set of initial values cannot determine an infinite sequence. As an example, the sequence 1, 2, 4, 8 is generated nicely by the function $f(n) = (n-1)(n-2)(n-3)(n-4) + 2^{n-1}$; the next term is 40, not 16! However, in many instances (including most applications) the “simplest” rule that fits the given data is a good one to explore first.

In the other direction, “build this graph” activities require student teams to try to build given graphs (perhaps visually modeling real-world data) from graphs of well-understood “simple” functions—perhaps monomials such as ax^b, perhaps also $\sin(x)$ and $\cos(x)$, or whatever set of “parent” functions is already understood. The graph to the right contains the graphs of $g(x) = -3x^2$ and $h(x) = \sin(9x)$, together with their sum $f(x) = -3x^2 + \sin(9x)$. This type of decomposition of a (graph of a) function is very important in many applied settings, in which (for example) different causal factors might act on very different time scales.

Discovering regularity in repeated reasoning and structure

To explore a context with an eye for algebraic structure is to consider the parts that make up or might make up an algebraic object such as a function, visual representation, graph, expression, or equation, and to try to build some understanding of the object as a whole from knowledge about its parts. Noticing regularity in repeated reasoning in an algebraic context often leads to discoveries that similar reasoning is required for different parameter values (e.g., comparing the processes of transforming the graph of x^2 into the graphs for the functions $3x^2 + 2$, $\frac{1}{2}x^2 - 4$, and $-2x^2 + 1$, leading to general statements about graphing functions of the form $ax^2 + b$).

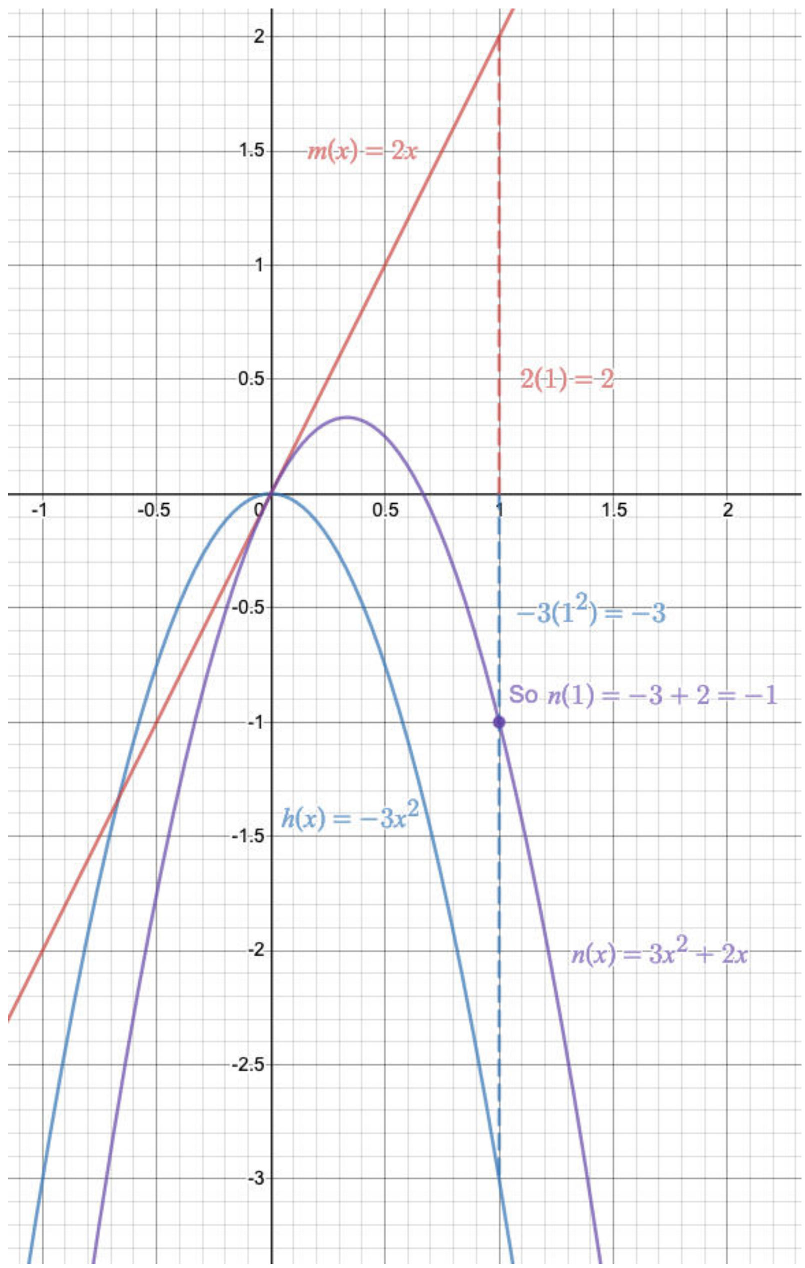


Euler's Polyhedron Formula
(https://commons.wikimedia.org/wiki/File:Euler%27s_Polyhedron_Formula.svg)

In a geometric context, structural exploration (SMP 7) examines the relationships between objects and their parts: polyhedra and their faces, edges, and vertices; circles and their radii, perimeters, and areas; areas in the plane and their bounding curves. Repeated reasoning occurs when exploring the sum of interior angles for polygons with different numbers of sides, discovering Euler's formula *V* – *E* + *F* = 2 (see figure), exploring possible tilings of the plane with regular polygons, and more.

For instance, a "guess my rule" game (for the sequence –6, –13, –26, –45,...), followed by "predict the 100th number in the sequence," can lead to a rich exploration of quadratics and the meaning and impact of the quadratic, linear, and constant terms—and eventually to the quadratic function $f(x) = -3x^2 + 2x - 5$. Carefully-designed prompts and/or a series of "guess my rule" constraints can help student teams discover the relationship between the coefficient of x^2 and the constant second difference of a sequence (here, the constant second difference of the sequence is –6, so the coefficient of x^2 is –3). Further exploration, perhaps graphical, can uncover the idea of finding a linear function to add to $-3x^2$ so that the sum generates the original sequence for whole-number inputs.

Exploring the general behavior of $f(x)$ could be motivated by comparing sequences, using questions like “which sequence will have a higher value in the long run? How do you know?”

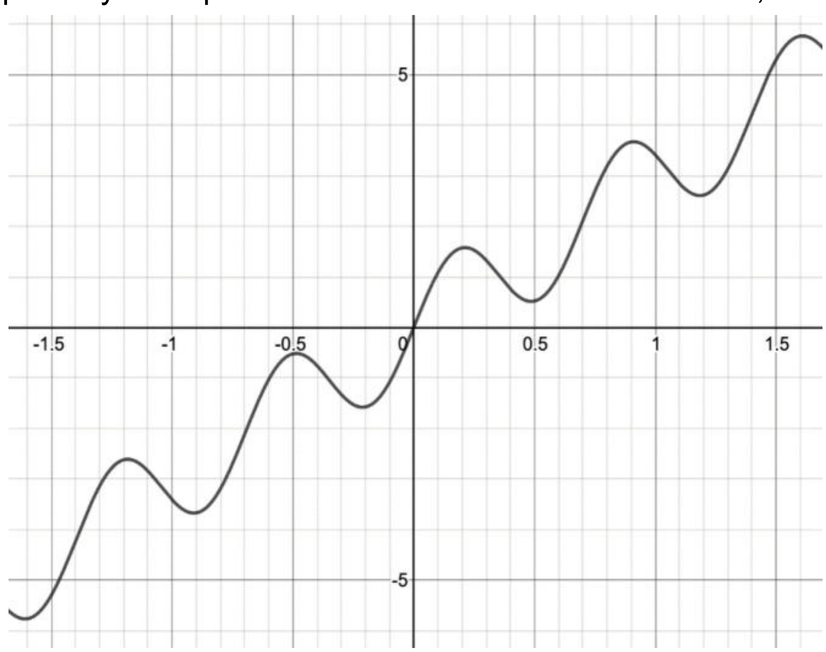


To try to predict the general behavior (that is, the shape of the graph) of $f(x)$, student teams should consider the known shape of the graph of $g(x) = x^2$, explore what happens to the graph if they multiply every output value by 3 and then take the opposite of every output, then perhaps sketch the two functions $h(x) = -3x^2$ and $m(x) = 2x$ both on a plane and add the output values for many sample values for x, to get a sense for the shape of $n(x) = -3x^2 + 2x$. Sharing strategies, and being accountable for understanding and using other teams’ strategies, will ensure that students have ample opportunity to connect across approaches and be prepared to notice patterns and repeated reasoning when tackling similar problems.

It is important to note that producing by hand a reasonably accurate graph of a function given by a formula is not a goal in its own right. Instead, it can be a means towards the end of deeply and flexibly understanding the meaning of a graph and the relationship between a function, its graph, the points on the graph, and the context that generated the function.

//callout box// Every student should also have easy access and frequent opportunities to use computer algebra systems to graph functions, thus focusing mental energy on interpretation and connection. //callout box//

Playing the "guess my rule" game several times (perhaps with a constraint of constant second differences) would have students noticing the similarity in what they are having to do each time. The point is not to become fast at sketching the graph of a quadratic, but to first notice, and then understand, the ways in which the different parts of the formula can be considered separately to help understand the whole. In other words, noticing repeated reasoning leads to the revealing of structure.



The "build this graph" example in the previous section may seem at first glance to be more difficult than understanding the structure of $f(x)$, since the parts are not necessarily as apparent as they are in the formula for $f(x)$. However, consider the graph to the right. If asked to describe the behavior of this function, students will offer ideas like "as x gets bigger, the function values generally get bigger; it wiggles up and down and generally goes up." A student team offering such a description has noted the two “parts” of this function’s behavior, and thus discovered some of its structure. They are well on their way to using graphing software in identifying $k(x) = 3x + \sin(9x)$ as a likely formula for this function.

Abstracting and generalizing from observed regularity and structure
Observing repetition in reasoning naturally leads to questions such as, “Do we have to keep doing the same thing with different numbers?” and, “What is the largest set of examples that we could apply this reasoning to?” Exploring either question involves examining structure. Students abstract an argument when they phrase it in terms of properties which might be shared by a number of objects or situations—thus paying attention to the structure of the objects or situations. They generalize when they extend an observation or known property to a larger class.

Several rounds of explorations such as the “guess my rule” example above could lead to any of the following abstractions and generalizations:

- The quadratic term in a quadratic function always dominates over time; that is, graphs of functions of the form $g(x) = ax^2 + bx + c$, where *a, b*, and *c* are real numbers, always have the shape of a parabola, and the parabola opens up or down depending on the sign of a.

- If g is as above and you compare $g(x), g(x+1)$, and $g(x+2)$, then the difference $g(x+2) - g(x+1)$ is $2a$ more than the difference $g(x+1) - g(x)$ (generalizing to non-integer “second differences”).
- To determine a quadratic function, you need to know at least four points on the graph because with just three you cannot decide whether the second differences are constant (note that this conjecture is not true, which means it raises a good opportunity for exploring possible justifications or critiques).
- When adding two functions, the *steepness (slope)* of the new function at each input value is also the sum of the two slopes (at that input) of the functions being added.

- When comparing two quadratics, the one with the faster-growing quadratic term (the larger a) always will be larger for large enough values of *x*, no matter what the linear and constant terms are.
- When comparing two polynomials, the one with the faster-growing quadratic term always wins in the long run (generalizing to polynomials from the smaller class of quadratics).

The “build this function” tasks above might lead to abstractions that are more along the lines of heuristics for understanding the structure of functions presented graphically:

- When trying to break down a graph, look at the largest-scale pattern you can see. If the graph generally goes in a straight line, like the $k(x) = 3x + \sin(9x)$ example, try to find that straight line and subtract it out.
- When trying to break down a graph, look at the most important pattern—the one that causes the biggest ups and/or downs (like the parabolic shape of the $f(x) = -3x^2 + 2x - 5$ example). Try to figure out the shape of that pattern, and subtract it out.
- If there is a periodic up-and-down in the graph, there’s probably a $\sin(ax)$ or $\cos(ax)$ in the formula.

Reasoning and communicating to share and justify

In many respects, mathematical knowledge and content understanding is developed and demonstrated socially; it is of little value to find a correct “solution” to a problem without the ability to communicate to others the validity and meaning of that solution, and we clarify our thinking through exchange with others. SMP 3 includes these aspects of the development of arguments: “They justify their conclusions, communicate them to others, and respond to the arguments of others.” In order to create an environment that makes mathematical practices such as SMP 3 accessible to all students, teachers should develop routines with students that support being able to communicate their thoughts and ideas, as well as work socially in a classroom of mixed language and math knowledge. Chapter 2 offers examples of such routines, including reflective discussions, peer revoicing routines, as well as teacher moves that support the creation of a mixed language mathematics community.

An important (implicit) aspect of SMP 3 is a recognition that the authority in mathematics lies within mathematical reasoning itself. Students come to own their understanding through constructing and critiquing arguments, and through this process increase their confidence and their sense of agency in mathematics. Classroom routines in which students must justify—or at least give evidence for—their abstractions or generalizations, and in which other students are responsible for questioning justifications and evidence, help to build the “am I convinced?” and “could I convince a reasonable skeptic?” meta-thinking that is at the heart of SMP 3. An example would be a mathematical implementation of the classroom routine “Claim, Evidence, and Reasoning (CER),” which is popular in science and writing instruction (McNeil & Martin, 2011; see https://my.nsta.org/collection/GBdqFKABr0U_E for science resources). Here, the different elements of an argument when investigating a problems are:

- Stating a claim
- Giving evidence for that claim
- Producing mathematical reasoning to support the claim

It is important to note that the mathematical reasoning here is of a different sort than scientific reasoning when CER is used in science: In science, the reasoning is for the purpose of connecting the evidence to the claim, explaining *why* the evidence supports the claim. On the other hand, the *mathematical* reasoning in the CER routine is expected to explain why (making use of structure) something is true *in general* (thus also explaining why the examples used as evidence are valid.

It is useful to name “giving evidence” and “producing reasoning” as separate processes, to distinguish between the noticing of pattern and structure (evidence) and the reasoning to support a general claim. For instance, in exploring a growth pattern, students might notice that the sum of three consecutive integers always seems to be divisible by three, and formulate that as a claim: “I think that whenever you add three numbers in a row, the answer is always a multiple of three.” When it’s clear the student means three consecutive *integers*, other students might check additional examples and contribute additional evidence. But the reasoning step requires something more: A numerical fluency argument (“If you take away one from the third number and add it to the first number, then you just have three times the middle number”), an algebraic argument (such as “if *a* is an integer, then $a + (a + 1) + (a + 2) = 3a + 3 = 3(a + 1)$”), or some other general argument.

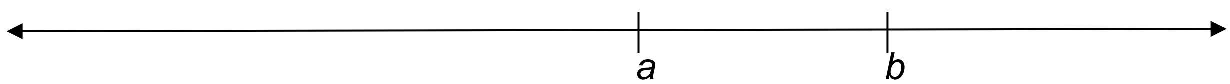
Carefully chosen number talks—well-known in the elementary math classroom—can be implemented in high school as a way to enable students to compare ideas and approaches with others in a low-stakes environment. They help to build SMP.1 (Make sense of problems and persevere in solving them) in addition to SMP.3. Well-chosen routine/tasks, such as number strings, can help build SMP.7 and SMP.8 by building from specific examples to thinking in terms of structure (abstraction) or larger classes (generalization).

For example, open number lines (blank, with no numbers marked), used with multiplication or division, can provide problems for number talks or strings that lead often to over-generalization—a great thing to happen, as it creates skepticism and forces a re-evaluation of evidence and a search for convincing justification.

High School snapshot: Number string on an open number line

Big Idea: What characteristics matter?

The teacher introduces the activity by drawing a long horizontal line on the board, with arrow heads at both ends, and placing two marks on the line, labeled a and b (with a to the left of b).



> I'd like you to think about where on the line I should place $a + b$. Should it go to the left of a, between a and b, or to the right of b?

After most students give thumbs-up in front of their chests (this signal for "I've got a strategy or explanation"), the teacher explores with the students and discovers that most students have tried several possible values for each variable, and concluded that $a + b$ must be to the right of b. A few students, however, are having trouble not blurting out. The teacher calls on one of these students:

> Teacher: Angel, you are shaking your head. Why is that?
>
> Angel: Because –1 + 2.

Quite a few students have an, "Oh, I didn't think about that" look on their faces. After further sharing, every student generates examples for each possible placement of $a + b$. Finally, the teacher moves from the number talk into a more-involved team activity, asking—given specific numbers a and b—how to tell where to place $a + b$. The class generates these generalizations (assuming a and b are real numbers, and $a < b$):

- If a and b are both positive, then $a + b$ is greater than b
- If a and b are both negative, then $a + b$ is less than a
- If a is negative and b is positive, then $a + b$ is between a and b

In pairs, students generate informal justifications for each of these (which are then refined whole-class; for instance, for the third one: b is positive, so adding it to a moves to the right of a. So, $a + b$ is greater than a. And a is negative, so adding it to b moves to the left of b. So, $a + b$ is less than b.

The students think they are done, but the teacher assures them that their list of possibilities is incomplete. One student volunteers the idea that perhaps *b* could be negative and *a* positive; other students point out that this is impossible given the original condition that *a* is to the left of *b* on the number line. Ultimately, one pair realizes that one of *a* or *b* could be 0, and students modify their list of statements to include these possibilities. The teacher asks: "Is there anything I could add to the number line that would make it possible to answer the original question?"

Students quickly agree that if they knew where zero was, they could answer the question. At the next math talk opportunity, the teacher again draws a number line with just *a* and *b* marked on it as before, and asks students this time to think about where $a \cdot b$ should go. After wait time and thumbs, the question is: "What different kinds of numbers do you expect to matter?"

Students discuss in pairs, and most believe that it matters whether *a* and *b* are positive or negative. Some share examples $-2 \cdot -4$ is greater than both –2 and –4; $-3 \cdot 5$ is less than both factors. A few pairs consider what happens if one factor is zero. After these considerations are offered and recorded, the teacher asks:

> So, if I tell you where zero is, you think you can place $a \cdot b$ on the line?

Many students say yes or nod; nobody disagrees. The teacher places zero on the number line to the left of *a*, and invites pairs of students to formulate statements about the relationship of $a \cdot b$ to *a* and *b*, along the lines of the previous session's statements about addition. Most pairs do not consider non-integer values for *a* and *b*, and generate statements such as:

- If *a* and *b* are both positive, then $a \cdot b$ is greater than *b.*

Some pairs have noticed that if $a = 1$, then the above statement is not true; the class modifies the statement to address this case (either by excluding $a = 1$ or by adding "or equal to" to the conclusion). If no pairs consider the possibility of *a* between 0 and 1, the teacher might prompt:

> There are some types of numbers I'm worried about that we haven't considered yet.

This work leads to a significant investigation of statements that can be made and justified about the relative locations on the number line of a, b, and $a + b$, $a \cdot b$, $a - b$, or $a \div b$.

Notice several important features of this number string (leading to extended investigation): The number line is a familiar mathematical representation that can be explored to a great depth. Students easily generate their own examples to engage in wondering about the posed questions, and these examples lead to tempting generalizations (conjectures). Some of those generalizations turn out to be false, forcing students to examine a broader set of examples and to look for structure to explain why they are false and how to fix them. Different generalizations will arise in different student teams, leading to a need to justify and to critique others' arguments.

Additional types of activities can create in students the need to reason and communicate as ways to support explanations and justifications. These include producing reports, videos, or materials to model for others (for example, to parents or to the next-younger class); prediction and estimation activities; and creating contexts. The last—creating real-life or puzzle-based contexts generating given mathematics such as a given function type—help to cultivate meta-thinking about structure (what are the parts of a quadratic function and how might I recreate them in a puzzle or find them in a real-life setting) and to develop a way of seeing the world through the lens of mathematics.

The CA CCSSM identify two particular proof methods in SMP 3.1: Proof by contradiction and proof by induction. The logic of proof by contradiction is straightforward to students: "No, that can't be, because if it was true, then…." The standard high school examples are proofs that $\sqrt{2}$ is irrational, and that there are infinitely many prime integers. These are both clear examples. Although the second of these two does not actually require a proof by contradiction, the proof below is most easily understood when worked out through the contradiction framework: "What would happen if there were only finitely many primes?"

The difficulty is to embed such proofs in a context that prompts a wondering, a need to know, on the part of students; and then to uncover the steps of the argument in such a way so as not to seem pulled out of thin air. Some approaches attempt to motivate with historical contexts, others with patterns. For example, suppose we already have established that every natural number greater than 1 is either prime or is a product of two or more prime factors. “Maybe 2, 3, 5, 7, 11, and 13 are all the primes we need to make all integers! No? Well, maybe if we add 17 to the set we have them all?” When students get tired of the repeated reasoning of finding an integer that is not a product of the given primes, either students or the teacher can ask whether there might always be a way of finding an integer that is not a product of integers in the given finite set. This gives an opening for a proof by contradiction: Let’s pretend (assume) that there are only finitely many primes—let’s say *n* of them. Why don’t we call them $p_1, p_2, p_3, \ldots, p_n$. Can you write down an expression for a natural number that is not divisible by any of these primes? To eventually arrive at a proof requires constructing an integer that can’t possibly be divisible by any of $p_1, p_2, \ldots, p_n$—Euclid’s choice (call it s) was the product of all of them, plus 1: $s = p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1$. Once an argument is found that s is not divisible by any of $p_1, p_2, p_3, \ldots, p_n$, then since s must be either a prime or a product of 2 or more prime factors that are not in the list $p_1, p_2, p_3, \ldots, p_n$, we have found a contradiction to our initial assumption that $p_1, p_2, p_3, \ldots, p_n$ contains all primes. Thus, the list of primes cannot be finite.

The logic of proof by induction is also straightforward when described informally: The first case is true, and whenever one case is true, the next one is true as well. Thus, the chain goes on forever. Such chains of statements, and wonder about whether they go on forever, might be easier to motivate from patterns than proof by contradiction. For instance, students might notice, in the context of exploring quadratic functions, that whenever they substitute an odd integer in for *x* in the function $f(x) = x^2 - 1$, they obtain an output that is a multiple of 8. This naturally leads to the questions, “Is this really true for all odd integers *x*?” and, “Could I use the fact that it’s true for $x = 5$ to show that it’s true for $x = 7$?” The formalism of representing “the next odd number” after x as $x + 2$ follows relatively naturally, and “using one case to prove the next” can proceed. This example should be accompanied by the question, “Why doesn’t the argument work for even integers?”

As described here, “proof” in high school does not originate with purely mathematical claims put forth by curriculum or by the teacher (“Prove that alternate interior angles are congruent”), nor with formal axioms and rules of logic. Rather, proof originates, like all mathematics, with a need to understand—in the case of proof, a need to understand why an observed phenomenon is true and that it is true for a defined range of cases. It is not enough that the curriculum writer or the teacher understand, and wishes for students to understand. The need to understand—and to understand why—must be authentic to students for learning to be deep and lasting. Thus, it is important that students’ experiences with constructing and critiquing arguments (SMP 3)—including their experiences with formal proof—be embedded as much as possible within a process beginning with wonder about a context and ending with a social and intellectual need to understand and justify:

1. Exploring authentic mathematical contexts
2. Discovering regularity in repeated reasoning and structure
3. Abstracting and generalizing from observed regularity and structure
4. Reasoning and communicating with and about mathematics in order to share and justify conclusions

Conclusion

This chapter focuses on key ideas that bring the Standards for Mathematical Practice to life. The content focuses on three interrelated practices: 1) Constructing viable arguments and critiquing the reasoning of others; 2) Looking for and making use of structure; and 3) Looking for and expressing regularity in repeated reasoning. By considering these practices together, the chapter focuses on the foundations of classroom experiences that center exploring, discovering, and reasoning with and about mathematics. While this chapter illustrates the integration of three mathematical practices, in fact *all* SMPs must be taught in an integrated way throughout the year. This vision for teaching and learning mathematics comes out of a several decades-long national push in mathematics education to pay more attention to supporting K–12 students in becoming powerful users of mathematics to help make sense of their world.

The chapter explores the practices across the elementary-, middle-, and high-school grade bands. Included below is an example tracing students' as they progress with the mathematical practices, including some ways in which contexts for learning and doing mathematics and the practices themselves might evolve over the grades. Note that socialization with these SMPs occurs through language, and so supports for developing language for reasoning and interacting with mathematics and others is central to these progressions.

Across the grades, students use everyday contexts and examples in order to explore, discover, and reason with and about mathematics. At the early grades, everyday contexts might come from familiar activities that children engage in at home, at school and within their community. These contexts might include imagined play or familiar celebrations with friends, siblings, or cousins; and familiar places such as a park, playground, zoo, or school itself. Meaningful contexts are also those that center notions of fairness and justice, such as issues related to the environment, social policies, or particular problems faced in the community. As teachers better know their students and the communities they represent and those create in classrooms, the contexts that matter to young children come to the fore.

In the middle grades, the contexts that are relevant to students continue to include—but increasingly go beyond—local, everyday activities and interactions. Middle-school students might begin to explore publicly available datasets on current events of interest, use familiar digital tools to explore the mathematics around them, and explore mathematical topics within everyday contexts like purchasing snacks with friends, playing or watching sports, or saving money. By the time they reach high school, students have acquired a wide array of contexts to explore, increasingly understanding society and the world around them through explorations in data, number, and space.

As noted in the CA CCSSM, the SMPs span the entirety of K–12. They develop in relation to progressions in mathematics content. At the elementary level, students work with numbers with which they are currently familiar, and begin to explore the structure of place value, patterns in our base-ten number system (such as even and odd numbers), and mathematical relationships (such as different ways to decompose numbers or relationships between addition and multiplication). Through these explorations, young students conjecture, explain, express agreement and disagreement, and come to make sense of data, number, and shapes.

Students in middle school build on these early experiences to deepen their interactions with mathematics and with others as they do mathematics together. During the elementary grades, students typically draw on contexts and on concrete manipulatives and representations in order to engage in mathematical reasoning and argumentation. At the middle school level, students continue to reason with such concrete referents, and also begin to draw on symbolic representations (such as expressions and equations), graphs, and other representations which have become familiar enough that students experience them as concrete. Middle-school students deepen their opportunities for sense-making as they move into ratios and proportional relationships, expressions and equations, geometric reasoning, and data.

In high school, students continue to build on earlier experiences as they make sense of functions and ways of representing functions, relationships between geometric objects and their parts, and data arising in contexts of interest. As students grow through years of making sense of and communicating about mathematics with one another and the teacher, the same practices that cut across grades K–12 emerge at developmentally and mathematically appropriate levels.

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