Short Response/Gridded Response

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

11. The speed a tsunami, or tidal wave, can travel is modeled by the equation $s = 356\sqrt{d}$, where s is the speed in kilometers per hour and d is the average depth of the water in kilometers. A tsunami is found to be traveling at 145 kilometers per hour. What is the average depth of the water? Round to the nearest hundredth.

12. GRIDDED RESPONSE Suppose you deposit $500 in an account paying 4.5% interest compounded semiannually. Find the dollar value of the account rounded to the nearest penny after 10 years.

13. In order to remain healthy, a horse requires 10 pounds of hay per day.

a. Write an equation to represent the amount of hay needed to sustain x horses for d days.

b. Is your equation a direct, joint, or inverse variation? Explain.

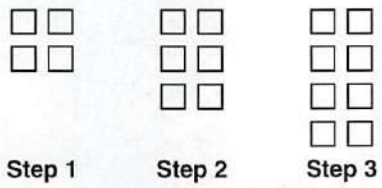
c. How much hay do three horses need for the month of July?

14. GRIDDED RESPONSE What is the radius of the circle with equation $x^2 + y^2 + 8x + 8y + 28 = 0$?

15. Anna is training to run a 10-kilometer race. The table below lists the times she received in different races. The times are listed in minutes. What was her mean time in minutes for a 10-kilometer race?

7.25	8.10
7.40	6.75
7.20	7.35
7.10	7.25
8.00	7.45

16. GRIDDED RESPONSE The pattern of squares below continues infinitely, with more squares being added at each step. How many squares are in the tenth step?



Extended Response

Record your answers on a sheet of paper. Show your work.

17. Amanda's hours at her summer job for one week are listed in the table below. She earns $6 per hour.

Amanda's Work Hours	
Sunday	0
Monday	6
Tuesday	4
Wednesday	0
Thursday	2
Friday	6
Saturday	8

a. Write an expression for Amanda's total weekly earnings.

b. Evaluate the expression from part **a** by using the Distributive Property.

c. Michael works with Amanda and also earns $6 per hour. If Michael's earnings were $192 this week, write and solve an equation to find how many more hours Michael worked than Amanda.

Need Extra Help?																	
If you missed Question...	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Go to Lesson or Page...	13-1	1-3	13-2	2-4	13-3	3-5	13-4	4-1	5-1	6-2	7-3	8-2	9-5	10-3	12-2	11-2	1-3

CHAPTER 14

Trigonometric Identities and Equations

Then

In Chapter 13, you graphed trigonometric functions and determined the period, amplitude, phase shifts, and vertical shifts.

Now

In Chapter 14, you will:

- Use and verify trigonometric identities.
- Use the sum and difference of angles identities.
- Use the double- and half-angle identities.
- Solve trigonometric equations.

Why?

ELECTRONICS Many aspects of electronics can be modeled by trigonometric functions. Radio, television, cellular telephones, and wireless Internet all communicate through radio waves that are modeled by trigonometric functions. The amount of power in an electronic gadget can be found by using a trigonometric equation.

Math *in Motion,* Animation glencoe.com

Get Ready for Chapter 14

Diagnose Readiness You have two options for checking Prerequisite Skills.

Text Option

Take the Quick Check below. Refer to the Quick Review for help.

QuickCheck

Factor completely. If the polynomial is not factorable, write *prime*. (Lesson 6-5)

1. $-16a^2 + 4a$
2. $5x^2 - 20$
3. $x^3 + 9$
4. $2y^2 - y - 15$
5. **GEOMETRY** The area of a rectangular piece of cardboard is $x^2 + 6x + 8$ square inches. If the cardboard has a length of $(x + 4)$ inches, what is the width?

Solve each equation by factoring. (Lesson 5-3)

6. $x^2 + 6x = 0$
7. $x^2 + 2x - 35 = 0$
8. $x^2 - 9 = 0$
9. $x^2 - 7x + 12 = 0$
10. **GARDENING** Peyton is building a flower bed in her back yard. The area of the flower bed will be 42 square feet. Find the possible values for x.

Find the exact value of each trigonometric function. (Lesson 13-3)

11. sin 45°
12. cos 225°
13. tan 150°
14. sin 120°
15. **RIDES** The distance from the highest point of a Ferris wheel to the ground can be found by multiplying 90 feet by sin 90°. What is the height of the Ferris wheel when it is halfway between the tallest point and the ground?

QuickReview

EXAMPLE 1

Factor $x^3 + 2x^2 - 24x$ completely.

$x^3 + 2x^2 - 24x = x(x^2 + 2x - 24)$

The product of the coefficients of the x terms must be -24, and their sum must be 2. The product of 6 and -4 is -24 and their sum is 2.

$x(x^2 + 2x - 24) = x(x + 6)(x - 4)$

EXAMPLE 2

Solve $x^2 + 6x + 5 = 0$ by factoring.

$x^2 + 6x + 5 = 0$ **Original equation**

$(x + 5)(x + 1) = 0$ **Factor.**

$x + 5 = 0$ or $x + 1 = 0$

$x = -5$ $\quad$ $x = -1$

The solution set is $\{-5, -1\}$.

EXAMPLE 3

Find the exact value of cos 135°.

The reference angle is 180° − 135° or 45°. cos 45° is $\frac{\sqrt{2}}{2}$. Since 135° is in the second quadrant, cos 135° $= -\frac{\sqrt{2}}{2}$.

Online Option

Math Online Take a self-check Chapter Readiness Quiz at glencoe.com.

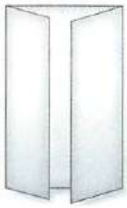
Get Started on Chapter 14

You will learn several new concepts, skills, and vocabulary terms as you study Chapter 14. To get ready, identify important terms and organize your resources. You may wish to refer to **Chapter 0** to review prerequisite skills.

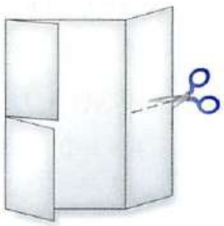
FOLDABLES® Study Organizer

Trigonometric Identities and Equations Make this Foldable to help you organize your Chapter 14 notes about trigonometric identities and equations. Begin with one sheet of 11″ × 17″ paper and four sheets of grid paper.

1. **Fold** the short sides of the 11″ × 17″ paper to meet in the middle.

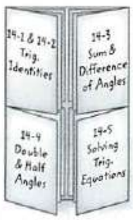


2. **Cut** each tab in half as shown.



3. **Cut** four sheets of grid paper in half and fold the half-sheets in half.

4. **Insert** two folded half-sheets under each of the four tabs and staple along the fold. Label each tab as shown.



New Vocabulary

English		Español
trigonometric identity	p. 891	identidad trigométrica
quotient identity	p. 891	identidad de cociente
reciprocal identity	p. 891	identidad recíproca
Pythagorean identity	p. 891	identidad Pitagórica
cofunction identity	p. 891	identidad de función conjunta
negative angle identity	p. 891	identidad negativa de ángulo
trigonometric equation	p. 919	ecuación trigométrica

Review Vocabulary

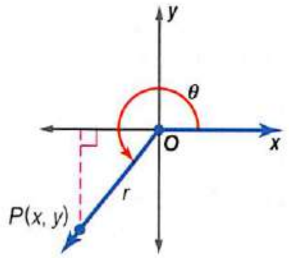
formula • p. 6 • fórmula a mathematical sentence that expresses the relationship between certain quantities

identity • p. 229 • identidad an equality that remains true regardless of the values of any variables that are in it

trigonometric functions • p. 808 • funciones trigonométricas For any angle, with measure θ, a point $P(x, y)$ on its terminal side, $r = \sqrt{x^2 + y^2}$, the trigonometric functions of θ are as follows.

$\sin \theta = \frac{y}{r}$ $\cos \theta = \frac{x}{r}$ $\tan \theta = \frac{y}{x}$

$\csc \theta = \frac{r}{y}$ $\sec \theta = \frac{r}{x}$ $\cot \theta = \frac{x}{y}$



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- Get extra help from your own **Personal Tutor**
- Use **Extra Examples** for additional help
- Take a **Self-Check Quiz**
- **Review Vocabulary** in fun ways

14-1 Trigonometric Identities

Then
You evaluated trigonometric functions. (Lesson 13-7)

Now
- Use trigonometric identities to find trigonometric values.
- Use trigonometric identities to simplify expressions.

New Vocabulary
trigonometric identity

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Why?

The amount of light that a source provides to a surface is called the *illuminance*. The illuminance E in foot candles on a surface is related to the distance R in feet from the light source.

The formula $\sec \theta = \frac{I}{ER^2}$, where I is the intensity of the light source measured in candles and θ is the angle between the light beam and a line perpendicular to the surface, can be used in situations in which lighting is important, as in photography.

Find Trigonometric Values The equation above can also be written as $E = \frac{I \cos \theta}{R^2}$. This is an example of a trigonometric identity. A **trigonometric identity** is an equation involving trigonometric functions that is true for all values for which every expression in the equation is defined.

If you can show that a specific value of the variable in an equation makes the equation false, then you have produced a *counterexample*. It only takes one counterexample to prove that an equation is not an identity.

Key Concept — Basic Trigonometric Identities (For Your FOLDABLE)

Quotient Identities

$\tan \theta = \frac{\sin \theta}{\cos \theta}$, $\cos \theta \neq 0$

$\cot \theta = \frac{\cos \theta}{\sin \theta}$, $\sin \theta \neq 0$

Reciprocal Identities

$\sin \theta = \frac{1}{\csc \theta}$, $\csc \theta \neq 0$

$\csc \theta = \frac{1}{\sin \theta}$, $\sin \theta \neq 0$

$\cos \theta = \frac{1}{\sec \theta}$, $\sec \theta \neq 0$

$\sec \theta = \frac{1}{\cos \theta}$, $\cos \theta \neq 0$

$\tan \theta = \frac{1}{\cot \theta}$, $\cot \theta \neq 0$

$\cot \theta = \frac{1}{\tan \theta}$, $\tan \theta \neq 0$

Pythagorean Identities

$\cos^2 \theta + \sin^2 \theta = 1$ $\quad$ $\tan^2 \theta + 1 = \sec^2 \theta$ $\quad$ $\cot^2 \theta + 1 = \csc^2 \theta$

Cofunction Identities

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$ $\quad$ $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$ $\quad$ $\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$

Negative Angle Identities

$\sin(-\theta) = -\sin \theta$ $\quad$ $\cos(-\theta) = \cos \theta$ $\quad$ $\tan(-\theta) = -\tan \theta$

The identity $\tan \theta = \frac{\sin \theta}{\cos \theta}$ is true except for angle measures such as 90°, 270°, … , $90° + k180°$, where k is an integer. The cosine of each of these angle measures is 0, so $\tan \theta$ is not defined when $\cos \theta = 0$. These identities are sometimes called *quotient identities*. An identity similar to this is $\cot \theta = \frac{\cos \theta}{\sin \theta}$.

You can use trigonometric identities to find exact values of trigonometric functions. You can find approximate values by using a graphing calculator.

EXAMPLE 1 **Use Trigonometric Identities**

a. **Find the exact value of $\cos\theta$ if $\sin\theta = \frac{1}{4}$ and $90° < \theta < 180°$.**

$\cos^2\theta + \sin^2\theta = 1$	Pythagorean identity
$\cos^2\theta = 1 - \sin^2\theta$	Subtract $\sin^2\theta$ from each side.
$\cos^2\theta = 1 - \left(\frac{1}{4}\right)^2$	Substitute $\frac{1}{4}$ for $\sin\theta$.
$\cos^2\theta = 1 - \frac{1}{16}$	Square $\frac{1}{4}$.
$\cos^2\theta = \frac{15}{16}$	Subtract: $\frac{16}{16} - \frac{1}{16} = \frac{15}{16}$.
$\cos\theta = \pm\frac{\sqrt{15}}{4}$	Take the square root of each side.

Since θ is in the second quadrant, $\cos\theta$ is negative. Thus, $\cos\theta = -\frac{\sqrt{15}}{4}$.

CHECK Use a calculator to find an approximate answer.

Step 1 Find Arcsin $\frac{1}{4}$.

$\sin^{-1}\frac{1}{4} \approx 14.48°$ Use a calculator.

Because $90° < \theta < 180°$, $\theta \approx 180° - 14.48°$ or about $165.52°$.

Step 2 Find $\cos\theta$.
Replace θ with $165.52°$.
$\cos 165.52° \approx -0.97$

Step 3 Compare with the exact value.

$-\frac{\sqrt{15}}{4} \stackrel{?}{\approx} 0.97$

$-0.968 \approx 0.97$ ✓

b. **Find the exact value of $\csc\theta$ if $\cot\theta = -\frac{3}{5}$ and $270° < \theta < 360°$.**

$\cot^2\theta + 1 = \csc^2\theta$	Pythagorean identity
$\left(-\frac{3}{5}\right)^2 + 1 = \csc^2\theta$	Substitute $-\frac{3}{5}$ for $\cot\theta$.
$\frac{9}{25} + 1 = \csc^2\theta$	Square $-\frac{3}{5}$.
$\frac{34}{25} = \csc^2\theta$	Add: $\frac{9}{25} + \frac{25}{25} = \frac{34}{25}$.
$\pm\frac{\sqrt{34}}{5} = \csc\theta$	Take the square root of each side.

Since θ is in the fourth quadrant, $\csc\theta$ is negative. Thus, $\csc\theta = -\frac{\sqrt{34}}{5}$.

Check Your Progress

1A. Find $\sin\theta$ if $\cos\theta = \frac{1}{3}$ and $270° < \theta < 360°$.

1B. Find $\sec\theta$ if $\sin\theta = -\frac{2}{7}$ and $180° < \theta < 270°$.

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StudyTip

Quadrants Here is a table to help you remember which ratios are positive and which are negative in each quadrant.

Function	+	−
$\sin\theta$	1, 2	3, 4
$\cos\theta$	1, 4	2, 3
$\tan\theta$	1, 3	2, 4
$\csc\theta$	1, 2	3, 4
$\sec\theta$	1, 4	2, 3
$\cot\theta$	1, 3	2, 4

Simplify Expressions Simplifying an expression that contains trigonometric functions means that the expression is written as a numerical value or in terms of a single trigonometric function, if possible.

StudyTip

Simplifying It is often easiest to write all expressions in terms of sine and/or cosine.

EXAMPLE 2 Simplify an Expression

Simplify $\frac{\sin\theta\csc\theta}{\cot\theta}$.

$\frac{\sin\theta\csc\theta}{\cot\theta} = \frac{\sin\theta\frac{1}{\sin\theta}}{\frac{1}{\tan\theta}}$ $\csc\theta = \frac{1}{\sin\theta}$ and $\cot\theta = \frac{1}{\tan\theta}$

$= \frac{1}{\frac{1}{\tan\theta}}$ $\frac{\sin\theta}{\sin\theta} = 1$

$= \frac{1}{1}\cdot\frac{\tan\theta}{1}$ or $\tan\theta$ $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b}\cdot\frac{d}{c}$

Check Your Progress

Simplify each expression.

2A. $\frac{\tan^2\theta\csc^2\theta - 1}{\sec^2\theta}$

2B. $\frac{\sec\theta}{\sin\theta}(1 - \cos^2\theta)$

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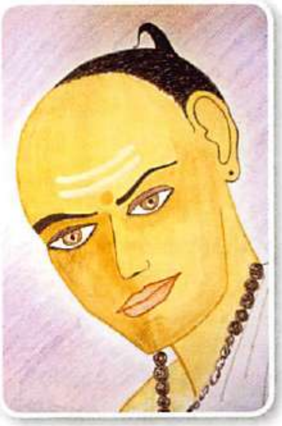

Math History Link

Aryabhatta (476–550 A.D.) Among Indian mathematicians, Aryabhatta is probably the most famous. His name is closely associated with trigonometry. He was the first to introduce inverse trig functions and spherical trigonometry. Aryabhatta also calculated approximations for pi and trig functions.

Simplifying trigonometric expressions can be helpful when solving real-world problems.

Real-World EXAMPLE 3 Simplify and Use an Expression

LIGHTING **Refer to the beginning of the lesson.**

a. Solve the formula in terms of E.

$\sec\theta = \frac{I}{ER^2}$ Original equation

$ER^2\sec\theta = I$ Multiply each side by ER^2.

$ER^2\frac{1}{\cos\theta} = I$ $\frac{1}{\cos\theta} = \sec\theta$

$\frac{E}{\cos\theta} = \frac{I}{R^2}$ Divide each side by R^2.

$E = \frac{I\cos\theta}{R^2}$ Multiply each side by $\cos\theta$.

b. Is the equation in part a equivalent to $R^2 = \frac{I\tan\theta\cos\theta}{E}$? Explain.

$R^2 = \frac{I\tan\theta\cos\theta}{E}$ Original equation

$ER^2 = I\tan\theta\cos\theta$ Multiply each side by E.

$E = \frac{I\tan\theta\cos\theta}{R^2}$ Divide each side by R^2.

$E = \frac{I\frac{\sin\theta}{\cos\theta}\cos\theta}{R^2}$ $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$E = \frac{I\sin\theta}{R^2}$ Simplify.

No; the equations are not equivalent. $R^2 = \frac{I\tan\theta\cos\theta}{E}$ simplifies to $E = \frac{I\sin\theta}{R^2}$.

Check Your Progress

3. Rewrite $\cot^2\theta - \tan^2\theta$ in terms of $\sin\theta$.

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Check Your Understanding

Example 1 p. 892

Find the exact value of each expression if $0° < \theta < 90°$.

1. If $\cot \theta = 2$, find $\tan \theta$.

2. If $\sin \theta = \frac{4}{5}$, find $\cos \theta$.

3. If $\cos \theta = \frac{2}{3}$, find $\sin \theta$.

4. If $\cos \theta = \frac{2}{3}$, find $\csc \theta$.

Example 2 p. 893

Simplify each expression.

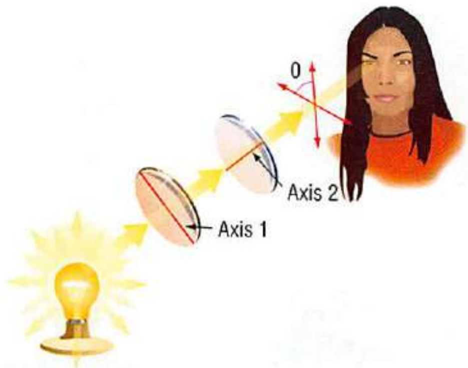
5. $\tan \theta \cos^2 \theta$

6. $\csc^2 \theta - \cot^2 \theta$

7. $\frac{\cos \theta \csc \theta}{\tan \theta}$

Example 3 p. 893

8. **OPTICS** When unpolarized light passes through polarized sunglass lenses, the intensity of the light is cut in half. If the light then passes through another polarized lens with its axis at an angle of θ to the first, the intensity of the light is again diminished. The intensity of the emerging light can be found by using the formula $I = I_0 - \frac{I_0}{\csc^2 \theta}$, where I_0 is the intensity of the light incoming to the second polarized lens, I is the intensity of the emerging light, and θ is the angle between the axes of polarization.



a. Simplify the formula in terms of $\cos \theta$.

b. Use the simplified formula to determine the intensity of light that passes through a second polarizing lens with axis at 30° to the original.

Practice and Problem Solving

● = **Step-by-Step Solutions** begin on page R20.
Extra Practice begins on page 947.

Example 1 p. 892

Find the exact value of each expression if $0° < \theta < 90°$.

9. If $\cos \theta = \frac{3}{5}$, find $\csc \theta$.

10. If $\sin \theta = \frac{1}{2}$, find $\tan \theta$.

11. If $\sin \theta = \frac{3}{5}$, find $\cos \theta$.

12. If $\tan \theta = 2$, find $\sec \theta$.

Find the exact value of each expression if $180° < \theta < 270°$.

13. If $\cos \theta = -\frac{3}{5}$, find $\csc \theta$.

14. If $\sec \theta = -3$, find $\tan \theta$.

15. If $\cot \theta = \frac{1}{4}$, find $\csc \theta$.

16. If $\sin \theta = -\frac{1}{2}$, find $\cos \theta$.

Find the exact value of each expression if $270° < \theta < 360°$.

17. If $\cos \theta = \frac{5}{13}$, find $\sin \theta$.

18. If $\tan \theta = -1$, find $\sec \theta$.

19. If $\sec \theta = \frac{5}{3}$, find $\cos \theta$.

20. If $\csc \theta = -\frac{5}{3}$, find $\cos \theta$.

Example 2 p. 893

Simplify each expression.

21. $\sec \theta \tan^2 \theta + \sec \theta$

22. $\cos\left(\frac{\pi}{2} - \theta\right)\cot \theta$

23. $\cot \theta \sec \theta$

24. $\sin \theta (1 + \cot^2 \theta)$

25. $\sin\left(\frac{\pi}{2} - \theta\right)\sec \theta$

26. $\frac{\cos(-\theta)}{\sin(-\theta)}$

Example 3
p. 893

27. **ELECTRONICS** When there is a current in a wire in a magnetic field, such as in a hairdryer, a force acts on the wire. The strength of the magnetic field can be determined using the formula $B = \frac{F \csc \theta}{I\ell}$, where F is the force on the wire, I is the current in the wire, ℓ is the length of the wire, and θ is the angle the wire makes with the magnetic field. Rewrite the equation in terms of $\sin \theta$. (*Hint:* Solve for F.)

Simplify each expression.

28. $\frac{1 - \sin^2 \theta}{\sin^2 \theta}$ **29.** $\tan \theta \csc \theta$ **30.** $\frac{1}{\sin^2 \theta} - \frac{\cos^2 \theta}{\sin^2 \theta}$

31. $2(\csc^2 \theta - \cot^2 \theta)$ **32.** $(1 + \sin \theta)(1 - \sin \theta)$ **33.** $1 - 2\sin^2 \theta$

34. **SUN** The ability of an object to absorb energy is related to a factor called the emissivity e of the object. The emissivity can be calculated by using the formula $e = \frac{W \sec \theta}{AS}$, where W is the rate at which a person's skin absorbs energy from the Sun, S is the energy from the Sun in watts per square meter, A is the surface area exposed to the Sun, and θ is the angle between the Sun's rays and a line perpendicular to the body.

a. Solve the equation for W. Write your answer using only $\sin \theta$ or $\cos \theta$.

b. Find W if $e = 0.80$, $\theta = 40°$, $A = 0.75 \text{ m}^2$, and $S = 1000 \text{ W/m}^2$. Round to the nearest hundredth.

Real-World Link

Research has shown that skin cancer is related to Sun exposure. If current trends continue, 1 in 5 Americans will develop skin cancer during their lifetimes.

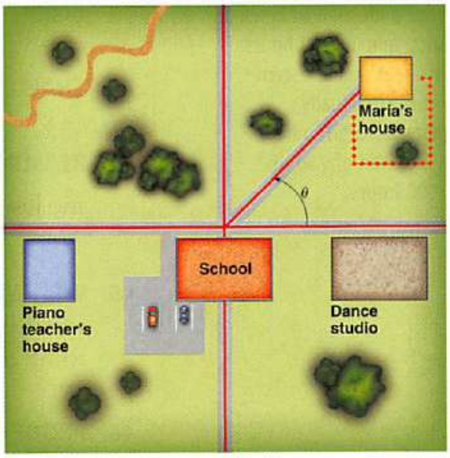
Source: Skin Cancer Net

35. **MAPS** The map shows some of the buildings in Maria's neighborhood that she visits on a regular basis. The sine of the angle θ formed by the roads connecting the dance studio, the school, and Maria's house is $\frac{4}{9}$.

a. What is the cosine of the angle?

b. What is the tangent of the angle?

c. What are the sine, cosine, and tangent of the angle formed by the roads connecting the piano teacher's house, the school, and Maria's house?



36. **MULTIPLE REPRESENTATIONS** In this problem, you will use a graphing calculator to determine whether an equation may be a trigonometric identity. Consider the trigonometric identity $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$.

a. **TABULAR** Complete the table below.

θ	0°	30°	45°	60°
$\tan^2 \theta - \sin^2$				
$\tan^2 \theta \sin^2 \theta$				

b. **GRAPHICAL** Use a graphing calculator to graph $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$ as two separate functions. Sketch the graph.

c. **ANALYTICAL** If the graphs of the two functions do not match, then the equation is not an identity. Do the graphs coincide?

d. **ANALYTICAL** Use a graphing calculator to determine whether the equation $\sec^2 x - 1 = \sin^2 x \sec^2 x$ may be an identity. (Be sure your calculator is in degree mode.)

37. SKIING A skier of mass m descends a θ-degree hill at a constant speed. When Newton's laws are applied to the situation, the following system of equations is produced: $F_n - mg\cos\theta = 0$ and $mg\sin\theta - \mu_k F_n = 0$, where g is the acceleration due to gravity, F_n is the normal force exerted on the skier, and μ_k is the coefficient of friction. Use the system to define μ_k as a function of θ.

Real-World Link

Compact fluorescent lights, or CFLs, typically have a lifespan of between 6000 and 15,000 hours, whereas incandescent lights are usually manufactured to have a life span of 750 to 1000 hours.

Source: Osram

Simplify each expression.

38. $\dfrac{\tan\left(\frac{\pi}{2} - \theta\right)\sec\theta}{1 - \csc^2\theta}$

39 $\dfrac{\cos\left(\frac{\pi}{2} - \theta\right) - 1}{1 + \sin(-\theta)}$

40. $\dfrac{\sec\theta\sin\theta + \cos\left(\frac{\pi}{2} - \theta\right)}{1 + \sec\theta}$

41. $\dfrac{\cot\theta\cos\theta}{\tan(-\theta)\sin\left(\frac{\pi}{2} - \theta\right)}$

H.O.T. Problems

Use Higher-Order Thinking Skills

42. FIND THE ERROR Clyde and Rosalina are debating whether an equation from their homework assignment is an identity. Clyde says that since he has tried ten specific values for the variable and all of them worked, it must be an identity. Rosalina argues that specific values could only be used as counterexamples to prove that an equation is not an identity. Is either of them correct? Explain your reasoning.

43. CHALLENGE Find a counterexample to show that $1 - \sin x = \cos x$ is *not* an identity.

44. REASONING Demonstrate how the formula about illuminance from the beginning of the lesson can be rewritten to show that $\cos\theta = \frac{ER^2}{I}$.

45. WRITING IN MATH Pythagoras is most famous for the Pythagorean Theorem. The identity $\cos^2\theta + \sin^2\theta = 1$ is an example of a Pythagorean identity. Why do you think that this identity is classified in this way?

46. PROOF Prove that $\tan(-a) = -\tan a$ by using the quotient and negative angle identities.

47. OPEN ENDED Write two expressions that are equivalent to $\tan\theta\sin\theta$.

48. REASONING Explain how you can use division to rewrite $\sin^2\theta + \cos^2\theta = 1$ as $1 + \cot^2\theta = \csc^2\theta$.

49. CHALLENGE Find $\cot\theta$ if $\sin\theta = \frac{3}{5}$ and $90° \leq \theta < 180°$.

50. FIND THE ERROR Jordan and Ebony are simplifying $\frac{\sin^2\theta}{\cos^2\theta + \sin^2\theta}$. Is either of them correct? Explain your reasoning.

Jordan

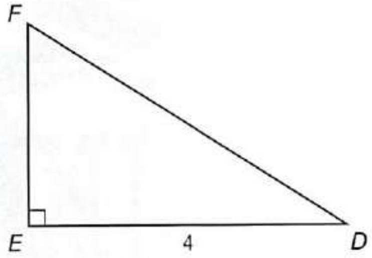
$$\frac{\sin^2\theta}{\cos^2\theta + \sin^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\sin^2\theta}$$
$$= \tan^2\theta + 1$$
$$= \sec^2\theta$$

Ebony

$$\frac{\sin^2\theta}{\cos^2\theta + \sin^2\theta} = \frac{\sin^2\theta}{1}$$
$$= \sin^2\theta$$

Standardized Test Practice

51. Refer to the figure below. If $\cos D = 0.8$, what is the length of $\overline{DF}$?



A 5

B 4

C 3.2

D $\frac{4}{5}$

52. PROBABILITY There are 16 green marbles, 2 red marbles, and 6 yellow marbles in a jar. How many yellow marbles need to be added to the jar in order to double the probability of selecting a yellow marble?

F 4

G 6

H 8

J 12

53. ACT/SAT Ella is 6 years younger than Amanda. Zoe is twice as old as Amanda. The total of their ages is 54. Which equation can be used to find Amanda's age?

A $x + (x - 6) + 2(x - 6) = 54$

B $x - 6x + (x + 2) = 54$

C $x - 6 + 2x = 54$

D $x + (x - 6) + 2x = 54$

54. Which of the following functions represents exponential decay?

F $y = 0.2(7)^x$

G $y = (0.5)^x$

H $y = 4(9)^x$

J $y = 5\left(\frac{4}{3}\right)^x$

Spiral Review

Find each value. Write angle measures in radians. Round to the nearest hundredth. (Lesson 13-9)

55. $\text{Cos}^{-1}\left(-\frac{1}{2}\right)$

56. $\text{Sin}^{-1}\frac{\pi}{2}$

57. $\text{Arctan}\,\frac{\sqrt{3}}{3}$

58. $\tan\left(\text{Cos}^{-1}\frac{6}{7}\right)$

59. $\sin\left(\text{Arctan}\,\frac{\sqrt{3}}{3}\right)$

60. $\cos\left(\text{Arcsin}\,\frac{3}{5}\right)$

61. TIDES The height of the water in a harbor rose to a maximum height of 15 feet at 6:00 P.M., and then dropped to a minimum level of 3 feet by 3:00 A.M. Assume that the water level can be modeled by the sine function. Write an equation that represents the height h of the water t hours after noon on the first day. (Lesson 13-8)

Evaluate the sum of each geometric series. (Lesson 11-3)

62. $\sum_{k=1}^{5} \frac{1}{4} \cdot 2^{k-1}$

63. $\sum_{k=1}^{7} 81\left(\frac{1}{3}\right)^{k-1}$

64. $\sum_{k=1}^{8} \frac{1}{3} \cdot 5^{k-1}$

Skills Review

Solve each equation. (Lesson 9-6)

65. $a + 1 = \frac{6}{a}$

66. $\frac{9}{t-3} = \frac{t-4}{t-3} + \frac{1}{4}$

67. $\frac{5}{x+1} - \frac{1}{3} = \frac{x+2}{x+1}$

14-2

Verifying Trigonometric Identities

Then

You used identities to find trigonometric values and simplify expressions. (Lesson 14-1)

Now

- Verify trigonometric identities by transforming one side of an equation into the form of the other side.
- Verify trigonometric identities by transforming each side of the equation into the same form.

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Why?

While running on a circular track, Lamont notices that his body is not perpendicular to the ground. Instead, it leans away from a vertical position. The nonnegative acute angle θ that Lamont's body makes with the vertical is called the *angle of incline* and is described by the equation $\tan \theta = \frac{v^2}{gR}$.

This is not the only equation that describes the angle of incline in terms of trigonometric functions. Another such equation is $\sin \theta = \cos \frac{v^2}{gR}\theta$, where $0 \leq \theta \leq 90°$. Are these two equations completely independent of one another or are they merely different versions of the same relationship?

Transform One Side of an Equation You can use the basic trigonometric identities along with the definitions of the trigonometric functions to verify identities. If you wish to show an identity, you need to show that it is true for all values of θ.

Key Concept — For Your FOLDABLE

Verifying Identities by Transforming One Side

Step 1 Simplify one side of an equation until the two sides of the equation are the same. It is often easier to work with the more complicated side of the equation.

Step 2 Transform that expression into the form of the simpler side.

EXAMPLE 1 Transform One Side of an Equation

Verify that $\frac{\sin^2 \theta}{1 - \cos \theta} = 1 + \cos \theta$ is an identity.

$$\frac{\sin^2 \theta}{1 - \cos \theta} \stackrel{?}{=} 1 + \cos \theta \quad \text{Original equation}$$

$$\frac{1 + \cos \theta}{1 + \cos \theta} \cdot \frac{\sin^2 \theta}{1 - \cos \theta} \stackrel{?}{=} 1 + \cos \theta \quad \text{Multiply the numerator and denominator by } 1 + \cos \theta.$$

$$\frac{\sin^2 \theta(1 + \cos \theta)}{1 - \cos^2 \theta} \stackrel{?}{=} 1 + \cos \theta \quad (1 + \cos \theta)(1 - \cos \theta) = 1 - \cos^2 \theta$$

$$\frac{\sin^2 \theta(1 + \cos \theta)}{\sin^2 \theta} \stackrel{?}{=} 1 + \cos \theta \quad \sin^2 \theta = 1 - \cos^2 \theta$$

$$1 + \cos \theta = 1 + \cos \theta \checkmark \quad \text{Divide the numerator and denominator by } \sin^2 \theta.$$

Check Your Progress

1. Verify that $\cot^2 \theta - \cos^2 \theta = \cot^2 \theta \cos^2 \theta$ is an identity.

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When you verify a trigonometric identity, you are really working backward. In Example 1, consider the last step $1 + \cos\theta = 1 + \cos\theta$. Since that step is clearly true, you can conclude that the next-to-last step is also true, and so on, all the way back to the original equation.

Watch Out!

Simplify Separately Verifying an identity is like checking the solution of an equation. You must simplify one or both sides separately until they are the same.

STANDARDIZED TEST EXAMPLE 2

$\dfrac{\cos\theta\csc\theta}{\tan\theta} =$

A $\cot\theta$ B $\csc\theta$ C $\cot^2\theta$ D $\csc^2\theta$

Read the Test Item

Find an expression that is always equal to the given expression. Notice that all of the answer choices involve either $\cot\theta$ or $\csc\theta$. So work toward eliminating the other trigonometric functions.

Solve the Test Item

Transform the given expression to match one of the choices.

$$\frac{\cos\theta\csc\theta}{\tan\theta} = \frac{\cos\theta\frac{1}{\sin\theta}}{\frac{\sin\theta}{\cos\theta}}$$ $\csc\theta = \frac{1}{\sin\theta}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$= \frac{\frac{\cos\theta}{\sin\theta}}{\frac{\sin\theta}{\cos\theta}}$$ Multiply.

$$= \frac{\cos\theta}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}$$ Invert the denominator and multiply.

$$= \cot\theta\cdot\cot\theta$$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

$$= \cot^2\theta$$ Multiply.

The answer is C.

Test-Taking Tip

Checking Answers Verify your answer by choosing values for θ. Then evaluate the original expression and compare to your answer choice.

Check Your Progress

2. $\tan^2\theta(\cot^2\theta - \cos^2\theta) =$

F $\cot^2\theta$ G $\tan^2\theta$ H $\cos^2\theta$ J $\sin^2\theta$

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Transform Each Side of an Equation Sometimes it is easier to transform each side of an equation separately into a common form. The following suggestions may be helpful as you verify trigonometric identities.

Key Concept Suggestions for Verifying Identities

For Your FOLDABLE

- Substitute one or more basic trigonometric identities to simplify the expression.
- Factor or multiply as necessary. You may have to multiply both the numerator and denominator by the same trigonometric expression.
- Write each side of the identity in terms of sine and cosine only. Then simplify each side as much as possible.
- The properties of equality do not apply to identities as with equations. Do not perform operations to the quantities from each side of an unverified identity.

EXAMPLE 3 Verify by Transforming Each Side

Verify that $1 - \tan^4 \theta = 2\sec^2 \theta - \sec^4 \theta$ is an identity.

$1 - \tan^4 \theta \stackrel{?}{=} 2 \sec^2 \theta - \sec^4 \theta$	Original equation
$(1 - \tan^2 \theta)(1 + \tan^2 \theta) \stackrel{?}{=} \sec^2 \theta (2 - \sec^2 \theta)$	Factor each side.
$[1 - (\sec^2 \theta - 1)] \sec^2 \theta \stackrel{?}{=} (2 - \sec^2 \theta) \sec^2 \theta$	$1 + \tan^2 \theta = \sec^2 \theta$
$(2 - \sec^2 \theta) \sec^2 \theta = (2 - \sec^2 \theta) \sec^2 \theta$ ✓	Simplify.

Check Your Progress

3. Verify that $\csc^2 \theta - \cot^2 \theta = \cot \theta \tan \theta$ is an identity.

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Check Your Understanding

Examples 1 and 3 pp. 898–900

Verify that each equation is an identity.

1. $\cot \theta + \tan \theta = \dfrac{\sec^2 \theta}{\tan \theta}$
2. $\cos^2 \theta = (1 + \sin \theta)(1 - \sin \theta)$
3. $\sin \theta = \dfrac{\sec \theta}{\tan \theta + \cot \theta}$
4. $\tan^2 \theta = \dfrac{1 - \cos^2 \theta}{\cos^2 \theta}$
5. $\tan^2 \theta \csc^2 \theta = 1 + \tan^2 \theta$
6. $\tan^2 \theta = (\sec \theta + 1)(\sec \theta - 1)$

Example 2 p. 899

7. **MULTIPLE CHOICE** Which expression can be used to form an identity with $\dfrac{\tan^2 \theta + 1}{\tan^2 \theta}$?

A $\sin^2 \theta$ B $\cos^2 \theta$ C $\tan^2 \theta$ D $\csc^2 \theta$

Practice and Problem Solving

● = **Step-by-Step Solutions** begin on page R20.
Extra Practice begins on page 947.

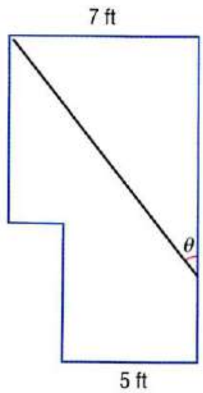
Example 1 p. 898

Verify that each equation is an identity.

8. $\cos^2 \theta + \tan^2 \theta \cos^2 \theta = 1$
9. $\cot \theta (\cot \theta + \tan \theta) = \csc^2 \theta$
10. $1 + \sec^2 \theta \sin^2 \theta = \sec^2 \theta$
11. $\sin \theta \sec \theta \cot \theta = 1$
12. $\dfrac{1 - \cos \theta}{1 + \cos \theta} = (\csc \theta - \cot \theta)^2$
13. $\dfrac{1 - 2\cos^2 \theta}{\sin \theta - \cos \theta} = \tan \theta - \cot \theta$
14. $\tan \theta = \dfrac{\sec \theta}{\csc \theta}$
15. $\cos \theta = \sin \theta \cot \theta$
16. $(\sin \theta - 1)(\tan \theta + \sec \theta) = -\cos \theta$
17. $\cos \theta \cos (-\theta) - \sin \theta \sin (-\theta) = 1$

Example 2 p. 899

18. **LADDER** Some students derived an expression for the length of a ladder that, when carried flat, could fit around a corner from a 5-foot-wide hallway into a 7-foot-wide hallway, as shown. They determined that the maximum length ℓ of a ladder that would fit was given by $\ell(\theta) = \dfrac{7 \sin \theta + 5 \cos \theta}{\sin \theta \cos \theta}$. When their teacher worked the problem, she concluded that $\ell(\theta) = 7 \sec \theta + 5 \csc \theta$. Are the two expressions equivalent?



Example 3
p. 900

Verify that each equation is an identity.

19. $\sec\theta - \tan\theta = \dfrac{1-\sin\theta}{\cos\theta}$

20. $\dfrac{1+\tan\theta}{\sin\theta+\cos\theta} = \sec\theta$

21. $\sec\theta\csc\theta = \tan\theta + \cot\theta$

22. $\sin\theta + \cos\theta = \dfrac{2\sin^2\theta - 1}{\sin\theta - \cos\theta}$

23. $(\sin\theta + \cos\theta)^2 = \dfrac{2+\sec\theta\csc\theta}{\sec\theta\csc\theta}$

24. $\dfrac{\cos\theta}{1-\sin\theta} = \dfrac{1+\sin\theta}{\cos\theta}$

25. $\csc\theta - 1 = \dfrac{\cot^2\theta}{\csc\theta+1}$

26. $\cos\theta\cot\theta = \csc\theta - \sin\theta$

27. $\sin\theta\cos\theta\tan\theta + \cos^2\theta = 1$

28. $(\csc\theta - \cot\theta)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$

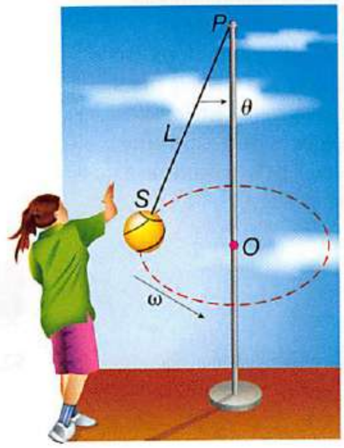
29. $\csc^2\theta = \cot^2\theta + \sin\theta\csc\theta$

30. $\dfrac{\sec\theta - \csc\theta}{\csc\theta\sec\theta} = \sin\theta - \cos\theta$

31. $\sin^2\theta + \cos^2\theta = \sec^2\theta - \tan^2\theta$

32. $\sec\theta - \cos\theta = \tan\theta\sin\theta$

33. TETHERBALL The diagram at the right represents a game of tetherball. As the ball rotates around the pole, a conical surface is swept out by line segment $\overline{SP}$. A formula for the relationship between the length L of the string and the angle θ that the string makes with the pole is given by the equation $L = \dfrac{g\sec\theta}{\omega^2}$. Is $L = \dfrac{g\tan\theta}{\omega^2\sin\theta}$ also an equation for the relationship between L and θ?



Real-World Link

Running games were organized in ancient Egypt as early as 3800 B.C. The first marathon race, which was 24 miles long, was held during the 1896 Olympic Games in Athens, Greece.

34. RUNNING A portion of a racetrack has the shape of a circular arc with a radius of 16.7 meters. As a runner races along the arc, the sine of her angle of incline θ is found to be $\frac{1}{4}$. Find the speed of the runner. Use the Angle of Incline Formula given at the beginning of the lesson, $\tan\theta = \dfrac{v^2}{gR}$, where $g = 9.8$ and R is the radius. (*Hint*: Find $\cos\theta$ first.)

When simplified, would the expression be equal to 1 or −1?

35. $\cot(-\theta)\tan(-\theta)$

36. $\sin\theta\csc(-\theta)$

37. $\sin^2(-\theta) + \cos^2(-\theta)$

38. $\sec(-\theta)\cos(-\theta)$

39 $\sec^2(-\theta) - \tan^2(-\theta)$

40. $\cot(-\theta)\cot\left(\frac{\pi}{2} - \theta\right)$

Simplify the expression to either a constant or a basic trigonometric function.

41. $\dfrac{\tan\left(\frac{\pi}{2}-\theta\right)\csc\theta}{\csc^2\theta}$

42. $\dfrac{1+\tan\theta}{1+\cot\theta}$

43. $(\sec^2\theta + \csc^2\theta) - (\tan^2\theta + \cot^2\theta)$

44. $\dfrac{\sec^2\theta - \tan^2\theta}{\cos^2 x + \sin^2 x}$

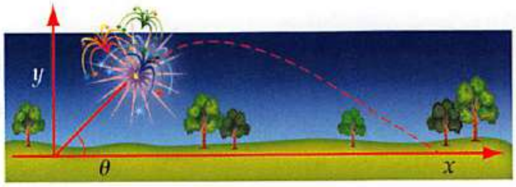
45. $\tan\theta\cos\theta$

46. $\cot\theta\tan\theta$

47. $\sec\theta\sin\left(\frac{\pi}{2} - \theta\right)$

48. $\dfrac{1+\tan^2\theta}{\csc^2\theta}$

49. PHYSICS When a firework is fired from the ground, its height y and horizontal displacement x are related by the equation $y = \dfrac{-gx^2}{2{v_0}^2\cos^2\theta} + \dfrac{x\sin\theta}{\cos\theta}$, where v_0 is the initial velocity of the projectile, θ is the angle at which it was fired, and g is the acceleration due to gravity. Rewrite this equation so that $\tan\theta$ is the only trigonometric function that appears in the equation.



Real-World Career

Electrician

An electrician specializes in the wiring of electrical components. Electricians serve an apprenticeship lasting 3–5 years. Schooling in electrical theory and building codes is required. Certification requires work experience and a passing score on a written test.

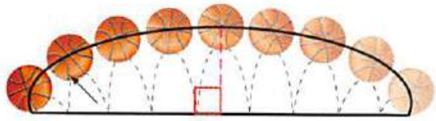
50. ELECTRONICS When an alternating current of frequency f and peak current I_0 passes through a resistance R, then the power delivered to the resistance at time t seconds is $P = I_0^2 R \sin^2 2\pi ft$.

a. Write an expression for the power in terms of $\cos^2 2\pi$ ft.

b. Write an expression for the power in terms of $\csc^2 2\pi$ ft.

51. THROWING A BALL In this problem, you will investigate the path of a ball represented by the equation $h = \frac{v_0^2 \sin^2 \theta}{2g}$, where θ is the measure of the angle between the ground and the path of the ball, v_0 is its initial velocity in meters per second, and g is the acceleration due to gravity. The value of g is 9.8 m/s^2.

a. If the initial velocity of the ball is 47 meters per second, find the height of the ball at 30°, 45°, 60°, and 90°. Round to the nearest tenth.



b. Graph the equation on a graphing calculator.

c. Show that the formula $h = \frac{v_0^2 \tan^2 \theta}{2g \sec^2 \theta}$ is equivalent to the one given above.

H.O.T. Problems

Use Higher-Order Thinking Skills

52. WHICH ONE DOESN'T BELONG? Identify the equation that does not belong with the other three. Explain your reasoning.

$\sin^2 \theta + \cos^2 \theta = 1$	$1 + \cot^2 \theta = \csc^2 \theta$
$\sin^2 \theta - \cos^2 \theta = 2 \sin^2 \theta$	$\tan^2 \theta + 1 = \sec^2 \theta$

53. CHALLENGE Transform the right side of $\tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta}$ to show that $\tan^2 \theta = \sec^2 \theta - 1$.

54. WRITING IN MATH Explain why you cannot square each side of an equation when verifying a trigonometric identity.

55. REASONING Explain why $\sin^2 \theta + \cos^2 \theta = 1$ is an identity, but $\sin \theta = \sqrt{1 - \cos \theta}$ is not.

56. WRITE A QUESTION A classmate is having trouble trying to verify a trigonometric identity involving multiple trigonometric functions to multiple degrees. Write a question to help her work through the problem.

57. WRITING IN MATH Write about why you think terms of a trigonometric identity are often rewritten in terms of sine and cosine.

58. CHALLENGE Let $x = \frac{1}{2} \tan \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Write $f(x) = \frac{x}{\sqrt{1 + 4x^2}}$ in terms of a single trigonometric function of θ.

59. REASONING Justify the three basic Pythagorean identities.

Standardized Test Practice

60. ACT/SAT A small business owner must hire seasonal workers as the need arises. The following list shows the number of employees hired monthly for a 5-month period.

5, 14, 6, 8, 12

If the mean of these data is 9, what is the population standard deviation for these data? (Round your answer to the nearest tenth.)

A 2.6
B 5.7
C 8.6
D 12.3

61. Find the center and radius of the circle with equation $(x-4)^2 + y^2 - 16 = 0$.

F $C(-4, 0)$; $r = 4$ units
G $C(-4, 0)$; $r = 16$ units
H $C(4, 0)$; $r = 4$ units
J $C(4, 0)$; $r = 16$ units

62. GEOMETRY The perimeter of a right triangle is 36 inches. Twice the length of the longer leg minus twice the length of the shorter leg is 6 inches. What are the lengths of all three sides?

A 3 in., 4 in., 5 in.
B 6 in., 8 in., 10 in.
C 9 in., 12 in., 15 in.
D 12 in., 16 in., 20 in.

63. Simplify $128^{\frac{1}{4}}$.

F $2\sqrt[4]{2}$
G $2\sqrt[4]{8}$
H 4
J $4\sqrt[4]{2}$

Spiral Review

Find the exact value of each expression. (Lesson 14-1)

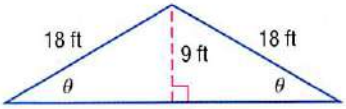
64. $\tan\theta$, if $\cot\theta = 2$; $0° \le \theta < 90°$

65. $\sin\theta$, if $\cos\theta = \frac{2}{3}$; $0° \le \theta < 90°$

66. $\csc\theta$, if $\cos\theta = -\frac{3}{5}$; $90° < \theta < 180°$

67. $\cos\theta$, if $\sec\theta = \frac{5}{3}$; $270° < \theta < 360°$

68. ARCHITECTURE The support for a roof is shaped like two right triangles, as shown at the right. Find θ. (Lesson 13-9)



69. PROBABILITY An administrative assistant has 4 blue file folders, 3 red folders, and 3 yellow folders on her desk. Each folder contains different information, so two folders of the same color should be viewed as being different. She puts the file folders randomly in a box to take to a meeting. Find each probability. (Lesson 12-3)

a. P(4 blue, 3 red, 3 yellow, in that order)

b. P(first 2 blue, last 2 blue)

Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbolas with the given equations. Then graph the hyperbola. (Lesson 10-5)

70. $\frac{y^2}{18} - \frac{x^2}{20} = 1$

71. $\frac{(y+6)^2}{20} - \frac{(x-1)^2}{25} = 1$

72. $x^2 - 36y^2 = 36$

Skills Review

Simplify. (Lesson 7-5)

73. $\frac{2+\sqrt{2}}{5-\sqrt{2}}$

74. $\frac{x+1}{\sqrt{x^2-1}}$

75. $\frac{x-1}{\sqrt{x}-1}$

76. $\frac{-2-\sqrt{3}}{1+\sqrt{3}}$

14-3 Sum and Difference of Angles Identities

Then

You found values of trigonometric functions for general angles. (Lesson 13-3)

Now

- Find values of sine and cosine by using sum and difference identities.
- Verify trigonometric identities by using sum and difference identities.

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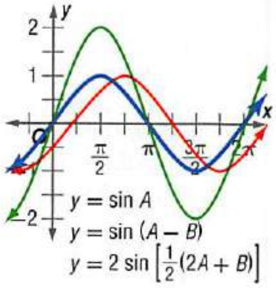
Why?

Have you ever been using a wireless Internet provider and temporarily lost the signal? Waves that pass through the same place at the same time cause interference. Interference occurs when two waves combine to have a greater, or smaller, amplitude than either of the component waves.



Sum and Difference Identities Notice that the third equation shown above involves the sum of A and B. It is often helpful to use formulas for the trigonometric values of the difference or sum of two angles. For example, you could find the exact value of sin 15° by evaluating $\sin(60° - 45°)$. Formulas exist that can be used to evaluate expressions like $\sin(A - B)$ or $\cos(A + B)$.

Key Concept

For Your FOLDABLE

Sum Identities	Difference Identities
$\sin(A + B) = \sin A \cos B + \cos A \sin B$	$\sin(A - B) = \sin A \cos B - \cos A \sin B$
$\cos(A + B) = \cos A \cos B - \sin A \sin B$	$\cos(A - B) = \cos A \cos B + \sin A \sin B$
$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$	$\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

EXAMPLE 1 Find Trigonometric Values

Find the exact value of each expression.

a. $\sin 105°$

Use the identity $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

$$\sin 105° = \sin(60° + 45°) \quad A = 60° \text{ and } B = 45°$$
$$= \sin 60° \cos 45° + \cos 60° \sin 45° \quad \text{Sum identity}$$
$$= \left(\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2} \cdot \frac{\sqrt{2}}{2}\right) \quad \text{Evaluate each expression.}$$
$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \text{ or } \frac{\sqrt{6} + \sqrt{2}}{4} \quad \text{Multiply.}$$

b. $\cos(-120°)$

Use the identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

$$\cos(-120) = \cos(60° - 180°) \quad A = 60° \text{ and } B = 180°$$
$$= \cos 60° \cos 180° + \sin 60° \sin 180° \quad \text{Difference identity}$$
$$= \frac{1}{2} \cdot (-1) + \frac{\sqrt{3}}{2} \cdot 0 \quad \text{Evaluate each expression.}$$
$$= -\frac{1}{2} \quad \text{Multiply.}$$

Check Your Progress

1A. $\sin 15°$

1B. $\cos(-15°)$

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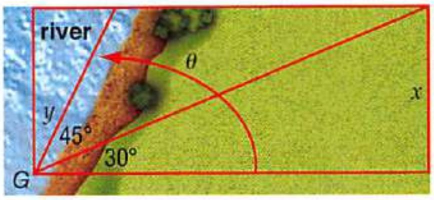
You can use the sum and difference of angles identities to solve real-world applications.

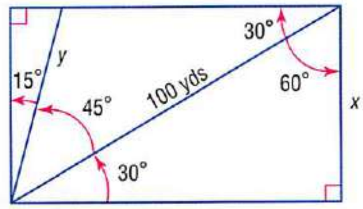
Real-World EXAMPLE 2 Sum and Difference of Angles Identities

A geologist measures the angle between one side of a rectangular lot and the line from her position to the opposite corner of the lot as 30°. She then measures the angle between that line and the line to the point on the property where a river crosses as 45°. She stands 100 yards from the opposite corner of the property. How far is she from the point at which the river crosses the property line?



Understand The question asks for the distance between the geologist and the point where the river crosses the property line, or y.

Plan Draw a picture that labels all the things that you know from the information given.



Solve Solve for x.

$\sin 30° = \frac{x}{100}$ **Definition of sine**

$x = 100 \sin 30°$

$x = 50$ **Since the lot is rectangular, opposite sides are equal.**

Now look at the triangle on the far left and solve for y.

$\cos 15° = \frac{50}{y}$ **Definition of cosine**

$\cos (45° - 30°) = \frac{50}{y}$ **15 = 45 − 30**

$\cos 45° \cos 30° + \sin 45° \sin 30° = \frac{50}{y}$ **Difference identity**

$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{50}{y}$ **Evaluate.**

$\frac{\sqrt{6} + \sqrt{2}}{4} = \frac{50}{y}$ **Simplify.**

$(\sqrt{6} + \sqrt{2})y = 200$ **Cross products**

$$y = \frac{200}{(\sqrt{6} + \sqrt{2})} \cdot \frac{(\sqrt{6} - \sqrt{2})}{(\sqrt{6} - \sqrt{2})}$$

$$y = 50(\sqrt{6} - \sqrt{2})$$

$$y = 50\sqrt{6} - 50\sqrt{2} \text{ or about } 51.8$$

The geologist is about 51.8 yards from the point where the river crosses the property line.

Check Use a calculator to find the Arccos of $\frac{50}{51.8} \approx 15°$. ✓

Check Your Progress

2. Use the expression $E \sin (113.5° - \phi)$, where ϕ is the latitude of the location, and E is the amount of light energy. Determine the amount of light energy in West Hollywood, California, which is located at latitude of 34.1° N.

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Problem-Solving Tip

Make a Model Make a model to visualize a problem situation. A model can be a drawing or a figure made of different objects, such as algebra tiles or folded paper.

Reading Math

Greek Letters The Greek letter phi, ϕ, is used in Check Your Progress 2 to represent latitude.

StudyTip

Make a List Make a list of the trigonometric values for the angles between 0° and 360° for which the sum and difference identities can be easily used. Use your list as a reference.

Verify Trigonometric Identities You can also use the sum and difference identities to verify identities.

EXAMPLE 3 Verify Trigonometric Identities

Verify that each equation is an identity.

a. $\cos(90° - \theta) = \sin\theta$

$\cos(90° - \theta) \stackrel{?}{=} \sin\theta$	Original equation
$\cos 90° \cos\theta + \sin 90° \sin\theta \stackrel{?}{=} \sin\theta$	Sum identity
$0 \cdot \cos\theta + 1 \cdot \sin\theta \stackrel{?}{=} \sin\theta$	Evaluate each expression.
$\sin\theta = \sin\theta$ ✓	Simplify.

b. $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$

$\sin\left(\theta + \frac{\pi}{2}\right) \stackrel{?}{=} \cos\theta$	Original equation
$\sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2} \stackrel{?}{=} \cos\theta$	Sum identity
$\sin\theta \cdot 0 + \cos\theta \cdot 1 \stackrel{?}{=} \cos\theta$	Evaluate each expression.
$\cos\theta = \cos\theta$ ✓	Simplify.

Check Your Progress

Verify that each equation is an identity.

3A. $\sin(90° - \theta) = \cos\theta$

3B. $\cos(90° + \theta) = -\sin\theta$

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Check Your Understanding

Example 1 p. 904

Find the exact value of each expression.

1. $\cos 165°$
2. $\cos 105°$
3. $\cos 75°$
4. $\sin(-30°)$
5. $\sin 135°$
6. $\sin(-210°)$

Example 2 p. 905

7. **ELECTRONICS** Refer to the beginning of the lesson. *Constructive interference* occurs when two waves combine to have a greater amplitude than either of the component waves. *Destructive interference* occurs when the component waves combine to have a smaller amplitude. The first signal can be modeled by the equation $y = 20\sin(3\theta + 45°)$. The second signal can be modeled by the equation $y = 20\sin(3\theta + 225°)$.

a. Find the sum of the two functions.

b. What type of interference results when signals modeled by the two equations are combined?

Example 3 p. 906

Verify that each equation is an identity.

8. $\sin(90° + \theta) = \cos\theta$
9. $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$
10. $\tan\left(\theta + \frac{\pi}{2}\right) = -\cot\theta$
11. $\sin(\theta + \pi) = -\sin\theta$

Practice and Problem Solving

● = **Step-by-Step Solutions** begin on page R20.
Extra Practice begins on page 947.

Example 1 p. 904

Find the exact value of each expression.

12. $\sin 165°$ **13.** $\cos 135°$ **14.** $\cos \frac{7\pi}{12}$

15. $\sin \frac{\pi}{12}$ **16.** $\tan 195°$ **17.** $\cos\left(-\frac{\pi}{12}\right)$

Example 2 p. 905

18. ELECTRONICS In a certain circuit carrying alternating current, the formula $c = 2\sin(120t)$ can be used to find the current c in amperes after t seconds.

a. Rewrite the formula using the sum of two angles.

b. Use the sum of angles formula to find the exact current at $t = 1$ second.

Example 3 p. 906

Verify that each equation is an identity.

19. $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$ **20.** $\cos(60° + \theta) = \sin(30° - \theta)$

21. $\cos(180° + \theta) = -\cos\theta$ **22.** $\tan(\theta + 45°) = \frac{1 + \tan\theta}{1 - \tan\theta}$

Real-World Link

The most snow that Minneapolis, Minnesota, received in one year was 101.5 inches in 1983. On average, Minneapolis receives 40 inches of snow per year.

Source: Minnesota Climatology

23. WEATHER The monthly high temperatures for Minneapolis, Minnesota, can be modeled by the equation $y = 31.65\sin\left(\frac{\pi}{6}x - 2.09\right) + 52.35$, where the months x are represented by January = 1, February = 2, and so on. The monthly low temperatures for Minneapolis can be modeled by the equation $y = 30.15\sin\left(\frac{\pi}{6}x - 2.09\right) + 32.95$.

a. Write a new function by adding the expressions on the right side of each equation and dividing the result by 2.

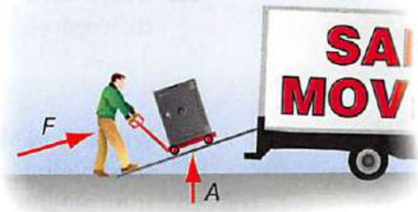
b. What is the meaning of the function you wrote in part **a**?

Find the exact value of each expression.

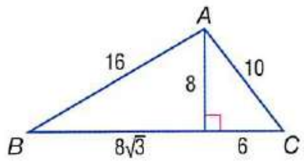
24. $\tan 165°$ **25.** $\sec 1275°$ **26.** $\sin 735°$

27. $\tan \frac{23\pi}{12}$ **28.** $\csc \frac{5\pi}{12}$ **29.** $\cot \frac{113\pi}{12}$

30. FORCE In the figure at the right, the effort F necessary to hold a safe in position on a ramp is given by $F = \frac{W(\sin A + \mu\cos A)}{\cos A - \mu\sin A}$, where W is the weight of the safe and $\mu = \tan\theta$. Show that $F = W\tan(A + \theta)$.



31. QUILTING As part of a quilt that is being made, the quilter places two right triangular swatches together to make a new triangular piece. One swatch has sides 6 inches, 8 inches, and 10 inches long. The other swatch has sides 8 inches, $8\sqrt{3}$ inches, and 16 inches long. The pieces are placed with the sides of eight inches against each other, as shown in the figure, to form triangle ABC.



a. What is the exact value of the sine of angle BAC?

b. What is the exact value of the cosine of angle BAC?

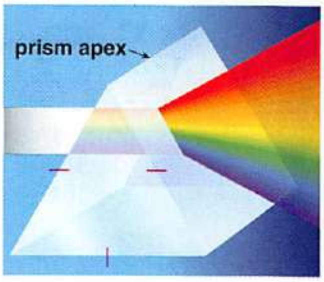
c. What is the measure of angle BAC?

d. Is the new triangle formed from the two triangles also a right triangle?

Real-World Link

Dispersive prisms break light into its constituent spectral colors. White light entering a prism is a mixture of different frequencies, each of which gets bent differently. Blue light is slowed down more than red light and will therefore be bent more than red light.

32. OPTICS When light passes symmetrically through a prism, the index of refraction n of the glass with respect to air is $n = \dfrac{\sin\left[\frac{1}{2}(a+b)\right]}{\sin\frac{b}{2}}$, where a is the measure of the deviation angle and b is the measure of the prism apex angle.



a. Show that for the prism shown, $n = \sqrt{3}\sin\frac{a}{2} + \cos\frac{a}{2}$.

b. Find n for the prism shown.

33. MULTIPLE REPRESENTATIONS In this problem, you will disprove the hypothesis that $\sin(A+B) = \sin A + \sin B$.

a. TABULAR Complete the table.

A	B	$\sin A$	$\sin B$	$\sin(A+B)$	$\sin A + \sin B$
30°	90°				
45°	60°				
60°	45°				
90°	30°				

b. GRAPHICAL Assume that B is always 15° less than A. Use a graphing calculator to graph $\sin(x + x - 15)$ and $\sin x + \sin(x - 15)$ on the same screen.

c. ANALYTICAL Determine whether $\cos(A+B) = \cos A + \cos B$ is an identity. Explain your reasoning.

Verify that each equation is an identity.

34. $\sin(A+B) = \dfrac{\tan A + \tan B}{\sec A \sec B}$

35. $\cos(A+B) = \dfrac{1 - \tan A \tan B}{\sec A \sec B}$

36. $\sec(A-B) = \dfrac{\sec A \sec B}{1 + \tan A \tan B}$

37. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

H.O.T. Problems Use Higher-Order Thinking Skills

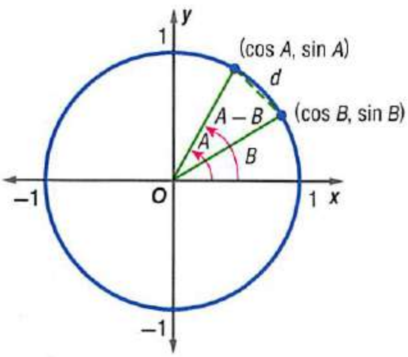
38. REASONING Simplify the following expression without expanding any of the sums or differences.

$$\sin\left(\frac{\pi}{3} - \theta\right)\cos\left(\frac{\pi}{3} + \theta\right) - \cos\left(\frac{\pi}{3} - \theta\right)\sin\left(\frac{\pi}{3} + \theta\right)$$

39. WRITING IN MATH Use the information at the beginning of the lesson and in Exercise 7 to explain how the sum and difference identities are used to describe wireless Internet interference. Include an explanation of the difference between constructive and destructive interference.

40. CHALLENGE Derive an identity for $\cot(A+B)$ in terms of $\cot A$ and $\cot B$.

41. PROOF The figure at the right shows two angles A and B in standard position on the unit circle. Use the Distance Formula to find d, where $(x_1, y_1) = (\cos B, \sin B)$ and $(x_2, y_2) = (\cos A, \sin A)$.



42. OPEN ENDED Consider the following theorem. *If A, B, and C are the angles of an oblique triangle, then tan A + tan B + tan C = tan A tan B tan C.* Choose values for A, B, and C. Verify that the conclusion is true for your specific values.

Standardized Test Practice

43. GRIDDED RESPONSE The mean of seven numbers is 0. The sum of three of the numbers is −9. What is the sum of the remaining four numbers?

44. The variables a, b, c, d, and f are integers in a sequence, where $a = 2$ and $b = 12$. To find the next term, double the last term and add that result to one less than the next-to-last term. For example, $c = 25$, because $2(12) = 24$, $2 - 1 = 1$, and $24 + 1 = 25$. What is the value of f?

A 74
B 144
C 146
D 256

45. ACT/SAT Solve $x^2 - 5x < 14$.

F $\{x \mid -7 < x < 2\}$
G $\{x \mid -7 < x > 2\}$
H $\{x \mid -2 < x < 7\}$
J $\{x \mid -2 < x > 7\}$

46. PROBABILITY A math teacher is randomly distributing 15 yellow pencils and 10 green pencils. What is the probability that the first pencil she hands out will be yellow and the second pencil will be green?

A $\frac{1}{24}$
B $\frac{1}{4}$
C $\frac{2}{5}$
D $\frac{23}{25}$

Spiral Review

Verify that each equation is an identity. (Lesson 14-2)

47. $\frac{\sin \theta}{\tan \theta} + \frac{\cos \theta}{\cot \theta} = \cos \theta + \sin \theta$

48. $\sec \theta (\sec \theta - \cos \theta) = \tan^2 \theta$

Simplify each expression. (Lesson 14-1)

49. $\sin \theta \csc \theta - \cos^2 \theta$

50. $\cos^2 \theta \sec \theta \csc \theta$

51. $\cos \theta + \sin \theta \tan \theta$

52. GUITAR When a guitar string is plucked, it is displaced from a fixed point in the middle of the string and vibrates back and forth, producing a musical tone. The exact tone depends on the frequency, or number of cycles per second, that the string vibrates. To produce an A, the frequency is 440 cycles per second, or 440 hertz (Hz). (Lesson 13-6)

a. Find the period of this function.

b. Graph the height of the fixed point on the string from its resting position as a function of time. Let the maximum distance above the resting position have a value of 1 unit, and let the minimum distance below this position have a value of 1 unit.

Prove that each statement is true for all positive integers. (Lesson 11-7)

53. $4^n - 1$ is divisible by 3.

54. $5^n + 3$ is divisible by 4.

Skills Review

Solve each equation. (Lesson 7-7)

55. $7 + \sqrt{4x + 8} = 9$

56. $\sqrt{y + 21} - 1 = \sqrt{y + 12}$

57. $\sqrt{4z + 1} = 3 + \sqrt{4z - 2}$

CHAPTER 14 Mid-Chapter Quiz

Lessons 14-1 through 14-3

Simplify each expression. (Lesson 14-1)

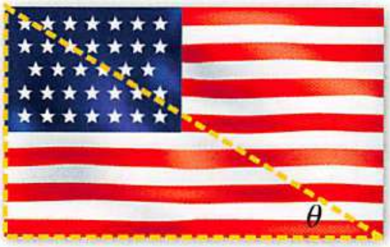
1. $\cot\theta\sec\theta$

2. $\dfrac{1-\cos^2\theta}{\sin^2\theta}$

3. $\dfrac{1}{\cos\theta}-\dfrac{\sin^2\theta}{\cos\theta}$

4. $\cos\left(\frac{\pi}{2}-\theta\right)\csc\theta$

5. **HISTORY** In 1861, the United States 34-star flag was adopted. For this flag, $\tan\theta=\frac{31.5}{51}$. Find $\sin\theta$.



Find the value of each expression. (Lesson 14-1)

6. $\sin\theta$, if $\cos\theta=\frac{3}{5}$; $0^\circ<\theta<90^\circ$

7. $\csc\theta$, if $\cot\theta=\frac{1}{2}$; $270^\circ<\theta<360^\circ$

8. $\tan\theta$, if $\sec\theta=\frac{4}{3}$; $0^\circ<\theta<90^\circ$

9. **MULTIPLE CHOICE** Which of the following is equivalent to $\frac{\cos\theta}{1-\sin^2\theta}$? (Lesson 14-1)

A $\cos\theta$

B $\csc\theta$

C $\tan\theta$

D $\sec\theta$

10. **AMUSEMENT PARKS** Suppose a child on a merry-go-round is seated on an outside horse. The diameter of the merry-go-round is 16 meters. The angle of inclination is represented by the equation $\tan\theta=\frac{v^2}{gR}$, where R is the radius of the circular path, v is the speed in meters per second, and g is 9.8 meters per second squared. (Lesson 14-1)

a. If the sine of the angle of inclination of the child is $\frac{1}{5}$, what is the angle of inclination made by the child?

b. What is the velocity of the merry-go-round?

c. If the speed of the merry-go-round is 3.6 meters per second, what is the value of the angle of inclination of a rider?

Verify that each of the following is an identity. (Lesson 14-2)

11. $\cot^2\theta+1=\dfrac{\cot\theta}{\cos\theta\cdot\sin\theta}$

12. $\dfrac{\cos\theta\csc\theta}{\cot\theta}=1$

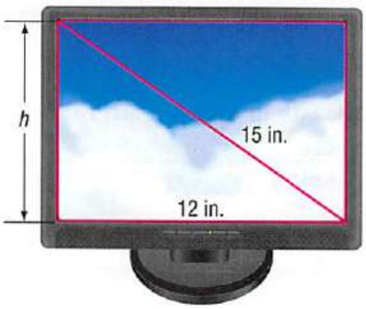
13. $\dfrac{\sin\theta\tan\theta}{1-\cos\theta}=(1+\cos\theta)\sec\theta$

14. $\tan\theta(1-\sin\theta)=\dfrac{\cos\theta\sin\theta}{1+\sin\theta}$

15. **COMPUTER** The front of a computer monitor is usually measured along the diagonal of the screen as shown below. (Lesson 14-2)

a. Find h.

b. Using the diagram shown, show that $\cot\theta=\frac{\cos\theta}{\sin\theta}$.



Verify that each of the following is an identity. (Lesson 14-2)

16. $\tan^2\theta+1=\dfrac{\tan\theta}{\cos\theta\cdot\sin\theta}$

17. $\dfrac{\sin\theta\cdot\sec\theta}{\sec\theta-1}=(\sec\theta+1)\cot\theta$

18. $\sin^2\theta\cdot\tan^2=\tan^2\theta-\sin^2\theta$

19. $\cot\theta(1-\cos\theta)=\dfrac{\cos\theta\cdot\sin\theta}{1+\cos\theta}$

Find the exact value of each expression. (Lesson 14-3)

20. $\cos 105^\circ$

21. $\sin(-135^\circ)$

22. $\tan 15^\circ$

23. $\cot 75^\circ$

24. **MULTIPLE CHOICE** What is the exact value of $\cos\frac{5\pi}{12}$? (Lesson 14-3)

F $\sqrt{2}$

G $\dfrac{\sqrt{6}+\sqrt{2}}{2}$

H $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

J $\dfrac{\sqrt{6}+\sqrt{2}}{4}$

25. Verify that $\cos(30^\circ-\theta)=\sin(60^\circ+\theta)$ is an identity. (Lesson 14-3)

14-4

Double-Angle and Half-Angle Identities

Then

You found values of sine and cosine by using sum and difference identities. (Lesson 14-3)

Now

- Find values of sine and cosine by using double-angle identities.
- Find values of sine and cosine by using half-angle identities.

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- Extra Examples
- Personal Tutor
- Self-Check Quiz
- Homework Help

Why?

Chicago's Buckingham Fountain contains jets placed at specific angles that shoot water into the air to create arcs. When a stream of water shoots into the air with velocity v at an angle of θ with the horizontal, the model predicts that the water will travel a horizontal distance of $D = \frac{v^2}{g} \sin 2\theta$ and reach a maximum height of $H = \frac{v^2}{2g} \sin^2 \theta$. The ratio of H to D helps determine the total height and width of the fountain. Express $\frac{H}{D}$ as a function of θ.

Double-Angle Identities It is sometimes useful to have identities to find the value of a function of twice an angle or half an angle.

Key Concept — Double-Angle Identities

For Your FOLDABLE

The following identities hold true for all values of θ.

$\sin 2\theta = 2 \sin \theta \cos \theta$

$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$
$\cos 2\theta = 2 \cos^2 \theta - 1$
$\cos 2\theta = 1 - 2 \sin^2 \theta$

$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$

EXAMPLE 1 Double-Angle Identities

Find the exact value of $\sin 2\theta$ if $\sin \theta = \frac{2}{3}$ and θ is between 0° and 90°.

Step 1 Use the identity $\sin 2\theta = 2 \sin \theta \cos \theta$ to find the value of $\cos \theta$.

$\cos^2 \theta = 1 - \sin^2 \theta$ — **$\cos^2 \theta + \sin^2 \theta = 1$**

$\cos^2 \theta = 1 - \left(\frac{2}{3}\right)^2$ — **$\sin \theta = \frac{2}{3}$**

$\cos^2 \theta = \frac{5}{9}$ — **Subtract.**

$\cos \theta = \pm\frac{\sqrt{5}}{3}$ — **Take the square root of each side.**

Since θ is in the first quadrant, cosine is positive. Thus, $\cos \theta = \frac{\sqrt{5}}{3}$.

Step 2 Find $\sin 2\theta$.

$\sin 2\theta = 2 \sin \theta \cos \theta$ — **Double-angle identity**

$= 2\left(\frac{2}{3}\right)\left(\frac{\sqrt{5}}{3}\right)$ — **$\sin \theta = \frac{2}{3}$ and $\cos \theta = \frac{\sqrt{5}}{3}$**

$= \frac{4\sqrt{5}}{9}$ — **Multiply.**

Check Your Progress

1. Find the exact value of $\sin 2\theta$ if $\cos \theta = -\frac{1}{3}$ and $90° < \theta < 180°$.

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EXAMPLE 2 Double-Angle Identities

Find the exact value of each expression if $\sin \theta = \frac{2}{3}$ and θ is between 0° and 90°.

a. $\cos 2\theta$

Since we know the values of $\cos \theta$ and $\sin \theta$, we can use any of the double-angle identities for cosine. We will use the identity $\cos 2\theta = 1 - 2\sin^2 \theta$.

$\cos 2\theta = 1 - 2\sin^2 \theta$	Double-angle identity
$= 1 - 2\left(\frac{2}{3}\right)^2$ or $\frac{1}{9}$	$\sin \theta = \frac{2}{3}$

b. $\tan 2\theta$

Step 1 Find $\tan \theta$ to use the double-angle identity for $\tan 2\theta$.

$\tan \theta = \frac{\sin \theta}{\cos \theta}$	Definition of tangent
$= \frac{\frac{2}{3}}{\frac{\sqrt{5}}{3}}$	$\sin \theta = \frac{2}{3}$ and $\cos \theta = \frac{\sqrt{5}}{3}$
$= \frac{2}{\sqrt{5}}$ or $\frac{2\sqrt{5}}{5}$	Rationalize the denominator.

Step 2 Find $\tan 2\theta$.

$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$	Double-angle identity
$= \frac{2\left(\frac{2\sqrt{5}}{5}\right)}{1 - \left(\frac{2\sqrt{5}}{5}\right)^2}$	$\tan \theta = \frac{2\sqrt{5}}{5}$
$= \frac{2\left(\frac{2\sqrt{5}}{5}\right)}{\frac{25}{25} - \frac{20}{25}}$	Square the denominator.
$= \frac{\frac{4\sqrt{5}}{5}}{\frac{1}{5}}$	Simplify.
$= \frac{4\sqrt{5}}{5} \cdot \frac{5}{1}$ or $4\sqrt{5}$	$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$

Check Your Progress

Find the exact value of each expression if $\cos \theta = -\frac{1}{3}$ and $90° < \theta < 180°$.

2A. $\cos 2\theta$

2B. $\tan 2\theta$

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StudyTip

Deriving Formulas You can use the identity for $\sin (A + B)$ to find the sine of twice an angle θ, $\sin 2\theta$, and the identity for $\cos (A + B)$ to find the cosine of twice an angle θ, $\cos 2\theta$.

Half-Angle Identities It is sometimes useful to have identities to find the value of a function of half an angle.

Key Concept Half-Angle Identities

For Your FOLDABLE

The following identities hold true for all values of θ.

$\sin \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos \theta}{2}}$ $\quad\cos \frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos \theta}{2}}$ $\quad\tan \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}$, $\cos \theta \neq -1$

StudyTip

Choosing the Sign In the first step of the solution, you may want to determine the quadrant in which the terminal side of $\frac{\theta}{2}$ will lie. Then you can use the correct sign from that point on.

ReadingMath

Plus or Minus The first sign of the half-angle identity is read *plus or minus*. Unlike with the double-angle identities, you must determine the sign.

EXAMPLE 3 Half-Angle Identities

a. Find the exact value of $\cos \frac{\theta}{2}$ if $\sin \theta = -\frac{4}{5}$ and θ is in the third quadrant.

$\cos^2 \theta = 1 - \sin^2 \theta$ — Use a Pythagorean identity to find $\cos \theta$.

$\cos^2 \theta = 1 - \left(-\frac{4}{5}\right)^2$ — $\sin \theta = -\frac{4}{5}$

$\cos^2 \theta = 1 - \frac{16}{25}$ — Evaluate exponent.

$\cos^2 \theta = \frac{9}{25}$ — Subtract.

$\cos \theta = \pm\frac{3}{5}$ — Take the square root of each side.

Since θ is in the third quadrant, $\cos \theta = -\frac{3}{5}$.

$\cos \frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos \theta}{2}}$ — Half-angle identity

$= \pm\sqrt{\frac{1 - \frac{3}{5}}{2}}$ — $\cos \theta = -\frac{3}{5}$

$= \pm\sqrt{\frac{1}{5}}$ — Simplify.

$= \pm\frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$ or $\pm\frac{\sqrt{5}}{5}$ — Rationalize the denominator.

If θ is between 180° and 270°, $\frac{\theta}{2}$ is between 90° and 135°. So, $\cos \frac{\theta}{2}$ is $-\frac{\sqrt{5}}{5}$.

b. Find the exact value of cos 67.5°.

$\cos 67.5° = \cos \frac{135°}{2}$ — $67.5° = \frac{135°}{2}$

$= \sqrt{\frac{1 + \cos 135°}{2}}$ — $\cos \frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos \theta}{2}}$

$= \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}}$ — 67.5° is in Quadrant I; the value is positive.

$= \sqrt{\frac{\frac{2}{2} - \frac{\sqrt{2}}{2}}{2}}$ — $1 = \frac{2}{2}$

$= \sqrt{\frac{\frac{2 - \sqrt{2}}{2}}{2}}$ — Subtract fractions.

$= \sqrt{\frac{2 - \sqrt{2}}{2} \cdot \frac{1}{2}}$ — $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$

$= \sqrt{\frac{2 - \sqrt{2}}{4}}$ — Multiply.

$= \frac{\sqrt{2 - \sqrt{2}}}{\sqrt{4}}$ — $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

$= \frac{\sqrt{2 - \sqrt{2}}}{2}$ — Simplify.

Check Your Progress

3. Find the exact value of $\sin \frac{\theta}{2}$ if $\sin \theta = \frac{2}{3}$ and θ is in the second quadrant.

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Real-World Link

The City Hall Park Fountain in New York City is located in the heart of Manhattan in front of City Hall.

Source: *Fodor's*

Real-World EXAMPLE 4 Simplify Using Double-Angle Identities

FOUNTAIN Refer to the beginning of the lesson. Find $\frac{H}{D}$.

$$\frac{H}{D} = \frac{\frac{v^2}{2g}\sin^2\theta}{\frac{v^2}{g}\sin 2\theta}$$ Original equation

$$= \frac{\frac{v^2\sin^2\theta}{2g}}{\frac{v^2\sin 2\theta}{g}}$$ Simplify the numerator and denominator.

$$= \frac{v^2\sin^2\theta}{2g}\cdot\frac{g}{v^2\sin 2\theta}$$ $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$

$$= \frac{\sin^2\theta}{2\sin 2\theta}$$ Simplify.

$$= \frac{\sin^2\theta}{4\sin\theta\cos\theta}$$ $\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{1}{4}\cdot\frac{\sin\theta}{\cos\theta}$$ Simplify.

$$= \frac{1}{4}\tan\theta$$ $\frac{\sin\theta}{\cos\theta} = \tan\theta$

Check Your Progress

Find each value.

4A. $\sin 135°$

4B. $\cos\frac{7\pi}{8}$

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Recall that you can use the sum and difference identities to verify identities. Double- and half-angle identities can also be used to verify identities.

EXAMPLE 5 Verify Identities

Verify that $\frac{\cos 2\theta}{1+\sin 2\theta} = \frac{\cot\theta - 1}{\cot\theta + 1}$ is an identity.

$$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\cot\theta - 1}{\cot\theta + 1}$$ Original equation

$$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\frac{\cos\theta}{\sin\theta} - 1}{\frac{\cos\theta}{\sin\theta} + 1}$$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

$$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$$ Multiply numerator and denominator by $\sin\theta$.

$$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\cdot\frac{\cos\theta + \sin\theta}{\cos\theta + \sin\theta}$$ Multiply the right side by 1.

$$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta}$$ Multiply.

$$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\cos^2\theta - \sin^2\theta}{1 + 2\cos\theta\sin\theta}$$ Simplify.

$$\frac{\cos 2\theta}{1+\sin 2\theta} = \frac{\cos 2\theta}{1+\sin 2\theta}\ ✓$$ $\cos^2\theta - \sin^2\theta = \cos 2\theta$; $2\cos\theta\sin\theta = \sin 2\theta$

Check Your Progress

5. Verify that $4\cos^2 x - \sin^2 2x = 4\cos^4 x$.

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Check Your Understanding

Examples 1–3 pp. 911–913

Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$.

1. $\sin \theta = \frac{1}{4}$; $0° < \theta < 90°$

2. $\sin \theta = \frac{4}{5}$; $90° < \theta < 180°$

3. $\cos \theta = -\frac{5}{13}$; $\frac{\pi}{2} < \theta < \pi$

4. $\cos \theta = \frac{3}{5}$; $270° < \theta < 360°$

5. $\tan \theta = -\frac{8}{15}$; $90° < \theta < 180°$

6. $\tan \theta = \frac{5}{12}$; $\pi < \theta < \frac{3\pi}{2}$

Find the exact value of each expression.

7. $\sin \frac{\pi}{8}$

8. $\cos 15°$

Example 4 p. 914

9. **SOCCER** A soccer player kicks a ball at an angle of 37° with the ground with an initial velocity of 52 feet per second. The distance d that the ball will go in the air if it is not blocked is given by $d = \frac{2v^2 \sin \theta \cos \theta}{g}$. In this formula, g is the acceleration due to gravity and is equal to 32 feet per second squared, and v is the initial velocity.

a. Simplify this formula by using a double-angle identity.

b. Using the simplified formula, how far will this ball go?

Example 5 p. 914

Verify that each equation is an identity.

10. $\tan \theta = \frac{1 - \cos 2\theta}{\sin 2\theta}$

11. $(\sin \theta + \cos \theta)^2 = 1 + 2 \sin \theta \cos \theta$

Practice and Problem Solving

● = **Step-by-Step Solutions** begin on page R20.
Extra Practice begins on page 947.

Examples 1–3 pp. 911–913

Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$.

12. $\sin \theta = \frac{2}{3}$; $90° < \theta < 180°$

13. $\sin \theta = -\frac{15}{17}$; $\pi < \theta < \frac{3\pi}{2}$

14. $\cos \theta = \frac{3}{5}$; $\frac{3\pi}{2} < \theta < 2\pi$

15. $\cos \theta = \frac{1}{5}$; $270° < \theta < 360°$

16. $\tan \theta = \frac{4}{3}$; $180° < \theta < 270°$

17. $\tan \theta = -2$; $\frac{\pi}{2} < \theta < \pi$

Find the exact value of each expression.

18. $\sin 75°$

19. $\sin \frac{3\pi}{8}$

20. $\cos \frac{7\pi}{12}$

21. $\tan 165°$

22. $\tan \frac{5\pi}{12}$

23. $\tan 22.5°$

24. **GEOGRAPHY** The Mercator projection of the globe is a projection on which the distance between the lines of latitude increases with their distance from the equator. The calculation of the location of a point on this projection involves the expression $\tan\left(45° + \frac{L}{2}\right)$, where L is the latitude of the point.

a. Write this expression in terms of a trigonometric function of L.

b. The latitude of Tallahassee, Florida, is 30° north. Find the value of the expression if $L = 30°$.

Example 4 p. 914

25. ELECTRONICS Consider an AC circuit consisting of a power supply and a resistor. If the current I_0 in the circuit at time t is $I_0 \sin t\theta$, then the power delivered to the resistor is $P = I_0^2 R \sin^2 t\theta$, where R is the resistance. Express the power in terms of $\cos 2t\theta$.

Example 5 p. 914

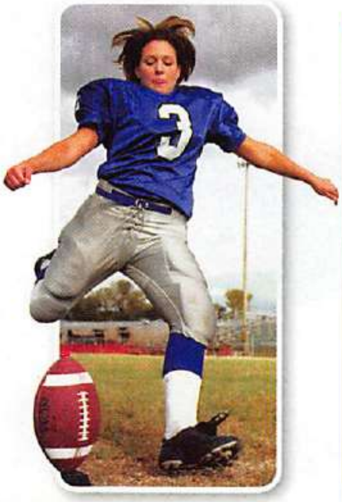
Verify that each equation is an identity.

26. $\tan 2\theta = \dfrac{2}{\cot \theta - \tan \theta}$

27. $1 + \dfrac{1}{2}\sin 2\theta = \dfrac{\sec \theta + \sin \theta}{\sec \theta}$

28. $\sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2} = \dfrac{\sin \theta}{2}$

29. $\tan \dfrac{\theta}{2} = \dfrac{\sin \theta}{1 + \cos \theta}$



Real-World Link

On average, a place kick travels about 15 yards farther than a punt, assuming the kicker is at least 13 years old.

30. FOOTBALL Suppose a place kicker consistently kicks a football with an initial velocity of 95 feet per second. Prove that the horizontal distance the ball travels in the air will be the same for $\theta = 45° + A$ as for $\theta = 45° - A$. Use the formula given in Exercise 9.

Find the exact values of sin 2θ, cos 2θ, and tan 2θ.

31. $\cos \theta = \frac{4}{5}$; $0° < \theta < 90°$

32. $\sin \theta = \frac{1}{3}$; $0 < \theta < \frac{\pi}{2}$

33. $\tan \theta = -3$; $90° < \theta < 180°$

34. $\sec \theta = -\frac{4}{3}$; $90° < \theta < 180°$

35. $\csc \theta = -\frac{5}{2}$; $\frac{3\pi}{2} < \theta < 2\pi$

36. $\cot \theta = \frac{3}{2}$; $180° < \theta < 270°$

H.O.T. Problems

Use **H**igher-**O**rder **T**hinking Skills

37. FIND THE ERROR Teresa and Nathan are calculating the exact value of sin 15°. Is either of them correct? Explain your reasoning.

Teresa

$\sin (A - B) = \sin A \cos B - \cos A \sin B$

$\sin (45 - 30) = \sin 45 \cos 30 - \cos 45 \sin 30$

$= \dfrac{\sqrt{2}}{2} \cdot \dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{2}}{2} \cdot \dfrac{1}{2}$

$= \dfrac{\sqrt{4}}{4}$

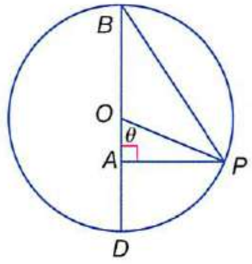
Nathan

$\sin \dfrac{A}{2} = \pm\sqrt{\dfrac{1 - \cos A}{2}}$

$\sin \dfrac{30}{2} = \pm\sqrt{\dfrac{1 - \frac{1}{2}}{2}}$

$= 0.5$

38. CHALLENGE Circle O is a unit circle. Use the figure to prove that $\tan \frac{1}{2}\theta = \dfrac{\sin \theta}{1 + \cos \theta}$.



39. WRITING IN MATH Write a short paragraph about the conditions under which you would use each of the three identities for $\cos 2\theta$.

40. PROOF Use the formula for $\sin (A + B)$ to derive the formula for $\sin 2\theta$, and use the formula for $\cos (A + B)$ to derive the formula for $\cos 2\theta$.

41. REASONING Derive the half-angle identities from the double-angle identities.

42. OPEN ENDED Suppose a golfer consistently hits the ball so that it leaves the tee with an initial velocity of 115 feet per second and $d = \dfrac{2v^2 \sin \theta \cos \theta}{g}$. Explain why the maximum distance is attained when $\theta = 45°$.

Standardized Test Practice

43. SHORT RESPONSE Perry drove to the gym at an average rate of 30 miles per hour. It took him 45 minutes. Going home, he took the same route, but drove at a rate of 45 miles per hour. How many miles is it to his house from the gym?

44. ACT/SAT Ms. Romero has a list of the yearly salaries of the staff members in her department. Which measure of data describes the middle income value of the salaries?

A mean

B median

C mode

D range

45. Identify the domain and range of the function $f(x) = |4x + 1| - 8$.

F $D = \{x \mid -3 \leq x \leq 1\}$, $R = \{y \mid y \geq -8\}$

G $D = \{\text{all real numbers}\}$, $R = \{y \mid y \geq -8\}$

H $D = \{x \mid -3 \leq x \leq 1\}$,
$R = \{\text{all real numbers}\}$

J $D = \{\text{all real numbers}\}$,
$R = \{\text{all real numbers}\}$

46. GEOMETRY Angel is putting a stone walkway around a circular pond. He has enough stones to make a walkway 144 feet long. If he uses all of the stones to surround the pond, what is the radius of the pond?

A $\frac{144}{\pi}$ ft

B $\frac{72}{\pi}$ ft

C 144π ft

D 72π ft

Spiral Review

Find the exact value of each expression. (Lesson 14-3)

47. $\sin 135°$

48. $\cos 105°$

49. $\sin 285°$

50. $\cos(-30°)$

51. $\sin(-240°)$

52. $\cos(-120°)$

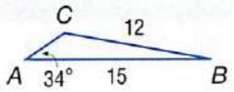
Verify that each equation is an identity. (Lesson 14-2)

53. $\cot\theta + \sec\theta = \frac{\cos^2\theta + \sin\theta}{\sin\theta\cos\theta}$

54. $\sin^2\theta + \tan^2\theta = (1 - \cos^2\theta) + \frac{\sec^2\theta}{\csc^2\theta}$

Determine whether each triangle should be solved by beginning with the Law of *Sines* or Law of *Cosines*. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. (Lesson 13-5)

55.



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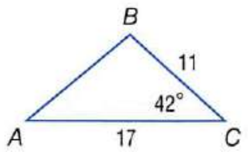

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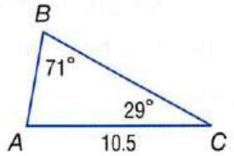

Skills Review

Solve each equation by factoring. (Lesson 5-3)

58. $x^2 + 5x - 24 = 0$

59. $x^2 - 3x - 28 = 0$

60. $x^2 - 4x = 21$

EXPLORE

14-5 Graphing Technology Lab: Solving Trigonometric Equations

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Other Calculator Keystrokes

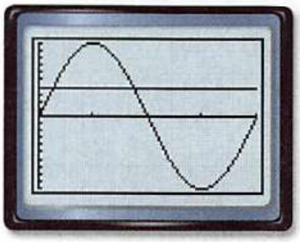
The graph of a trigonometric function is made up of points that represent all values that satisfy the function. To solve a trigonometric equation, you need to find all values of the variable that satisfy the equation. You can use a TI-83/84 Plus graphing calculator to solve trigonometric equations by graphing each side of the equation as a function and then locating the points of intersection.

ACTIVITY 1 Real Solutions

Use a graphing calculator to solve $\sin x = 0.4$ if $0° \leq x < 360°$.

Step 1 Enter and graph related equations. Rewrite the equation as two equations, Y1 = $\sin x$ and Y2 = 0.4. Then graph the two equations. Because the interval is in degrees, set your calculator to degree mode.

KEYSTROKES: [MODE] [▼] [▼] [▶] [ENTER]
[Y=] [SIN] [X,T,θ,n] [)]
[ENTER] 0.4 [ENTER] [GRAPH]


[0, 360] scl: 90 by [−15, 15] scl: 1

Step 2 Approximate the solutions. Based on the graph, you can see that there are two points of intersection in the interval $0° \leq x < 360°$. Use the CALC feature to determine the x-values at which the two graphs intersect.

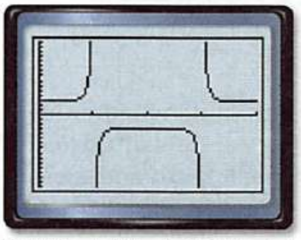
The solutions are $x \approx 23.57°$ and $x \approx 156.4°$.

ACTIVITY 2 No Real Solutions

Use a graphing calculator to solve $\tan^2 x \cos x + 3 \cos x = 0$ if $0° \leq x < 360°$.

Step 1 Enter and graph related equations. The related equations to be graphed are $y_1 = \tan^2 x \cos x + 3 \cos x$ and $y_2 = 0$.

KEYSTROKES: [Y=] [TAN] [X,T,θ,n] [)] [x^2]
[+] 3 [COS] [X,T,θ,n] [)]
[ENTER] 0 [ENTER]


[0, 360] scl: 90 by [−15, 15] scl: 1

Step 2 These two functions do not intersect.

Therefore, the equation $\tan^2 x \cos x + 3 \cos x = 0$ has no real solutions.

Exercises

Use a graphing calculator to solve each equation for the values of x indicated.

1. $\sin x = 0.7$; $0° \leq x < 360°$
2. $\tan x = \cos x$; $0° \leq x < 360°$
3. $3 \cos x + 4 = 0.5$; $0° \leq x < 360°$
4. $0.25 \cos x = 3.4$; $-720° \leq x < 720°$
5. $\sin 2x = \sin x$; $0° \leq x < 360°$
6. $\sin 2x - 3 \sin x = 0$ if $-360° \leq x < 360°$

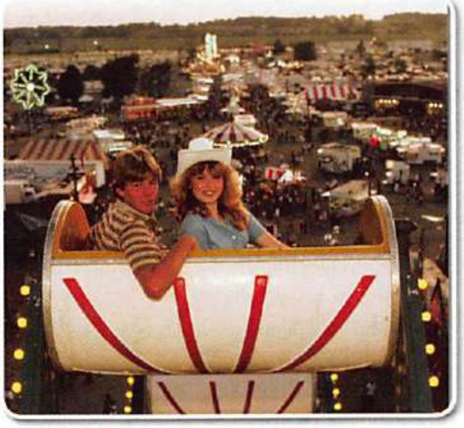
14-5 Solving Trigonometric Equations

Why?

When you ride a Ferris wheel that has a diameter of 40 meters and turns at a rate of 1.5 revolutions per minute, the height above the ground, in meters, of your seat after t minutes can be modeled by the equation

$$h = 21 - 20 \cos 3\pi t.$$

After the ride begins, how long is it before your seat is 31 meters above the ground for the first time?



Then
You verified trigonometric identities. (Lessons 14-2 through 14-4)

Now
- Solve trigonometric equations.
- Find extraneous solutions from trigonometric equations.

New Vocabulary
trigonometric equations

Math Online
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- Extra Examples
- Personal Tutor
- Self-Check Quiz
- Homework Help

Solve Trigonometric Equations So far in this chapter, we have studied a special type of trigonometric equation called an identity. Trigonometric identities are equations that are true for all values of the variable for which both sides are defined. In this lesson, we will examine **trigonometric equations** that are true for only certain values of the variable. Solving these equations resembles solving algebraic equations.

EXAMPLE 1 Solve Equations for a Given Interval

Solve $\sin \theta \cos \theta - \frac{1}{2} \cos \theta = 0$ if $0 \leq \theta \leq 180°$.

$\sin \theta \cos \theta - \frac{1}{2} \cos \theta = 0$ — **Original equation**

$\cos \theta \left(\sin \theta - \frac{1}{2}\right) = 0$ — **Factor.**

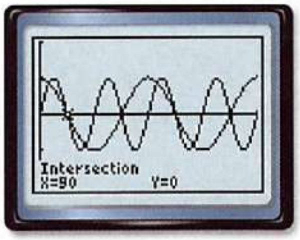
$\cos \theta = 0$ or $\sin \theta - \frac{1}{2} = 0$ — **Zero Product Property**

$\theta = 90°$ or $270°$ $\qquad$ $\sin \theta = \frac{1}{2}$

$\theta = 30°$ or $150°$

The solutions are 30°, 90°, and 150°.

CHECK You can check the answer by graphing $y = \sin \theta \cos \theta$ and $y = \frac{1}{2} \cos \theta$ in the same coordinate plane on a graphing calculator. Then find the points where the graphs intersect. You can see that there are infinitely many such points, but we are only interested in the points between 0° and 180°.



Check Your Progress

1. Find all solutions of $\sin 2\theta = \cos \theta$ if $0 \leq \theta \leq 2\pi$.

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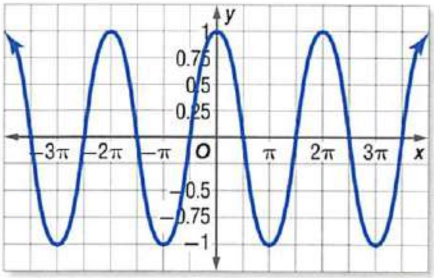
Trigonometric equations are usually solved for values of the variable between 0° and 360° or between 0 radians and 2π radians. There are solutions outside that interval. These other solutions differ by integral multiples of the period of the function.

EXAMPLE 2 Infinitely Many Solutions

Solve $\cos \theta + 1 = 0$ for all values of θ if θ is measured in radians.

$$\cos \theta + 1 = 0$$
$$\cos \theta = -1$$

Look at the graph of $y = \cos \theta$ to find solutions of $\cos \theta = -1$.



The solutions are π, 3π, 5π, and so on, and $-\pi$, -3π, -5π, and so on. The only solution in the interval 0 radians to 2π radians is π. The period of the cosine function is 2π radians. So the solutions can be written as $\pi + 2k\pi$, where k is any integer.

StudyTip

Expressing Solutions as Multiples The expression $\pi + 2k\pi$ includes 3π and its multiples, so it is not necessary to list them separately.

Check Your Progress

2A. Solve $\cos 2\theta + \cos \theta + 1 = 0$ for all values of θ if θ is measured in degrees.

2B. Solve $2 \sin \theta = -1$ for all values of θ if θ is measured in radians.

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Trigonometric equations are often used to solve real-world problems.

Real-World EXAMPLE 3 Solve Trigonometric Equations

AMUSEMENT PARKS **Refer to the beginning of the lesson. How long after the Ferris wheel starts will your seat first be 31 meters above the ground?**

$h = 21 - 20 \cos 3\pi t$ **Original equation**

$31 = 21 - 20 \cos 3\pi t$ **Replace h with 31.**

$10 = -20 \cos 3\pi t$ **Subtract 21 from each side.**

$-\frac{1}{2} = \cos 3\pi t$ **Divide each side by −20.**

$\cos^{-1} -\frac{1}{2} = 3\pi t$ **Take the Arccosine.**

$\frac{2\pi}{3} = 3\pi t$ or $\frac{4\pi}{3} = 3\pi t$ **The Arccosine of $-\frac{1}{2}$ is $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$.**

$\frac{2\pi}{3} + 2\pi k = 3\pi t$ or $\frac{4\pi}{3} + 2\pi k = 3\pi t$ **k is any integer.**

$\frac{2}{9} + \frac{2}{3}k = t$ $\quad$ $\frac{4}{9} + \frac{2}{3}k = t$ **Divide each term by 3π.**

The least positive value for t is obtained by letting $k = 0$ in the first expression. Therefore, $t = \frac{2}{9}$ of a minute or about 13 seconds.

Check Your Progress

3. How long after the Ferris wheel starts will your seat first be 41 meters above the ground?

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Problem-Solving Tip

Look for a Pattern Look for patterns in your solutions. Look for pairs of solutions that differ by exactly π or 2π and write your solutions with the simplest possible pattern.

Extraneous Solutions Some trigonometric equations have no solution. For example, the equation $\cos \theta = 4$ has no solution because all values of $\cos \theta$ are between -1 and 1, inclusive. Thus, the solution set for $\cos \theta = 4$ is empty.

EXAMPLE 4 Determine Whether a Solution Exists

Solve each equation.

a. $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$ **if** $0 \leq \theta \leq 2\pi$

$2 \sin^2 \theta - 3 \sin \theta - 2 = 0$ — Original equation

$(\sin \theta - 2)(2 \sin \theta + 1) = 0$ — Factor.

$\sin \theta - 2 = 0$ or $2 \sin \theta + 1 = 0$ — Zero Product Property

$\sin \theta = 2$ $\qquad 2 \sin \theta = -1$

This is not a solution since all values of $\sin \theta$ are between -1 and 1, inclusive.

$\sin \theta = -\frac{1}{2}$

$\theta = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

The solutions are $\frac{7\pi}{6}$ or $\frac{11\pi}{6}$.

CHECK

$2 \sin \theta - 3 \sin \theta - 2 = 0$

$2 \sin^2 \left(\frac{7\pi}{6}\right) - 3 \sin \left(\frac{7\pi}{6}\right) - 2 \stackrel{?}{=} 0$

$2\left(\frac{1}{4}\right) - 3\left(-\frac{1}{2}\right) - 2 \stackrel{?}{=} 0$

$\frac{1}{2} + \frac{3}{2} - 2 \stackrel{?}{=} 0$

$0 = 0$ ✓

$2 \sin^2 \theta - 3 \sin \theta - 2 = 0$

$2 \sin^2 \left(\frac{11\pi}{6}\right) - 3 \sin \left(\frac{11\pi}{6}\right) - 2 \stackrel{?}{=} 0$

$2\left(\frac{1}{4}\right) - 3\left(-\frac{1}{2}\right) - 2 \stackrel{?}{=} 0$

$\frac{1}{2} + \frac{3}{2} - 2 \stackrel{?}{=} 0$

$0 = 0$ ✓

b. $\sin \theta = 1 + \cos \theta$ **if** $0° \leq \theta < 360°$

$\sin \theta = 1 + \cos \theta$ — Original equation

$\sin^2 \theta = (1 + \cos \theta)^2$ — Square each side.

$1 - \cos^2 \theta = 1 + 2 \cos \theta + \cos^2 \theta$ — $\sin^2 \theta = 1 - \cos^2 \theta$

$0 = 2 \cos \theta + 2 \cos^2 \theta$ — Set the left side equal to 0.

$0 = 2 \cos \theta (1 + \cos \theta)$ — Factor.

$1 + \cos \theta = 0$ or $2 \cos \theta = 0$ — Zero Product Property

$\cos \theta = -1$ $\qquad \cos \theta = 0$

$\theta = 180$ $\qquad \theta = 90°$ or $270°$

CHECK

$\sin \theta = 1 + \cos \theta$

$\sin 90° \stackrel{?}{=} 1 + \cos 90°$

$1 \stackrel{?}{=} 1 + 0$

$1 = 1$ ✓

$\sin \theta = 1 + \cos \theta$

$\sin 180° \stackrel{?}{=} 1 + \cos 180°$

$0 \stackrel{?}{=} 1 + (-1)$

$0 = 0$ ✓

$\sin \theta = 1 + \cos \theta$

$\sin 270° \stackrel{?}{=} 1 + \cos 270°$

$-1 \stackrel{?}{=} 1 + 0$

$-1 \neq 1$ ✗

The solutions are 90° and 180°.

Check Your Progress

Solve each equation.

4A. $\sin^2 \theta + 2 \cos^2 \theta = 4$

4B. $\cos^2 \theta + 3 = 4 - \sin^2 \theta$

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If an equation cannot be solved easily by factoring, try rewriting the expression using trigonometric identities. However, using identities and some algebraic operations, such as squaring, may result in extraneous solutions. So, it is necessary to check your solutions using the original equation.

StudyTip

Solving Trigonometric Equations Remember that *solving a trigonometric equation* means solving for all values of the variable.

EXAMPLE 5 Solve Trigonometric Equations by Using Identities

Solve $2\sec^2\theta - \tan^4\theta = -1$ for all values of θ if θ is measured in degrees.

$2\sec^2\theta - \tan^4\theta = -1$ — **Original equation**

$2(1 + \tan^2\theta) - \tan^4\theta = -1$ — **$\sec^2\theta = 1 + \tan^2\theta$**

$2 + 2\tan^2\theta - \tan^4\theta = -1$ — **Distributive Property**

$\tan^4\theta - 2\tan^2\theta - 3 = 0$ — **Set one side of the equation equal to 0.**

$(\tan^2\theta - 3)(\tan^2\theta + 1) = 0$ — **Factor.**

$\tan^2\theta - 3 = 0$ or $\tan^2\theta + 1 = 0$ — **Zero Product Property**

$\tan^2\theta = 3$ $\qquad$ $\tan^2\theta = -1$

$\tan\theta = \pm\sqrt{3}$ $\qquad$ This part gives no solutions since $\tan^2\theta$ is never negative.

$\theta = 60° + 180°k$ and $\theta = -60° + 180°k$, where k is any integer. The solutions are $60° + 180°k$ and $-60° + 180°k$.

Check Your Progress

Solve each equation.

5A. $\sin\theta\cot\theta - \cos^2\theta = 0$

5B. $\dfrac{\cos\theta}{\cot\theta} + 2\sin^2\theta = 0$

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Check Your Understanding

Example 1 p. 919

Solve each equation if $0° \le \theta \le 360°$.

1. $2\sin\theta + 1 = 0$

2. $\cos^2\theta + 2\cos\theta + 1 = 0$

3. $\cos 2\theta + \cos\theta = 0$

4. $2\cos\theta = 1$

5. $\cos\theta = -\dfrac{\sqrt{3}}{2}$

6. $\sin 2\theta = -\dfrac{\sqrt{3}}{2}$

7. $\cos 2\theta = 8 - 15\sin\theta$

8. $\sin\theta + \cos\theta = 1$

Example 2 p. 920

Solve each equation for all values of θ if θ is measured in radians.

9. $4\sin^2\theta - 1 = 0$

10. $2\cos^2\theta = 1$

11. $\cos 2\theta\sin\theta = 1$

12. $\sin\dfrac{\theta}{2} + \cos\dfrac{\theta}{2} = \sqrt{2}$

13. $\cos 2\theta + 4\cos\theta = -3$

14. $\sin\dfrac{\theta}{2} + \cos\theta = 1$

Solve each equation for all values of θ if θ is measured in degrees.

15. $\cos 2\theta - \sin^2\theta + 2 = 0$

16. $\sin^2\theta - \sin\theta = 0$

17. $2\sin^2\theta - 1 = 0$

18. $\cos\theta - 2\cos\theta\sin\theta = 0$

19. $\cos 2\theta\sin\theta = 1$

20. $\sin\theta\tan\theta - \tan\theta = 0$

Example 3 p. 920

21. **LIGHT** The number of hours of daylight d in Hartford, Connecticut, may be approximated by the equation $d = 3\sin\frac{2\pi}{365}t + 12$, where t is the number of days after March 21.

a. On what days will Hartford have exactly $10\frac{1}{2}$ hours of daylight?

b. Using the results in part **a**, tell what days of the year have at least $10\frac{1}{2}$ hours of daylight. Explain how you know.

Examples 4 and 5
pp. 921–922

Solve each equation.

22. $\sin^2 2\theta + \cos^2 \theta = 0$

23. $\tan^2 \theta + 2 \tan \theta + 1 = 0$

24. $\cos^2 \theta + 3 \cos \theta = -2$

25. $\sin 2\theta - \cos \theta = 0$

26. $\tan \theta = 1$

27. $\cos 8\theta = 1$

28. $\sin \theta + 1 = \cos 2\theta$

29. $2 \cos^2 \theta = \cos \theta$

Practice and Problem Solving

= **Step-by-Step Solutions** begin on page R20.
Extra Practice begins on page 947.

Example 1
p. 919

Solve each equation for the given interval.

30. $\cos^2 \theta = \frac{1}{4}$; $0° \leq \theta \leq 360°$

31. $2 \sin^2 \theta = 1$; $90° < \theta < 270°$

32. $\sin 2\theta - \cos \theta = 0$; $0 \leq \theta \leq 2\pi$

33. $3 \sin^2 \theta = \cos^2 \theta$; $0 \leq \theta \leq \frac{\pi}{2}$

34. $2 \sin \theta + \sqrt{3} = 0$; $180° < \theta < 360°$

35. $4 \sin^2 \theta - 1 = 0$; $180° < \theta < 360°$

Example 2
p. 920

Solve each equation for all values of θ if θ is measured in radians.

36. $\cos 2\theta + 3 \cos \theta = 1$

37. $2 \sin^2 \theta = \cos \theta + 1$

38. $\cos^2 \theta - \frac{3}{2} = \frac{5}{2} \cos \theta$

39. $3 \cos \theta - \cos \theta = 2$

Solve each equation for all values of θ if θ is measured in degrees.

40. $\sin \theta - \cos \theta = 0$

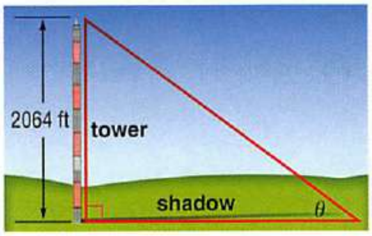
41. $\tan \theta - \sin \theta = 0$

42. $\sin^2 \theta = 2 \sin \theta + 3$

43. $4 \sin^2 \theta = 4 \sin \theta - 1$

Example 3
p. 920

44. **ELECTRONICS** One of the tallest structures in the world is a television transmitting tower located near Fargo, North Dakota, with a height of 2064 feet. What is the measure of θ if the length of the shadow is 1 mile?



Examples 4 and 5
pp. 921–922

Solve each equation.

45. $2 \sin^2 \theta = 3 \sin \theta + 2$

46. $2 \cos^2 \theta + 3 \sin \theta = 3$

47. $\sin^2 \theta + \cos 2\theta = \cos \theta$

48. $2 \cos^2 \theta = -\cos \theta$

49. **RIVERS** Due to ocean tides, the depth y in meters of the River Thames in London varies as a sine function of x, the hour of the day. On a certain day that function was $y = 3 \sin \left[\frac{\pi}{6}(x - 4)\right] + 8$, where $x = 0, 1, 2, \ldots, 24$ corresponds to 12:00 midnight, 1:00 A.M., 2:00 A.M., …, 12:00 midnight the next night.

a. What is the maximum depth of the River Thames on that day?

b. At what times does the maximum depth occur?

Solve each equation if θ is measured in radians.

50. $(\cos \theta)(\sin 2\theta) - 2 \sin \theta + 2 = 0$

51. $2 \sin^2 \theta + (\sqrt{2} - 1) \sin \theta = \frac{\sqrt{2}}{2}$

Solve each equation if θ is measured in degrees.

52. $\sin 2\theta + \frac{\sqrt{3}}{2} = \sqrt{3} \sin \theta + \cos \theta$

53. $1 - \sin^2 \theta - \cos \theta = \frac{3}{4}$

Solve each equation.

54. $2 \sin \theta = \sin 2\theta$

55. $\cos \theta \tan \theta - 2 \cos^2 \theta = -1$

Real-World Link

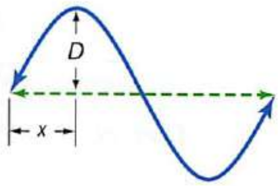
The sparkle of a diamond is created by refracted light. Light travels at different speeds in various media. When light rays pass from one medium to another in which they travel at a different velocity, the light is bent or refracted.

56. DIAMONDS According to Snell's Law, $n_1 \sin i = n_2 \sin r$, where n_1 is the index of refraction of the medium the light is exiting, n_2 is the index of refraction of the medium the light is entering, i is the degree measure of the angle of incidence, and r is the degree measure of the angle of refraction.

a. The index of refraction of a diamond is 2.42, and the index of refraction of air is 1.00. If a beam of light strikes a diamond at an angle of 35°, what is the angle of refraction?

b. Explain how a gemologist might use Snell's Law to determine whether a diamond is genuine.

57. MUSIC A wave traveling in a guitar string can be modeled by the equation $D = 0.5 \sin (6.5x) \sin (2500t)$, where D is the displacement in millimeters at the position x meters from the left end of the string at time t seconds. Find the first positive time when the point 0.5 meter from the left end has a displacement of 0.01 millimeter.



H.O.T. Problems

Use **H**igher-**O**rder **T**hinking Skills

58. FIND THE ERROR Jennifer and Tat are solving $2 \sin \theta \cos \theta = \sin \theta$ for $0° \leq \theta \leq 360°$. Is either of them correct? Explain your reasoning.

Jennifer

$$2 \sin \theta \cos \theta = \sin \theta$$
$$\frac{2 \sin \theta \cos \theta}{\sin \theta} = \frac{\sin \theta}{\sin \theta}$$
$$2 \cos \theta = 1$$
$$\cos \theta = \frac{1}{2}$$
$$\theta = 0°, 60°, 180°, 300°$$

Tat

$$2 \sin \theta \cos \theta = \sin \theta$$
$$-\sin \theta = -\sin \theta$$
$$2 \cos \theta = 0$$
$$\cos \theta = 0$$
$$\theta = 90°, 270°$$

59. CHALLENGE Solve $\sin 2x < \sin x$ for $0 \leq x \leq 2\pi$ without a calculator.

60. WRITING IN MATH Compare and contrast solving trigonometric equations with solving linear and quadratic equations. What techniques are the same? What techniques are different? How many solutions do you expect?

61. REASONING Explain why many trigonometric equations have infinitely many solutions.

62. OPEN ENDED Write an example of a trigonometric equation that has exactly two solutions if $0° \leq \theta \leq 360°$.

63. CHALLENGE How many solutions in the interval $0° \leq \theta \leq 360°$ should you expect for $a \sin (b\theta + c) + d = d\left(\frac{a}{2}\right)$, if $a \neq 0$ and b is a positive integer?

Standardized Test Practice

64. EXTENDED RESPONSE Charles received \$2500 for a graduation gift. He put it into a savings account in which the interest rate was 5.5% per year.

a. How much did he have in his savings account after 5 years if he made no deposits or withdrawals?

b. After how many years will the amount in his savings account have doubled?

65. PROBABILITY Find the probability of rolling three 3s if a number cube is rolled three times.

A $\frac{1}{216}$ C $\frac{1}{6}$

B $\frac{1}{36}$ D $\frac{1}{4}$

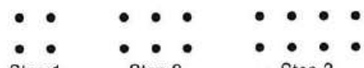
66. Use synthetic substitution to find $f(-2)$ for the function below.

$$f(x) = x^4 + 10x^2 + x + 8$$

F 62 H 30

G 38 J 8

67. ACT/SAT The pattern of dots below continues infinitely, with more dots being added at each step.



Which expression can be used to determine the number of dots in the nth step?

A $2n$ C $n(n+1)$

B $n(n+2)$ D $2(n+1)$

Spiral Review

Find the exact value of each expression. (Lesson 14-4)

68. $\cos 165°$

69. $\sin 22\frac{1}{2}°$

70. $\sin \frac{7\pi}{8}$

71. $\cos \frac{7\pi}{12}$

Verify that each equation is an identity. (Lesson 14-3)

72. $\sin(270° - \theta) = -\cos\theta$

73. $\cos(90° + \theta) = -\sin\theta$

74. $\cos(90° - \theta) = \sin\theta$

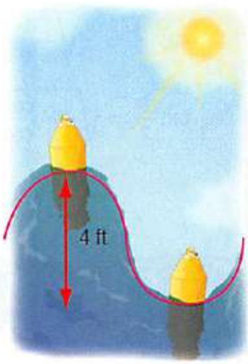
75. $\sin(90° - \theta) = \cos\theta$

76. WATER SAFETY A harbor buoy bobs up and down with the waves. The distance between the highest and lowest points is 4 feet. The buoy moves from its highest point to its lowest point and back to its highest point every 10 seconds. (Lesson 13-7)

a. Write an equation for the motion of the buoy. Assume that it is at equilibrium at $t = 0$ and that it is on the way up from the normal water level.

b. Draw a graph showing the height of the buoy as a function of time.

c. What is the height of the buoy after 12 seconds?



Find the first three terms of each arithmetic series described. (Lesson 11-2)

77. $a_1 = 17, a_n = 197, S_n = 2247$

78. $a_1 = -13, a_n = 427, S_n = 18{,}423$

79. $n = 31, a_n = 78, S_n = 1023$

80. $n = 19, a_n = 103, S_n = 1102$

Graph each rational function. (Lesson 9-4)

81. $f(x) = \frac{1}{(x+3)^2}$

82. $f(x) = \frac{x+4}{x-1}$

83. $f(x) = \frac{x+2}{x^2 - x - 6}$

CHAPTER 14

Study Guide and Review

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- STUDY TO GO
- Vocabulary Review

Chapter Summary

Key Concepts

Trigonometric Identities (Lessons 14-1, 14-2, and 14-5)

- Trigonometric identities describe the relationships between trigonometric functions.
- Trigonometric identities can be used to simplify, verify, and solve trigonometric equations and expressions.

Sum and Difference of Angles Identities (Lesson 14-3)

- For all values of A and B:
  $\cos(A \pm B) = \cos A \cos B \pm \sin A \sin B$
  $\sin(A \pm B) = \sin A \sin B \pm \cos A \sin B$

Double-Angle and Half-Angle Identities (Lesson 14-4)

- Double-angle identities:

  $\sin 2\theta = 2 \sin \theta \cos \theta$
  $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$
  $\cos 2\theta = 1 - 2 \sin^2 \theta$
  $\cos 2\theta = 2 \sin^2 \theta - 1$

- Half-angle identities:

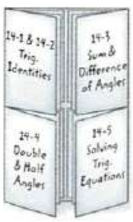
  $\sin \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos \theta}{2}}$

  $\cos \frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos \theta}{2}}$

  $\tan \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}, \cos \theta \neq -1$

FOLDABLES® Study Organizer

Be sure the Key Concepts are noted in your Foldable.



Key Vocabulary

cofunction identity (p. 891)
negative angle identity (p. 891)
Pythagorean identity (p. 891)
quotient identity (p. 891)
reciprocal identity (p. 891)
trigonometric equation (p. 919)
trigonometric identity (p. 891)

Vocabulary Check

Choose the correct term to complete each sentence.

1. The __________ can be used to find the sine or cosine of 75° if the sine and cosine of 90° and 15° are known.
2. The identities $\tan \theta = \frac{\sin \theta}{\cos \theta}$ and $\cot \theta = \frac{\cos \theta}{\sin \theta}$ are examples of __________.
3. A __________ is an equation involving trigonometric functions that is true for all values for which every expression in the equation is defined.
4. The __________ can be used to find sin 60° using 30° as a reference.
5. A __________ is true for only certain values of the variable.
6. The __________ formula can be used to find $\cos 33\frac{1}{2}°$.
7. The identities $\csc \theta = \frac{1}{\sin \theta}$ and $\sec \theta = \frac{1}{\cos \theta}$ are examples of __________.
8. The __________ can be used to find the sine or cosine of 120° if the sine and cosine of 90° and 30° are known.
9. $\cos^2 \theta + \sin^2 \theta = 1$ is an example of a __________.

MIXED PROBLEM SOLVING
For mixed problem-solving practice, see page 992.

Lesson-by-Lesson Review

14-1 Trigonometric Identities (pp. 891–897)

Find the value of each expression.

10. $\sin \theta$, if $\cos \theta = \frac{\sqrt{2}}{2}$ and $270° < \theta < 360°$

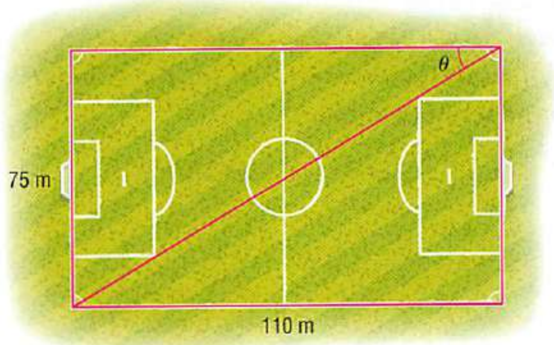
11. $\sec \theta$, if $\cot \theta = \frac{\sqrt{2}}{2}$ and $90° < \theta < 180°$

12. $\tan \theta$, if $\cot \theta = 2$ and $0° < \theta < 90°$

13. $\cos \theta$, if $\sin \theta = -\frac{3}{5}$ and $180° < \theta < 270°$

14. $\csc \theta$, if $\cot \theta = \frac{4}{5}$ and $270° < \theta < 360°$

15. SOCCER For international matches, the maximum dimensions of a soccer field are 110 meters by 75 meters. Find $\sin \theta$.



Simplify each expression.

16. $1 - \tan \theta \sin \theta \cos \theta$

17. $\tan \theta \csc \theta$

18. $\sin \theta + \cos \theta \cot \theta$

19. $\cos \theta (1 + \tan^2 \theta)$

EXAMPLE 1

Find $\sin \theta$ if $\cos \theta = \frac{3}{4}$ and $0° < \theta < 90°$.

$\cos^2 \theta + \sin^2 \theta = 1$	Trigonometric identity
$\sin^2 \theta = 1 - \cos^2 \theta$	Subtract $\cos^2 \theta$ from each side.
$\sin^2 \theta = 1 - \left(\frac{3}{4}\right)^2$	Substitute $\frac{3}{4}$ for $\cos \theta$.
$\sin^2 \theta = 1 - \frac{9}{16}$	Square $\frac{3}{4}$.
$\sin^2 \theta = \frac{7}{16}$	Subtract.
$\sin \theta = \pm\frac{\sqrt{7}}{4}$	Take the square root of each side.

Because θ is in the first quadrant, $\sin \theta$ is positive. Thus, $\sin \theta = \frac{\sqrt{7}}{4}$.

EXAMPLE 2

Simplify $\cos \theta \sec \theta \cot \theta$.

$$\cos \theta \sec \theta \cot \theta = \cos \theta \left(\frac{1}{\cos \theta}\right)\left(\frac{\cos \theta}{\sin \theta}\right)$$
$$= \cot \theta$$

14-2 Verifying Trigonometric Identities (pp. 898–903)

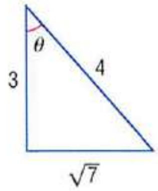
Verify that each of the following is an identity.

20. $\tan \theta \cos \theta + \cot \theta \sin \theta = \sin \theta + \cos \theta$

21. $\frac{\cos \theta}{\cot \theta} + \frac{\sin \theta}{\tan \theta} = \sin \theta + \cos \theta$

22. $\sec^2 \theta - 1 = \frac{\sin^2 \theta}{1 - \sin^2 \theta}$

23. GEOMETRY The right triangle shown at the right is used in a special quilt. Use the measures of the sides of the triangle to show that $\tan^2 \theta + 1 = \sec^2 \theta$.



EXAMPLE 3

Verify that $\frac{\cos \theta + 1}{\sin \theta} = \cot \theta + \csc \theta$ is an identity.

$\frac{\cos \theta + 1}{\sin \theta} \stackrel{?}{=} \cot \theta + \csc \theta$	Original equation
$\frac{\cos \theta}{\sin \theta} + \frac{1}{\sin \theta} \stackrel{?}{=} \cot \theta + \csc \theta$	Simplify.
$\cot \theta + \csc \theta = \cot \theta + \csc \theta$ ✓	Simplify.

Study Guide and Review

14-3 Sum and Difference of Angles Identities (pp. 904–909)

Find the exact value of each expression.

24. $\cos(-135°)$

25. $\cos 15°$

26. $\sin 210°$

27. $\sin 105°$

28. $\tan 75°$

29. $\cos 105°$

Verify that each of the following is an identity.

30. $\sin(\theta + 90) = \cos\theta$

31. $\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$

32. $\tan(\theta - \pi) = \tan\theta$

EXAMPLE 4

Find the exact value of sin 75°.

Use $\sin(A + B) = \sin A \sin B + \cos A \sin B$.

$$\begin{aligned}\sin 75° &= \sin(30° + 45°)\\ &= \sin 30° \sin 45° + \cos 30° \sin 45°\\ &= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{4} \text{ or } \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

14-4 Double-Angle and Half-Angle Identities (pp. 911–917)

Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin\frac{\theta}{2}$, and $\cos\frac{\theta}{2}$ for each of the following.

33. $\cos\theta = \frac{4}{5}$; $0° < \theta < 90°$

34. $\sin\theta = -\frac{1}{4}$; $180° < \theta < 270°$

35. $\cos\theta = \frac{2}{3}$; $\frac{\pi}{2} < \theta < \pi$

36. **BASEBALL** The infield of a baseball diamond is a square with side length 90 feet.

- **a.** Find the length of the diagonal.
- **b.** Write the ratio for sin 45° using the lengths of the baseball diamond.
- **c.** Use the formula $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$ to verify the ratio you wrote in part **b**.

EXAMPLE 5

Find the exact value of $\sin\frac{\theta}{2}$ if $\cos\theta = \frac{3}{5}$ and θ is in the second quadrant.

$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$ — **Half-angle identity**

$= \pm\sqrt{\frac{1-\frac{3}{5}}{2}}$ — **$\cos\theta = \frac{3}{5}$**

$= \pm\sqrt{\frac{\frac{2}{5}}{2}}$ — **Subtract.**

$= \pm\sqrt{\frac{1}{5}}$ — **Divide.**

$= \pm\frac{\sqrt{5}}{5}$ — **Simplify.**

Since θ is in the second quadrant, $\sin\frac{\theta}{2} = \frac{\sqrt{5}}{5}$.

14-5 Solving Trigonometric Equations (pp. 919–925)

Find all solutions of each equation for the given interval.

37. $2\cos\theta - 1 = 0$; $0° \le \theta < 360°$

38. $4\cos^2\theta - 1 = 0$; $0 \le \theta < 2\pi$

39. $\sin 2\theta + \cos\theta = 0$; $0° \le \theta < 360°$

40. $\sin^2\theta = 2\sin\theta + 3$; $0° \le \theta < 360°$

41. $4\cos^2\theta - 4\cos\theta + 1 = 0$; $0 \le \theta < 2\pi$

EXAMPLE 6

Find all solutions of $\sin 2\theta - \cos\theta = 0$ if $0 \le \theta < 2\pi$.

$\sin 2\theta - \cos\theta = 0$ — **Original equation**

$2\sin\theta\cos\theta - \cos\theta = 0$ — **Double-angle identity**

$\cos\theta(2\sin\theta - 1) = 0$ — **Factor.**

$\cos\theta = 0$ or $2\sin\theta - 1 = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad$ $\sin\theta = \frac{1}{2}$; $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Practice Test

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Chapter Test

1. **MULTIPLE CHOICE** Which expression is equivalent to $\sin\theta + \cos\theta\cot\theta$?

   A $\cot\theta$ C $\sec\theta$

   B $\tan\theta$ D $\csc\theta$

2. Verify that $\cos(30° - \theta) = \sin(60° + \theta)$ is an identity.

3. Verify that $\cos(\theta - \pi) = -\cos\theta$.

4. **MULTIPLE CHOICE** What is the exact value of $\sin\theta$, if $\cos\theta = -\frac{3}{5}$ and $90° < \theta < 180°$?

   F $\frac{5}{3}$

   G $\frac{\sqrt{34}}{8}$

   H $-\frac{4}{5}$

   J $-\frac{24}{25}$

Find the value of each expression.

5. $\cot\theta$, if $\sec\theta = \frac{4}{3}$; $270° < \theta < 360°$

6. $\tan\theta$, if $\cos\theta = -\frac{1}{2}$; $90° < \theta < 180°$

7. $\sec\theta$, if $\csc\theta = 2$; $180° < \theta < 270°$

8. $\cot\theta$, if $\csc\theta = -\frac{5}{3}$; $270° < \theta < 360°$

9. $\sec\theta$, if $\sin\theta = \frac{1}{2}$; $0° \le \theta < 90°$

Verify that each of the following is an identity.

10. $\sin\theta(\cot\theta + \tan\theta) = \sec\theta$

11. $\frac{\cos^2\theta}{1 - \sin\theta} = \frac{\cos\theta}{\sec\theta - \tan\theta}$

12. $(\tan\theta + \cot\theta)^2 = \csc^2\theta\sec^2\theta$

13. $\frac{1 + \sec\theta}{\sec\theta} = \frac{\sin^2\theta}{1 - \cos\theta}$

14. $\frac{\sin\theta}{1 - \cos\theta} = \csc\theta + \cot\theta$

15. **MULTIPLE CHOICE** What is the exact value of $\tan\frac{\pi}{8}$?

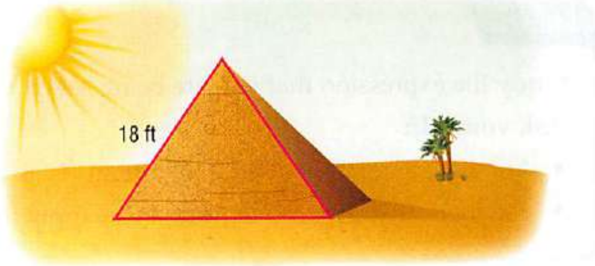
   A $\frac{\sqrt{2 - \sqrt{3}}}{2}$

   B $\sqrt{2} - 1$

   C $1 - \sqrt{2}$

   D $-\frac{\sqrt{2 - \sqrt{3}}}{2}$

16. **HISTORY** Some researchers believe that the builders of ancient pyramids, such as the Great Pyramid of Khufu, may have tried to build the faces as equilateral triangles. Later they had to change to other types of triangles. Suppose a pyramid is built such that a face is an equilateral triangle of side length 18 feet.



   a. Find the height of the equilateral triangle.

   b. Use the formula $\sin 2\theta = 2\sin\theta\cos\theta$ and the measures of the equilateral triangle and its height to show that $\sin 2(30°) = \sin 60°$. Find the exact values.

Find the exact value of each expression.

17. $\cos(-225°)$

18. $\sin 480°$

19. $\cos 75°$

20. $\sin 165°$

21. **ROCKETS** A model rocket is launched with an initial velocity of 20 meters per second. The range of a projectile is given by the formula $R = \frac{v^2}{g}\sin 2\theta$, where R is the range, v is the initial velocity, g is acceleration due to gravity or 9.8 meters per second squared, and θ is the launch angle. What angle is needed in order for the rocket to reach a range of 25 meters?

Solve each equation for all values of θ if θ is measured in radians.

22. $2\cos^2\theta - 3\cos\theta - 2 = 0$

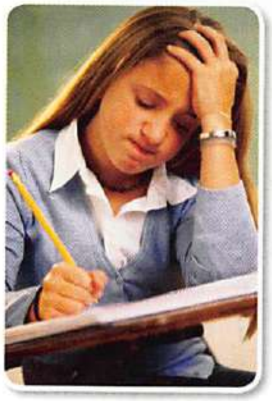
23. $2\sin 3\theta - 1 = 0$

Solve each equation for $0° \le \theta \le 360°$ if θ is measured in degrees.

24. $\cos 2\theta + \cos\theta = 2$

25. $\sin\theta\cos\theta - \frac{1}{2}\sin\theta = 0$

Simplify Expressions



Some standardized test questions will require you to use the properties of algebra to simplify expressions. Follow the steps below to help prepare to solve these kinds of problems.

Strategies for Simplifying Expressions

Step 1

Study the expression that you are being asked to simplify.

Ask yourself:

- Are there any mathematical operations I can apply to help simplify the expression?
- Are there any laws or identities I can apply to help simplify the expression?

Step 2

Solve the problem and check your solution.

- Use the order of operations.
- Combine terms and factor as appropriate.
- Apply laws and identities.

Step 3

Check your solution if time permits.

- Retrace the steps in your work to make sure you answered the question thoroughly and accurately.
- If needed, sometimes you can use your scientific calculator to help you check your solution. Evaluate the original expression and your answer for some value and make sure they are the same.

EXAMPLE

Solve the problem below. Responses will be graded using the short-response scoring rubric shown.

Simplify the trigonometric expression shown below by writing it in terms of $\sin\theta$. Show your work to receive full credit.

$$\frac{\cos\theta}{\sec\theta + \tan\theta}$$

Scoring Rubric

Criteria	Score
Full Credit: The answer is correct and a full explanation is provided that shows each step.	2
Partial Credit: • The answer is correct, but the explanation is incomplete. • The answer is incorrect, but the explanation is correct.	1
No Credit: Either an answer is not provided or the answer does not make sense.	0

Read the problem statement carefully. You are given a trigonometric expression and asked to simplify it by writing it in terms of sin θ. So, your final answer must contain only numbers and terms involving the sin θ. Show your work to receive full credit.

Example of a 2-point response:

Use trigonometric identities to simplify the expression.

$$\frac{\cos\theta}{\sec\theta+\tan\theta}=\frac{\cos\theta}{\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}$$ **Definition of sec θ and tan θ**

$$=\frac{\cos\theta}{\frac{1+\sin\theta}{\cos\theta}}$$ **Simplify the denominator.**

$$=\frac{\cos^2\theta}{1+\sin\theta}$$ **Simplify the complex fraction.**

$$=\frac{1-\sin^2\theta}{1+\sin\theta}$$ **Pythagorean identity**

$$=\frac{(1+\sin\theta)(1-\sin\theta)}{1+\sin\theta}$$ **Factor.**

$$=1-\sin\theta$$ **Simplify.**

The simplified expression is $1-\sin\theta$.

The steps, calculations, and reasoning are clearly stated. The student also arrives at the correct answer. So, this response is worth the full 2 points.

Exercises

Solve each problem. Show your work. Responses will be graded using the short-response scoring rubric given at the beginning of the lesson.

1. Simplify $\frac{\sec\theta}{\cot\theta+\tan\theta}$ by writing it in terms of sin θ.

2. What is $\frac{10a^{-3}}{29b^4}\div\frac{5a^{-5}}{16b^{-7}}$?

3. Write $\frac{y+1}{y-1}+\frac{y+2}{y-2}+\frac{y}{y^2-3y+2}$ in simplest form.

4. Simplify $\frac{\cot^2\theta-\csc^2\theta}{\tan^2\theta-\sec^2\theta}$ by writing it as a constant.

5. Multiply $(-5+2i)(6-i)(4+3i)$.

6. Simplify $(\cot\theta+1)^2-2\cot\theta$ by writing it in terms of csc θ.

7. Express $\frac{4-\sqrt{7}}{3+\sqrt{7}}$ in simplest form.

CHAPTER 14

Standardized Test Practice

Cumulative, Chapters 1 through 14

Multiple Choice

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. The profit p that Selena's Shirt Store makes in a day can be represented by the inequality $10t + 200 < p < 15t + 250$, where t represents the number of shirts sold. If the store sold 45 shirts on Friday, which of the following is a reasonable amount that the store made?

A \$200   B \$625   C \$850   D \$950

2. Use a sum or difference of angles identity to find the exact value of cos 75°.

F $\frac{\sqrt{2}-\sqrt{6}}{4}$ H $\frac{\sqrt{6}-\sqrt{2}}{4}$

G $\frac{\sqrt{2}+\sqrt{6}}{2}$ J $\frac{\sqrt{6}+\sqrt{2}}{4}$

Test-Taking Tip

Question 2 You can check your answer using a scientific calculator. Find cos 75° and compare it to the value of your answer.

3. Use the table to determine the expression that best represents the sum of the degree measures of the interior angles of a polygon with n sides.

Polygon	Number of Sides	Sum of Measures
triangle	3	180
quadrilateral	4	360
pentagon	5	540
hexagon	6	720
heptagon	7	900
octagon	8	1080

A $180 + n$ C $180(n - 2)$

B $180n$ D $60n$

4. Which of the following best describes the graphs of $y = 3x - 5$ and $4y = 12x + 16$?

F The lines have the same y-intercept.

G The lines have the same x-intercept.

H The lines are perpendicular.

J The lines are parallel.

5. What is the product of $\begin{bmatrix} 5 & -2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$?

A $\begin{bmatrix} 11 \\ -1 \end{bmatrix}$ C $\begin{bmatrix} 5 & -10 \\ 0 & -6 \\ 6 & -15 \end{bmatrix}$

B $\begin{bmatrix} 11 & -1 \end{bmatrix}$ D undefined

6. Which quadratic equation has roots $\frac{1}{2}$ and $\frac{1}{3}$?

F $5x^2 - 5x - 2 = 0$

G $5x^2 - 5x + 1 = 0$

H $6x^2 + 5x - 1 = 0$

J $6x^2 - 5x + 1 = 0$

7. How can you express $\cos\theta \csc\theta \cot\theta$ in terms of $\sin\theta$?

A $\frac{1-\sin^2\theta}{\sin^2\theta}$ C $\frac{\sin^2\theta}{2}$

B $\frac{1+\sin^2\theta}{\sin^2\theta}$ D $\frac{1-\sin^2\theta}{\sin\theta}$

8. The area of a rectangle is $25a^4 - 16b^2$. Which factors could represent the length times width?

F $(5a^2 + 4b)(5a^2 + 4b)$ H $(5a - 4b)(5a - 4b)$

G $(5a^2 + 4b)(5a^2 - 4b)$ J $(5a + 4b)(5a - 4b)$

9. What is the domain of $f(x) = \sqrt{5x - 3}$?

A $\left\{x \,\middle|\, x > \frac{3}{5}\right\}$ C $\left\{x \,\middle|\, x \geq \frac{3}{5}\right\}$

B $\left\{x \,\middle|\, x > -\frac{3}{5}\right\}$ D $\left\{x \,\middle|\, x \geq -\frac{3}{5}\right\}$

10. If the equation $y = 3^x$ is graphed, which of the following values of x would produce a point closest to the x-axis?

F $\frac{3}{4}$ H 0

G $\frac{1}{4}$ J $-\frac{3}{4}$