Introduction to the 2021 Mathematics Framework

Contents

Note to reader: The use of the non-binary, singular pronouns *they*, *them*, *their*, *theirs*, *themself*, and *themselves* in this framework is intentional.

Introduction

A society without mathematical affection is like a city without concerts, parks, or museums. To miss out on mathematics is to live without an opportunity to play with beautiful ideas and see the world in a new light.

—Francis Su (2020)

Welcome to the 2021 *Mathematics Framework for California Public Schools, Transitional Kindergarten Through Grade Twelve* (*Math Framework*). This framework serves as a guide to implementing the California Common Core State Standards for Mathematics (CA CCSSM or the Standards). Built upon underlying and updated principles of *focus*, *coherence*, and *rigor*, the Standards hold the promise of enabling all California students to become powerful users of mathematics in order to better understand and positively impact the world—in their careers, in college, and in civic life.

Mathematics as a Gatekeeper or a Launchpad?

Be careful how you interpret the world: It is like that.

—Erich Heller (1952)

Mathematics provides a set of lenses that provide important ways to understand many situations and ideas. The ability to use this mathematical lens flexibly and accurately enables the people of California to apply mathematical understandings in ways that influence their communities and the larger world in many important ways. In this way, math continuous to play a role in how we conceive of our careers, evidence-based civic discourse and policy-making, and the examination of assumptions and principles underlying action. All students are capable of making these contributions and achieving these abilities to very high levels. As a guide to implementing the Standards, this framework lays out mathematical learning experiences that can move California closer to the goal of mathematical power for all.

Unfortunately, the subject and community of mathematics has a history of exclusion and filtering, rather than inclusion and welcoming. There persists a mentality that some children are “bad in math” (or otherwise don’t belong) from many sources and at many levels. Girls and Black and Brown children, notably, represent groups that more often receive messages that they are not good in math compared to White and male counterparts (Shah & Leonardo, 2017). As early as preschool and kindergarten, research and policy documents use deficit-oriented labels to describe Black and Latinx and poor children’s mathematical learning and position them as already behind their white and middle-class peers (NCSM & TODOS, 2016).

Students internalize many of those messages to such a degree that switching their self-identity from “bad at math” to “love math” is rare. Students also self-select out of mathematics because they perceive that mathematics lacks relevance for them, and no longer recognize the inherent value or purpose in learning math. The fixed mindset about math ability reflected in these beliefs helps to explain the exclusionary role that mathematics plays in students' opportunities, and leads to widespread inequities in the discipline of mathematics such as:

- Students who are perceived as "weak" in math are often informally tracked before grade seven in ways that severely limit their experiences with and approaches to math (Butler, 2008) and their future options (Parker et al, 2014). See also Chapter Eight: *Grades 9–12*.
- Students who do not quickly and accurately perform rote procedures get discouraged and decide not to persist in mathematically-oriented studies.
- Students who are learning the English language are deemed incapable of handling, and denied access to, grade-level authentic mathematics.
- Students with learning differences that affect performance on computational tasks are denied access to richer mathematics, even when the learning differences might not affect other mathematical domains (Lambert, 2018).
- Students who do not have the opportunity to accelerate their math courses in middle and high school can be denied entry into prestigious colleges.

Many factors contribute to mathematics exclusion. As one example, consider a system described in more detail in chapters seven (Grades 6–8) and eight (Grades 9–12): Though many high schools offer integrated mathematics, high school mathematics courses are often structured in such a way (e.g., algebra-geometry-algebra 2-precalculus) that calculus is only available to students who are considered "advanced" in middle school—that is, taking algebra as a grade-eight student. In order to reach algebra in grade eight, students must take all of middle grades math, grades 6–8, in just two years (or else skip some foundational material). This means that many school systems are organized to effectively decide which students can reach calculus when they begin grade six. This reality is responsible for considerable racial- and gender-based inequities and to the majority of students being filtered out of a STEM pathway (Joseph, Hailu, Boston, 2017). Moreover, English learners have disproportionately less access, are placed more often in remedial classes and are steered away from STEM courses and pathways (National Academies of Sciences, Engineering, and Medicine, 2018).

When we consider the fact that many competitive colleges and universities (those that accept less than 25 percent of applicants) have calculus as an unstated requirement, the inequitable pathway becomes even more problematic. Many students remain unaware that their status at the end of fifth grade can help or hinder their ability to attend a top university; if they are not in the advanced math track and on a pathway to calculus in each of the subsequent six years of school, they will not meet this unstated admission requirement. This mathematics pathway system, typical of many school districts, counters the evidence that shows all fifth graders are capable of eventually learning calculus *when provided appropriate messaging, teaching, and support*. The system of providing pathways to calculus to only some students has resulted in too many potential STEM students—especially Latinx and African American students—being denied important opportunities. At the same time, too many students are blocked from pursuing non-STEM careers by arbitrary or irrelevant math hurdles. Mathematics education needs to support students whether they choose to pursue STEM disciplines, or other promising majors that prepare them for careers in other fields like law, politics, design, and the media. Math also needs to be relevant for students who pursue careers directly after high school, without attending college (Daro & Asturias, 2019). Schooling practices that lead to such race- and gender-correlated disparities can even lead to legal liabilities for districts and schools (Lawyers' Committee for Civil Rights of the San Francisco Bay Area, 2013). A fuller discussion of one example is included in Chapter 8: Grades 9–12. The middle- and high-school chapters (chapters 7 and 8) outline an approach that enables all students to move to calculus with grade level courses, 6, 7, and 8 in middle school.

Mathematics education can also serve as a launchpad to understanding and acting in the world through and with mathematics. While every level of schooling must focus on providing access to mathematical power for *all* students, a critical component needed to open mathematics doorways for all students is change at the high school level. In

Catalyzing Change in Middle School Mathematics, NCTM suggests that the purpose of school mathematics expand to include the development of positive mathematical identities and a strong sense of agency (see Aguirre, Mayfield-Ingram, & Martin, 2013). NCTM further urges educators to focus on dismantling structural obstacles that stand in the way of rich mathematical experiences for all students, and organize middle school mathematics along a common shared pathway grounded in the use of mathematical practices and processes that support mathematical understanding. Pathways that provide access to higher-level mathematics from a typical grade nine course are described in Chapter Eight: Grades 9–12. In local educational agencies where high school administrators commit to such pathways and vow to support communities of teachers and students in succeeding in grade-level appropriate mathematics, middle schools can avoid compressing or skipping important mathematical courses that can race students through fundamental content. Nor will they need to track students into different pathways. More fundamentally, all stakeholders need to work to shift the definition of mathematics success away from acceleration, and focus on depth of learning.

Learning Mathematics: for All

Students learn best when they are actively engaged in questioning, struggling, problem solving, reasoning, communicating, and explaining. The research is overwhelmingly clear that powerful mathematics classrooms thrive when students feel a sense of agency (a willingness to engage in the discipline, based in a belief in progress through engagement) and an understanding that the intellectual authority in mathematics rests in mathematical reasoning itself (in other words, that mathematics makes sense) (Boaler, 2019 a, b; Boaler, Cordero & Dieckmann, 2019; Anderson, Boaler & Dieckmann, 2018; Schoenfeld, 2014). These factors support students' as they develop their own identities as powerful math learners and users. Further, active-learning experiences enable students to engage in a full range of mathematical activity—exploring, noticing, questioning, solving, justifying, explaining, representing and analyzing—making clear that mathematics represents far more than calculating.

Research is also clear that *all* students are capable of becoming powerful math learners and users (Boaler, 2019a, c). This notion runs counter to many students' ideas about school math. Most adults can recall times when they received messages during their school or college years that they were not cut out for math-based fields. The race-, class-, and gender-based differences in those who pursue more advanced mathematics make it clear that the messages students receive about who belongs in math are biased along racial, socioeconomic status, language, and gender lines. This has led to considerable inequities in mathematics.

Sarah-Jane Leslie, Andrei Cimpian, and colleagues (2015) interviewed university professors in different subject areas to see how prevalent the idea of a "gift" was—the concept that people need a special ability to be successful in a particular field. The results were staggering; the more prevalent the idea of a gift was, in any academic field, the fewer women and people of color were in that field. This outcome held across all thirty subjects in the study. More math professors believed that students needed a gift than any other professor of STEM content. The study highlights the subtle ways that students are dissuaded from continuing in mathematics. It underscores the important role math teachers play in communicating messages that math success can only be achieved by a few students. This pervasive belief more often influences women and people of color to conclude they will not find success in classes or studies that rely on knowledge of mathematics.

Negative messages, either explicit ("I think you'd be happier if you didn't take that hard math class") or implicit ("I'm just not a math person"), both imply that only some people will succeed. Perceptions can also be personal ("math just doesn't seem to be your strength") or general ("this test isn't showing me that these students have what it takes in math. My other class aced this test"). And they can also be linked to labels ("low kids," "bubble kids," "slow kids"), which positions students in ways that lead to a differentiated and unjust mathematics education.

Thus, concerns about equity in mathematics learning are front and center throughout the framework. Some overarching principles that guide work towards equity in mathematics include the following:

- Access to an engaging and humanizing education—a socio-cultural, human endeavor—is a universal right, central among civil rights.
- All students deserve powerful mathematics; we reject ideas of natural gifts and talents (Cimpian et al, 2015; Boaler, 2019) and the “cult of the genius” (Ellenberg, 2015).
- “I treat everyone the same” is not enough: Active efforts in mathematics teaching are required in order to counter the cultural forces that have led to and continue to perpetuate current inequities (Langer-Osuna, 2011).
- Student engagement must be a design goal of mathematics curriculum design, co-equal with content goals.
- Mathematics pathways must open mathematics to all students, eliminating option-limiting tracking.
- Students’ cultural backgrounds, experiences, and language are resources for learning mathematics (González, Moll, & Amanti, 2006; Turner & Celedón-Pattichis, 2011; Moschkovich, 2013).
- All students, regardless of background, language of origin, differences, or foundational knowledge are capable and deserving of depth of understanding and engagement in rich math tasks.

Research on Mathematics Learning and Neuroscience

Hard work and persistence is more important for success in mathematics than natural ability. Actually, I would give this advice to anyone working in any field, but it’s especially important in mathematics and physics where the traditional view was that natural ability was the primary factor in success.”

—Maria Klawe, Mathematician, Harvey Mudd President (in Williams, 2018)

A strong cultural myth is the idea of a math brain—that people are born with a brain that is suited (or not) for math. But the last few decades have seen the emergence of technologies that have given researchers access into the workings of the mind and brain. Now scientists can study children and adults working on mathematics and watch their brain activity; they can look at brain growth and brain degeneration, and they can see the impact of different emotional conditions on brain activity. This work has shown—resoundingly—that we all have the capacity to learn mathematics to very high levels. Multiple studies have shown the incredible capacity of brains to grow and change within a short period of time (Huber et al, 2018; Luculano et al, 2015; Abiola & Dhindsa, 2011; Maguire, Woollett, & Spiers, 2006; Woollett & Maguire, 2011).

Every time we learn, our brains form, strengthen, or connect brain pathways in a process of almost constant change and adaptation (Doidge, 2007; Boaler, 2019a). Neuroscience research that has emerged in recent years shows the incredible potential for all people has been accompanied by cases of people accomplishing the highest levels of achievement in mathematics despite the reality that some started with significant disadvantages.

An important goal of this framework is to replace ideas of innate mathematics “talent” and “giftedness” with the recognition that every student is on a growth pathway. There is no cutoff determining when one child is gifted and another is not. Fixed-ability messages have contributed to the widespread myth of the math brain, underachievement in mathematics, and aversion to high-level study. The evidence that all students have the potential to reach high levels is particularly important for students diagnosed with special needs, many of whom are set on low-level pathways, even as research is showing the capacity of all brains to rewire and change (Boaler & LaMar, 2019).

A second important finding from the neuroscience research is the value of periods of struggle and its effect on the brain. Psychologist Jason Moser and his colleagues showed that when adults were taking tests, they experienced more brain growth and activity when they made mistakes than when they scored correctly (Moser, et al, 2011)—a conclusion illustrating how the process of mistake making can be a time when people are most challenged and engaged in struggle. The importance of struggle has been shown through both brain-based and behavior-based studies. Daniel Coyle

(2009), for example, studied the highest achieving people in different fields of work and found a characteristic shared by these achievers was a willingness to struggle—to work “at the edge of their understanding,” make mistakes, correct them, move on, and create more. This, he found, was the optimal approach to accelerate learning. This evidence becomes particularly important when we consider that students often struggle in math class, decide they do not have “a math brain,” and give up. It is important for teachers to share the research on the benefits of struggle for our brains; this is a liberating message for students that encourages them to persevere, rather than give up. Videos to u this underscore this message with students are available to share from Youcubed at https://www.youcubed.org/resource/videos/.

A third meaningful result from studies of the brain is the importance of brain connections. Vinod Menon (2015) and a team of researchers at Stanford Universityhave studied the interacting networks in the brain, particularly focusing on the ways the brain works when it is solving problems—including mathematics problems. They found that even when people are engaged with a simple arithmetic question, five different areas of the brain are involved, two of which are visual pathways. The dorsal visual pathway is the main brain region for representing quantity.

Menon and other neuroscientists have also found that communication between the different brain areas enhances learning and performance. Researchers Joonkoo Park and Elizabeth Brannon (2013) have reported a study in which they found that different areas of the brain were involved when people worked with symbols, such as numerals, than when they worked with visual and spatial information, such as an array of dots. The researchers also found that mathematics learning and performance were optimized when these two areas of the brain were communicating with each other. We can learn mathematical ideas through numbers, but we can also learn them through words, visuals, models, algorithms, multiple representations, tables, and graphs; from moving and touching; and from other representations. But when we learn by using two or more of these means and the different areas of the brain responsible for each communicate with each other, the learning experience is maximized.

For this reason, this framework highlights examples that are multi-dimensional, with mathematical experiences that are visual, physical, numerical, and more. These approaches are consistent with the principles of Universal Design for Learning (UDL), a framework designed to make learning more accessible, that helps all students. Visual and physical representations of mathematics are not only for young children, nor are they merely a prelude to abstraction or higher-level mathematics (Boaler et al, 2016). Some of the most important high-level mathematical work and thinking—such as the work of Fields medal winner Maryam Mirzakhani—is visual.

The three areas of neuroscientific research with evidence showing the potential of brains to grow and change, the importance of times of struggle, and the value in engaging with mathematics in multi-dimensional ways, should be shared with students. When messages such as these were shown in a free online class offered through a randomized controlled trial, students significantly increased their mathematics engagement in class and improved later achievement (Boaler et al, 2018). This information is shared through freely available lessons and videos on https://youcubed.org.

Research on Mindset

The neuroscientific evidence that shows the potential of all students to reach high levels in mathematics is the evidence base that underpins the importance of mindset messages. Stanford University psychologist Carol Dweck and her colleagues have conducted decades of research studies in different subjects and fields showing that what people believe about their potential changes the ways their brains operate and their actual achievement. One of the important studies Dweck and her colleagues conducted took place in mathematics classes at Columbia University (Carr et al., 2012), where researchers found that young women received messaging that they did not belong in the discipline. Moreover, when students with a fixed mindset heard the message that math was not for women, they dropped out. Those with a growth mindset, however, protected by the belief that anyone can learn anything, ultimately rejected the stereotype and persisted.

A related idea that teachers should challenge comes from social comparison. Students often believe that brains must be fixed, because some people appear to get ideas faster and to be naturally "gifted" at certain subjects. What these students do not realize is that brains grow and change every day. Each moment is an opportunity for brain growth and development and some students have developed stronger pathways on a different timeline. Teachers should strive to reinforce the idea that all students can develop those pathways at any time if they take the right approach to learning.

It is similarly important for teachers to start the first class of the year by sharing the science of brain growth and clarifying the idea that, although students are all unique, anyone can learn the content that is being taught, and productive learning is in part due to their thinking. This message is liberating, and overrides any prevailing messaging from teachers that success in math can only be achieved by a few students. When students learn about brain growth and mindset, they realize something critically important—no matter where they are in their learning, they can improve and eventually excel (Blackwell, Trzesniewski & Dweck, 2007). Various resources for sharing mindset messages and opportunities with students are provided here: https://www.youcubed.org/resource/mindset-boosting-videos/

Mathematics: Tools for Making Sense

Without mathematics, there's nothing you can do. Everything around you is mathematics. Everything around you is numbers.

—Shakuntala Devi, Author & "Human Calculator"

Mathematics grows out of curiosity about the world. Humans are born with an intuitive sense of numerical magnitude (Feigenson, Dehaene, & Spelke 2004), and this intuitive sense develops in early life into knowledge of number words, numerals, and the quantities they represent.

Give babies a set of blocks, and they will build and order them, fascinated by the ways the edges line up. Children will look up at the sky and be delighted by the V formations in which birds fly. Count a set of objects with a young child, move the objects and count them again, and they will be enchanted by the fact they still have the same number.

Human minds want to see and understand patterns (Devlin, 2006). But the joy and fascination young children experience with mathematics is quickly replaced by dread and dislike when mathematics is introduced as a dry set of methods they think they just have to accept and remember.

Young students’ work in mathematics is firmly rooted in their experiences in the world (Piaget and Cook, 1952). Numbers name quantities of objects or measurements such as time and distance, and operations such as addition and subtraction are represented by manipulations of such objects or measurements. Soon, the whole numbers themselves become a context that is concrete enough for students to grow curious about and to reason within—with real-world and visual representations always available to support reasoning.

Students who use mathematics powerfully can maintain this connection between mathematical ideas and meaningful contexts. Historically, too many students lose the connection at some point between primary grades and graduation from high school. The resulting experience creates students who see mathematics as an exercise in memorized procedures that match different problem types.

This framework takes as a given that all students are capable of accessing and mastering school mathematics in the ways envisioned in CA CCSSM. “Mastering” means becoming inclined and able to consider novel situations (arising either within or outside mathematics) through a variety of appropriate mathematical tools, using those tools to understand the situation and, when desired, to exert their own power to affect the situation. Thus, mathematical power is not reserved for a few, but available to all.

Translating this potential into reality requires a school mathematics system built to achieve this purpose. Current structures often reinforce existing factors that allow access for some while telling others they don’t belong; structures must instead challenge those factors by providing relevant, authentic mathematical experiences that make it clear to all students that mathematics is a powerful tool for making sense of and affecting their worlds. This will be an important contribution to equitable outcomes.

Audience

The *Math Framework* is intended to serve many different audiences, each of whom contribute to the shared mission of helping all students become powerful users of mathematics as envisioned in the CA CCSSM. First and foremost, the *Math Framework* is written for teachers and those educators who have the most direct relationship with students around their developing mastery of mathematics. As in every academic subject, developing powerful thinking requires contributions from many; and so this framework is also directed to:

- parents and caretakers of K–12 students who represent crucial partners in supporting their students' mathematical success;
- curricular materials designers and authors whose products help teachers to implement the Standards through engaging, authentic classrooms;
- educators leading pre-service and teacher preparation programs whose students face a daunting but exciting challenge of preparing to engage students in meaningful, coherent mathematics;
- in-service professional learning providers who can help teachers navigate deep mathematical and pedagogical questions as they strive to create coherent K–12 mathematical journeys for their students;
- instructional coaches and other key allies supporting teachers to improve students' experiences of mathematics;
- site, district, and county administrators who want to support improvement in mathematics experiences for their students;
- college and university instructors of California high school graduates who wish to use the framework in concert with the Standards to understand the types of knowledge, skills, and mindsets about mathematics that they can expect of incoming students; and
- assessment writers who create curriculum, state, and national tests that signal which content is important and the determine ways students should engage in the content.

Updating Coherence, Focus, and Rigor

The CA CCSSM were adopted by the State Board of Education in 2010 and modified in 2013. Over a decade of experiences have made evident the kinds of challenges the Standards posed for teachers, administrators, curriculum developers, professional learning providers, and others. When the Standards and the subsequent framework were each adopted, they both reflected an approach based on identifying major and minor standards--a recognition that it can be difficult for teachers to address all standards while maintaining a rich and deep learning experience for all students. This approach essentially eliminated key areas of content (such as data literacy). This framework reflects a revised approach, one that advocates for publishers and teachers avoiding the process of organizing around the detailed content standards, and instead establishing mathematics that reflect bigger ideas—those that connect many different standards in a more coherent whole. The *Math Framework* responds to challenges posed by each of the underlying principles.

Terms

Big Idea: Big ideas in math are central to the learning of mathematics, link numerous math understandings into a coherent whole, and provide focal points for students' investigations.

Drivers of Investigation: unifying reasons that both elicit curiosity and provide the motivation for deeply engaging with authentic mathematics

Content Connections: content themes that provide mathematical coherence through the grades

Authentic: An authentic problem, activity, or context is one in which students investigate or struggle with situations or questions about which they actually wonder. Lesson design should be built to elicit that wondering. In contrast, an activity is *inauthentic* if students recognize it as a straightforward practice of recently-learned techniques or procedures, including the repackaging of standard exercises in forced "real-world" contexts. Mathematical patterns and puzzles can be more authentic than such real-world settings.

Necessitate: An activity or task *necessitates* a mathematical idea or strategy if the attempt to understand the situation or task creates for students a need to understand or use the mathematical idea or strategy.

Coherence

I like crossing the imaginary boundaries people set up between different fields—it's very refreshing. There are lots of tools, and you don't know which one would work. It's about being optimistic and trying to connect things.

—Maryam Mirzakhani, Mathematician, 2014 Fields Medalist

Despite their differences and unique complexities, the Standards for Mathematical Practice (SMPs) and math content standards are intended to be equally important in planning, curriculum, and instruction (CA CCSSM [2013], p. 3). The content standards, however, are far more detailed at each grade level, and are more familiar to most educators; as a result, the content standards continue to provide the organizing structure for most curriculum and instruction. Because the content standards are more granular, curriculum developers and teachers find it easy when designing lessons to begin with one or two content standards and choose tasks and activities which develop that standard. Too often, this reinforces the concept as an isolated idea.

Because the Standards were then new to California educators (and to curriculum writers), the 2013 *California Mathematics Framework* was comprehensive in its treatment of the content standards; it included descriptions and examples throughout the framework for most. In the intervening years, many more examples, exemplars, and models of sample tasks representing illustrations of the mastery intended by each standard have emerged. Thus, the need is different in 2021: California teachers and students need mathematics experiences that provide access to the coherent body of understanding and strategies of the discipline.

Instructional materials should primarily involve tasks that invite students to make sense of these big ideas, elicit wondering in authentic contexts, and necessitate mathematical investigation. Big ideas in math are central to the learning of mathematics, link numerous mathematical understandings into a coherent whole, and provide focal points for students' investigations. An authentic activity or problem is one in which students investigate or struggle with situations or questions about which they actually wonder. Lesson design should be built to elicit that wondering.

> *Mathematical notation no more is mathematics than musical notation is music. A page of sheet music represents a piece of music, but the notation and the music are not the same; the music itself happens when the notes on the page are sung or performed on a musical instrument. It is in its performance that the music comes alive; it exists not on the page but in our minds. The same is true for mathematics.*
>
> *—Keith Devlin (2001)*

This framework sets out these organizing ideas to provide *coherence* and to help teachers avoid losing the forest for the trees. That is, discrete content standard mastery does not necessarily assemble in students' minds into a coherent big-picture view of mathematics.

This framework's response to the challenge posed by the principle of coherence are: focusing on big ideas, both as Drivers of Investigation (the reasons why we do math, see section below), and Content Connections (both within and across domains, see section below); progressions of learning across grades (thus, grade-band chapters rather than individual grade chapters); and relevance to students' lives. Principles guiding grade-band chapters include

- design from a smaller set of big ideas, spanning TK–12 in the forms of Drivers of Investigation and Content Connections (see below), within each grade band;
- a preponderance of student time spent on authentic problems through the lenses of DIs and CCs (see below) that engage multiple content and practice standards situated within one or more big ideas;
- a focus on connections: between students' lives and mathematical ideas and strategies; and between different mathematical ideas; and

- constant attention to opportunities for students to bring other aspects of their lives into the math classroom: How does this mathematical way of looking at this phenomenon compare with other ways to look at it? What problems do you see in our community that we might analyze? Teachers who relate aspects of mathematics to students' cultures often achieve more equitable outcomes (Hammond, 2014).

Focus

I didn't want to just know the names of things. I remember really wanting to know how it all worked.

—Elizabeth Blackburn, Winner of the 2009 Nobel Prize for Physiology or Medicine.

The principle of *focus* is closely tied to the goal of *depth* of understanding. The principle derives from a need to confront the mile-wide but inch-deep mathematics curriculum experienced by many.

Instructional design built on moving from one content standard to the next underscores the challenging reality that the Standards simply contain *too many* concepts and strategies to address comprehensively in this manner. Teachers often opt to choose between covering standards at an adequate depth (while skipping some topics), or including all topics from the Standards for their grade level and compromising opportunities to reach rich, deep understandings.

One common approach to the coverage-vs-depth challenge is to designate some content standards more important than others (for example, Student Achievement Partners). An unintentional result of this, in many schools, is that the standards deemed "less important" simply are not addressed.

The Standards, however, are *not* a design for instruction, and should not be used as such. The Standards lay out expected mastery of content at the grade levels, and expected mathematical practices at the conclusion of high school. They say little about how to achieve that mastery or build those practices.

This framework's answer to the coverage-vs-depth challenge posed by the principle of *focus* is to lay out principles for (and examples of) instructional design that make the Standards achievable. These principles include as follows:

- Focus on investigations and connections, not individual standards: class activities should be designed around big ideas, and typically should necessitate several clusters of content standards and multiple practice standards, as part of an investigation. Connections between those content standards then becomes an integral part of the class activity, and not an additional topic to cover. The twin focus on investigations and connections is reflected in titles and structure of the grade-banded chapters, chapters 6, 7, and 8, as well as in the Drivers of Investigation and Content Connections (see below).
- Tasks must be worthy of student engagement.
    - Problems (tasks which students do not already have the tools to solve) *precede* teaching of the focal mathematics which are necessitated by the problem. That is, the major point of a problem is to raise questions that can be answered, and promote students using their intuition, before learning new mathematical ideas (Deslauriers, McCarty, Miller, Callaghan, & Kestin, 2019).
    - Exercises (tasks which students already have the tools to solve) should either be embedded in a larger context which is motivating (such as the Drivers of Investigations, or exploration of patterns, or games), or should address strategies whose improvement will help students accomplish some motivating goal.
    - Students should learn to see their goal as investigating mathematical ideas, asking important questions, making conjectures and developing curiosity about mathematics and mathematical connections.

Rigor

True rigor is productive, being distinguished in this from another rigor which is purely formal and tiresome, casting a shadow over the problems it touches.

—Émile Picard (1905)

In this framework, *rigor* refers to an integrated way in which conceptual understanding, strategies for problem-solving and computation, and applications are learned, so that each supports the other. This definition is more specific and somewhat more demanding than the Common Core State Mathematics Standards' requirement that "*rigor* requires that conceptual understanding, procedural skill and fluency, and application be approached with equal intensity" (CA CCSSM, 2013, p. 2).

This definition expresses the basis of mathematical rigor: reasoning which enables understanding "all the way down to the bottom" (Ellenberg, 2014, p. 48), often expressed in terms of validity and soundness of arguments. According to the definition used here, conceptual understanding cannot be considered rigorous if it cannot be *used* to analyze a novel situation encountered in the world; computational speed and accuracy cannot be called rigorous unless it is accompanied by conceptual understanding of the strategy being used, including why it is appropriate in a given situation; and a correct answer to an application problem is not rigorous if the solver cannot explain to the client both the ideas of the model used and the methods of calculation.

In particular, rigor is *not* about abstraction. In fact, a push for premature abstraction leads, for many students, to an absence of rigor in the sense used in this framework. It is true that more advanced mathematics often occurs in more abstract contexts. This leads many to value more abstract subject matter as a marker of rigor. "Abstraction" in this case usually means "less connected to reality."

But mathematical abstraction is in fact *deeply* connected to reality: When second graders use a representation with blocks to argue that the sum of two odd numbers is even, in a way that other students can see would work for *any* two odd numbers (a representation-based proof; see Schifter, 2010), they have *abstracted* the idea of odd number, and they know that what they say about an odd number applies to one, three, five, etc. (Such an argument reflects Standard for Mathematical Practice 7: Look for and make use of structure.)

Abstraction must grow out of experiences in which students experience the same mathematical ideas and representations showing up and being useful in different contexts. When students figure out the size of a population, after 50 months, with a growth of three percent a month; their bank balance after 50 years if they can earn three percent interest per year; and the number of people after 50 days who have contracted a disease that is spreading at three percent per day, they will abstract the notion of a quantity growing by a certain percentage per time period, and recognize that they can use the same reasoning in each case to understand the changing quantity.

So the challenge posed by the principle of *rigor* is to provide all students with experiences that interweave concepts, problem-solving (including appropriate computation), and application, such that each supports the other. To meet this challenge, the *Math Framework* emphasizes these principles for designing instruction:

- Abstract formulations should *follow* experiences with multiple contexts that call forth similar mathematical models.
- Contexts for problem-solving should be chosen to provide representations for important concepts, so that students may later use those contexts to reason about the mathematical concepts raised. The Drivers of Investigation (see below) provide broad reasons to think rigorously ("all the way to the bottom") in ways that linkages between and through topics (Content Connections, see below) are recognized, valued and internalized.
- Computation should serve a genuine need for students to know, typically in a problem-solving or application context.
- Applications should be authentic to students and should be enacted in a way that requires students to explain or present solution paths and alternate ideas.

Designing for Coherence, Focus and Rigor: Drivers of Investigation and Content Connections

With motivating students to learn coherent, focused, and rigorous mathematics as the goal, this framework identifies three **Drivers of Investigation** (DIs), which provide the "why" of learning mathematics, to pair with four categories of **Content Connections** (CCs), which provide the "how and what" mathematics (CA-CCSSM) is to be learned in an activity. So, the DIs propel the learning of the ideas and actions framed in the CCs.

Drivers of Investigation (DIs)

The Content Connections should be developed through investigation of questions in authentic contexts; these investigations will naturally fall into one or more of the following Drivers of Investigation. The DIs are meant to serve a purpose similar to that of the Crosscutting Concepts in the CA NGSS, as unifying reasons that both elicit curiosity and provide the motivation for deeply engaging with authentic mathematics. In practical use, teachers can use these to frame questions or activities at the outset for the class period, the week, or longer; or refer to these in the middle of an investigation (perhaps in response to the "Why are we doing this again?" questions), or circle back to these at the conclusion of an activity to help students see "why it all matters." Their purpose is to pique interest and leverage students' innate wonder about the world, the future of the world, and their role in that future, in order to foster a deeper understanding of the Content Connections and grow into a perspective that mathematics itself is a lively, flexible endeavor by which we can appreciate and understand so much of the inner workings of our world. The DIs are:

- DI 1: Making Sense of the World (Understand and Explain)
- DI 2: Predicting What Could Happen (Predict)
- DI 3: Impacting the Future (Affect)

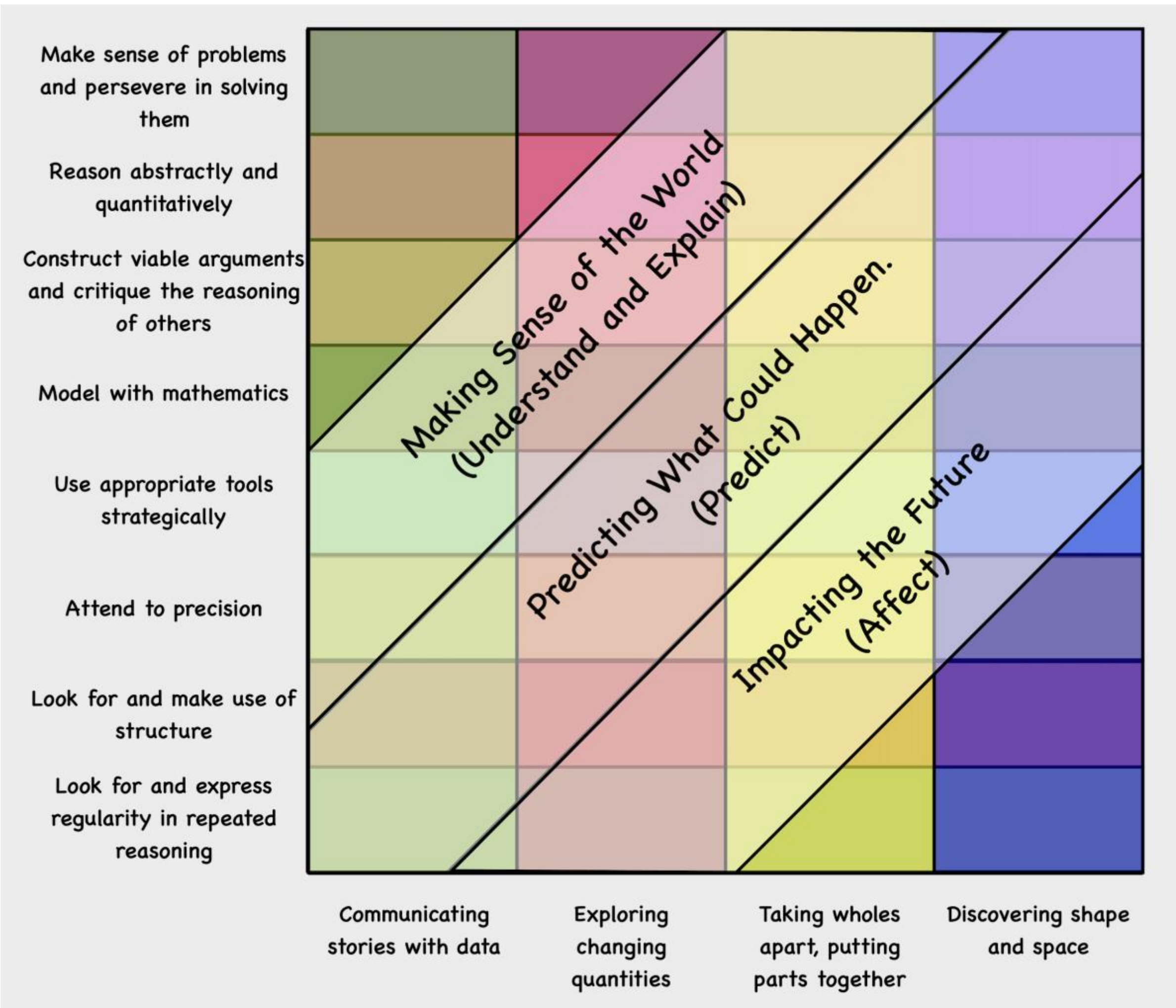
Content Connections (CCs)

The four Content Connections described in the framework organize content and provide mathematical coherence through the grades:

- CC1: Communicating Stories with Data
- CC2: Exploring Changing Quantities
- CC3: Taking Wholes Apart, Putting Parts Together
- CC4: Discovering Shape and Space

Big ideas that drive design of instructional activities will link one or more Content Connections, and Standards for Mathematical Practice, with a Driver of Investigation, so that students can Communicate Stories with Data **in order to** Predict What Could Happen, or Illuminate Changing Quantities **in order to** Impact the Future. The aim of the drivers of investigation is to ensure that there is always a reason to care about mathematical work, and that investigations allow students to make sense, predict, and/or affect the world. The following diagram is meant to illustrate the ways that the drivers of investigation relate to content connections and practices, as cross cutting themes. Any driver of investigation could go with any set of content and practices:

Figure 1: Content connections, Mathematical Practices and Drivers of Investigation



New to this Framework

To address the needs of California educators in 2021, the *Math Framework* includes several new emphases and types of chapters. Unlike 2013, when the framework featured two separate chapters—one on instruction and one on access—the 2021 framework offers a single chapter, Chapter Two: Teaching for Equity and Engagement, which promotes instruction that fosters equitable learning experiences for all, and challenges the deeply-entrenched policies and practices that lead to inequitable outcomes. While some people argue for a false dichotomy between equity and high achievement, this framework rejects that notion in favor of emphasizing ways good teaching leads to equitable and higher outcomes. Instruction and equity come together to create instructional designs that bring about equitable outcomes. Our commitment to equity extends throughout the framework, and every chapter highlights considerations and approaches designed to help mathematics educators create and maintain equitable opportunities for all.

Two chapters are devoted to exploring the development, across the TK–12 grade timeframe, of particular content areas. One such area is number sense across TK–12 (Chapter Three: Number Sense), a crucial foundation for all later mathematics and early predictor of mathematical perseverance. The other is data science (Chapter Five: Data Science), which has become tremendously important in the field since the last framework. The other new chapter, Chapter 4: Exploring, Discovering, and Reasoning With and About Mathematics, presents the development of a related cluster of SMPs across the entire TK–12 timeframe. While it is beyond the scope of the *Math Framework* to develop such a "progression" for all SMPs, this chapter can guide the careful work that is required to develop SMPs across the grades. The idea of learning progressions across multiple grade levels is emphasized further in the grade-banded chapters, Chapter Six: Grades TK–5, Chapter Seven: Grades 6–8, and Chapter Eight: Grades 9–12. The big ideas for each grade band, in the form of overarching Drivers of Investigation and Content Connections, provide a structure for promoting relevant and authentic activities for students, sample tasks, snapshots, and vignettes to illustrate the building of ideas across grades.

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