AP® CALCULUS BC
2010 SCORING GUIDELINES

Question 6

$$f(x) = \begin{cases} \dfrac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\dfrac{1}{2} & \text{for } x = 0 \end{cases}$$

The function f, defined above, has derivatives of all orders. Let g be the function defined by $g(x) = 1 + \int_0^x f(t)\,dt$.

(a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.

(c) Write the fifth-degree Taylor polynomial for g about $x = 0$.

(d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x = 0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

(a) $\cos(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \cdots + (-1)^n \dfrac{x^{2n}}{(2n)!} + \cdots$

$f(x) = -\dfrac{1}{2} + \dfrac{x^2}{4!} - \dfrac{x^4}{6!} + \cdots + (-1)^{n+1} \dfrac{x^{2n}}{(2n+2)!} + \cdots$

3 : { 1 : terms for $\cos x$; 2 : terms for f { 1 : first three terms; 1 : general term } }

(b) $f'(0)$ is the coefficient of x in the Taylor series for f about $x = 0$, so $f'(0) = 0$.

$\dfrac{f''(0)}{2!} = \dfrac{1}{4!}$ is the coefficient of x^2 in the Taylor series for f about $x = 0$, so $f''(0) = \dfrac{1}{12}$.

Therefore, by the Second Derivative Test, f has a relative minimum at $x = 0$.

2 : { 1 : determines $f'(0)$; 1 : answer with reason }

(c) $P_5(x) = 1 - \dfrac{x}{2} + \dfrac{x^3}{3 \cdot 4!} - \dfrac{x^5}{5 \cdot 6!}$

2 : { 1 : two correct terms; 1 : remaining terms }

(d) $g(1) \approx 1 - \dfrac{1}{2} + \dfrac{1}{3 \cdot 4!} = \dfrac{37}{72}$

Since the Taylor series for g about $x = 0$ evaluated at $x = 1$ is alternating and the terms decrease in absolute value to 0, we know $\left| g(1) - \dfrac{37}{72} \right| < \dfrac{1}{5 \cdot 6!} < \dfrac{1}{6!}$.

2 : { 1 : estimate; 1 : explanation }

Question 6

Let f be the function given by $f(x) = 6e^{-x/3}$ for all x.

(a) Find the first four nonzero terms and the general term for the Taylor series for f about $x = 0$.

(b) Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. Find the first four nonzero terms and the general term for the Taylor series for g about $x = 0$.

(c) The function h satisfies $h(x) = kf'(ax)$ for all x, where a and k are constants. The Taylor series for h about $x = 0$ is given by

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Find the values of a and k.

(a) $$f(x) = 6\left[1 - \frac{x}{3} + \frac{x^2}{2!3^2} - \frac{x^3}{3!3^3} + \cdots + \frac{(-1)^n x^n}{n!3^n} + \cdots\right]$$
$$= 6 - 2x + \frac{x^2}{3} - \frac{x^3}{27} + \cdots + \frac{6(-1)^n x^n}{n!3^n} + \cdots$$

3 :
- 1 : two of $6, -2x, \frac{x^2}{3}, -\frac{x^3}{27}$
- 1 : remaining terms
- 1 : general term

⟨−1⟩ missing factor of 6

(b) $g(0) = 0$ and $g'(x) = f(x)$, so
$$g(x) = 6\left[x - \frac{x^2}{6} + \frac{x^3}{3!3^2} - \frac{x^4}{4!3^3} + \cdots + \frac{(-1)^n x^{n+1}}{(n+1)!3^n} + \cdots\right]$$
$$= 6x - x^2 + \frac{x^3}{9} - \frac{x^4}{4(27)} + \cdots + \frac{6(-1)^n x^{n+1}}{(n+1)!3^n} + \cdots$$

3 :
- 1 : two terms
- 1 : remaining terms
- 1 : general term

⟨−1⟩ missing factor of 6

(c) $f'(x) = -2e^{-x/3}$, so $h(x) = -2ke^{-ax/3}$

$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = e^x$

$-2ke^{-ax/3} = e^x$

$\frac{-a}{3} = 1$ and $-2k = 1$

$a = -3$ and $k = -\frac{1}{2}$

OR

$f'(x) = -2 + \frac{2}{3}x + \cdots$, so

$h(x) = kf'(ax) = -2k + \frac{2}{3}akx + \cdots$

$h(x) = 1 + x + \cdots$

$-2k = 1$ and $\frac{2}{3}ak = 1$

$k = -\frac{1}{2}$ and $a = -3$

3 :
- 1 : computes $kf'(ax)$
- 1 : recognizes $h(x) = e^x$, or equates 2 series for $h(x)$
- 1 : values for a and k

Question 6

The function f is defined by the power series

$$f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \cdots + \frac{(-1)^n nx^n}{n+1} + \cdots$$

for all real numbers x for which the series converges. The function g is defined by the power series

$$g(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots + \frac{(-1)^n x^n}{(2n)!} + \cdots$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The graph of $y = f(x) - g(x)$ passes through the point $(0, -1)$. Find $y'(0)$ and $y''(0)$. Determine whether y has a relative minimum, a relative maximum, or neither at $x = 0$. Give a reason for your answer.

(a) $\left|\frac{(-1)^{n+1}(n+1)x^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n nx^n}\right| = \frac{(n+1)^2}{(n+2)(n)} \cdot |x|$

$\lim_{n\to\infty} \frac{(n+1)^2}{(n+2)(n)} \cdot |x| = |x|$

The series converges when $-1 < x < 1$.

When $x = 1$, the series is $-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \cdots$

This series does not converge, because the limit of the individual terms is not zero.

When $x = -1$, the series is $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

This series does not converge, because the limit of the individual terms is not zero.

Thus, the interval of convergence is $-1 < x < 1$.

5 :
- 1 : sets up ratio
- 1 : computes limit of ratio
- 1 : identifies radius of convergence
- 1 : considers both endpoints
- 1 : analysis/conclusion for both endpoints

(b) $f'(x) = -\frac{1}{2} + \frac{4}{3}x - \frac{9}{4}x^2 + \cdots$ and $f'(0) = -\frac{1}{2}$.

$g'(x) = -\frac{1}{2!} + \frac{2}{4!}x - \frac{3}{6!}x^2 + \cdots$ and $g'(0) = -\frac{1}{2}$.

$y'(0) = f'(0) - g'(0) = 0$

$f''(0) = \frac{4}{3}$ and $g''(0) = \frac{2}{4!} = \frac{1}{12}$.

Thus, $y''(0) = \frac{4}{3} - \frac{1}{12} > 0$.

Since $y'(0) = 0$ and $y''(0) > 0$, y has a relative minimum at $x = 0$.

4 :
- 1 : $y'(0)$
- 1 : $y''(0)$
- 1 : conclusion
- 1 : reasoning

AP® CALCULUS BC
2008 SCORING GUIDELINES (Form B)

Question 6

Let f be the function given by $f(x) = \frac{2x}{1+x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.

(c) The derivative of $\ln\left(1+x^2\right)$ is $\frac{2x}{1+x^2}$. Write the first four nonzero terms of the Taylor series for $\ln\left(1+x^2\right)$ about $x = 0$.

(d) Use the series found in part (c) to find a rational number A such that $\left|A - \ln\left(\frac{5}{4}\right)\right| < \frac{1}{100}$. Justify your answer.

(a) $\frac{1}{1-u} = 1 + u + u^2 + \cdots + u^n + \cdots$ $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + \left(-x^2\right)^n + \cdots$ $\frac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + \cdots + (-1)^n 2x^{2n+1} + \cdots$	$3: \begin{cases} 1: \text{two of the first four terms} \\ 1: \text{remaining terms} \\ 1: \text{general term} \end{cases}$
(b) No, the series does not converge when $x = 1$ because when $x = 1$, the terms of the series do not converge to 0.	1 : answer with reason
(c) $\ln\left(1+x^2\right) = \int_0^x \frac{2t}{1+t^2}\,dt$ $= \int_0^x \left(2t - 2t^3 + 2t^5 - 2t^7 + \cdots\right) dt$ $= x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \cdots$	$2: \begin{cases} 1: \text{two of the first four terms} \\ 1: \text{remaining terms} \end{cases}$
(d) $\ln\left(\frac{5}{4}\right) = \ln\left(1 + \frac{1}{4}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 + \frac{1}{3}\left(\frac{1}{2}\right)^6 - \frac{1}{4}\left(\frac{1}{2}\right)^8 + \cdots$ Let $A = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4 = \frac{7}{32}$. Since the series is a converging alternating series and the absolute values of the individual terms decrease to 0, $\left|A - \ln\left(\frac{5}{4}\right)\right| < \left|\frac{1}{3}\left(\frac{1}{2}\right)^6\right| = \frac{1}{3} \cdot \frac{1}{64} < \frac{1}{100}$.	$3: \begin{cases} 1: \text{uses } x = \frac{1}{2} \\ 1: \text{value of } A \\ 1: \text{justification} \end{cases}$

Question 6

The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$. The continuous function f is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function f has derivatives of all orders at $x = 1$.

(a) Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.

(b) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(c) Use the ratio test to find the interval of convergence for the Taylor series found in part (b).

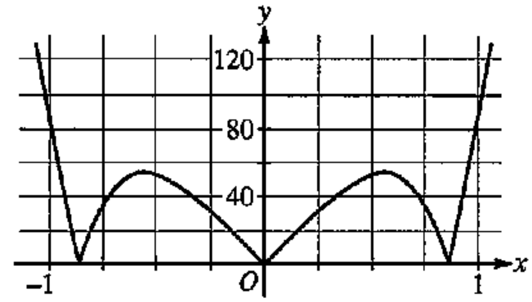
(d) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

(a) $1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{6} + \cdots + \frac{(x-1)^{2n}}{n!} + \cdots$	2 : { 1 : first four terms; 1 : general term }
(b) $1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{6} + \frac{(x-1)^6}{24} + \cdots + \frac{(x-1)^{2n}}{(n+1)!} + \cdots$	2 : { 1 : first four terms; 1 : general term }
(c) $\lim_{n\to\infty}\left\lvert \frac{\frac{(x-1)^{2n+2}}{(n+2)!}}{\frac{(x-1)^{2n}}{(n+1)!}} \right\rvert = \lim_{n\to\infty} \frac{(n+1)!}{(n+2)!}(x-1)^2 = \lim_{n\to\infty} \frac{(x-1)^2}{n+2} = 0$ Therefore, the interval of convergence is $(-\infty, \infty)$.	3 : { 1 : sets up ratio; 1 : computes limit of ratio; 1 : answer }
(d) $f''(x) = 1 + \frac{4\cdot 3}{6}(x-1)^2 + \frac{6\cdot 5}{24}(x-1)^4 + \cdots + \frac{2n(2n-1)}{(n+1)!}(x-1)^{2n-2} + \cdots$ Since every term of this series is nonnegative, $f''(x) \geq 0$ for all x. Therefore, the graph of f has no points of inflection.	2 : { 1 : $f''(x)$; 1 : answer }

AP® CALCULUS BC
2011 SCORING GUIDELINES

Question 6

Let $f(x) = \sin(x^2) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin(x^2)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin(x^2)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

(a) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$

$\sin(x^2) = x^2 - \dfrac{x^6}{3!} + \dfrac{x^{10}}{5!} - \dfrac{x^{14}}{7!} + \cdots$

$3 : \begin{cases} 1 : \text{series for } \sin x \\ 2 : \text{series for } \sin(x^2) \end{cases}$

(b) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$

$f(x) = 1 + \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \dfrac{121x^6}{6!} + \cdots$

$3 : \begin{cases} 1 : \text{series for } \cos x \\ 2 : \text{series for } f(x) \end{cases}$

(c) $\dfrac{f^{(6)}(0)}{6!}$ is the coefficient of x^6 in the Taylor series for f about $x = 0$. Therefore $f^{(6)}(0) = -121$.

1 : answer

(d) The graph of $y = \left|f^{(5)}(x)\right|$ indicates that $\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right| < 40$.

Therefore

$$\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| \le \frac{\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right|}{5!} \cdot \left(\frac{1}{4}\right)^5 < \frac{40}{120 \cdot 4^5} = \frac{1}{3072} < \frac{1}{3000}.$$

$2 : \begin{cases} 1 : \text{form of the error bound} \\ 1 : \text{analysis} \end{cases}$