Section 9.8

Infinite Series: "Maclaurin and Taylor Series; Power Series"

All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Introduction

- In the last section, the nth Taylor polynomial (x) at x= $_0$ for a function f was defined so its value and the values of its first n derivatives match those of f at $_0$.

- Also, for values of x near $_0$, the values of (x) appeared to be better and better approximations of f(x) as n increases, and may possibly converge to f(x) as n $\rightarrow +\infty$.

Maclaurin and Taylor Series

- If we take the nth Maclaurin polynomial for a function f

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

and the nth Taylor polynomial for f about x = x_0,

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

we can extend them beyond the nth index of summation.

9.8.1 DEFINITION If f has derivatives of all orders at x_0, then we call the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \cdots \qquad (1)$$

the *Taylor series for f about $x = x_0$*. In the special case where $x_0 = 0$, this series becomes

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots \qquad (2)$$

in which case we call it the *Maclaurin series for f*.

Example of Series vs. Polynomial

- Find the Maclaurin series for cos x.
- <u>Solution:</u>
- Start by finding several derivatives of cos x.
  - f(x) = cos x — f(0) = cos 0 = 1
  - f′(x) = -sin x — f′(0) = -sin 0 = 0
  - f′′(x) = -cos x — f′′(0) = -cos 0 = -1
  - f′′′(x) = sin x — f′′′(0) = sin 0 = 0
  - and the pattern (0,1,0,-1) continues to repeat for further derivatives at 0.
- This gives the Maclaurin ***<u>polynomials</u>***

$$_{2}(\ ) = \ _{2\ +1}(\ ) = 1 - \frac{^{2}}{2!} + \frac{^{4}}{4!} - \frac{^{6}}{6!} + \dots + (-1)\frac{^{2}}{(2\ )!} \quad (\ = 0,1,2,\dots).$$

- Therefore, the Maclaurin ***<u>series</u>*** for cos x is

$$\sum_{=0}^{\infty}(-1)\frac{^{2}}{(2\ )!} = 1 - \frac{^{2}}{2!} + \frac{^{4}}{4!} - \frac{^{6}}{6!} + \dots + (-1)\frac{^{2}}{(2\ )!} + \dots$$

Power Series in x

- Maclaurin and Taylor series are different than the series we worked with in sections 9.3-9.6.
- Instead of constants, they have variables in the series.
- These are examples of power series.

- If we use x as a variable, we get a **power series in x**:

$$\sum_{=0}^{\infty} \quad = {}_{0} + {}_{1} + {}_{2}{}^{2} + \dots + \quad + \dots$$

Examples of Power Series in x

- $\sum_{=0}^{\infty} \quad = 1 + \quad + \;{}^{2} + \;{}^{3} + \ldots$ Maclaurin series for $\frac{1}{1 -}$
- $\sum_{=0}^{\infty} \frac{}{!} = 1 + \quad + \frac{{}^{2}}{2!} + \frac{{}^{3}}{3!} + \ldots$ Maclaurin series for
- $\sum_{=0}^{\infty} (-1) \; \frac{{}^{2}}{(2\;)!} = 1 - \frac{{}^{2}}{2!} + \frac{{}^{4}}{4!} + \ldots$ Maclaurin series for cos x
- NOTE: Every Maclaurin series is a power series in x.

Radius and Interval of Convergence

- If a numerical value is substituted for x in a power series $\sum$, then the resulting series of numbers may either converge or diverge.
- We need to determine the set of x-values for which a given power series converges.
- This is called its convergence set.

- NOTE: Every power series in x converges at x=0, since substituting 0 into the Maclaurin series gives $0 + 0 + 0 + 0 + \ldots + 0 + \ldots$ whose sum is 0. For some, x=0 will be the only number in the convergence set.

Radius and Interval of Convergence Theorem

9.8.2 THEOREM *For any power series in x, exactly one of the following is true:*

(a) *The series converges only for $x = 0$.*

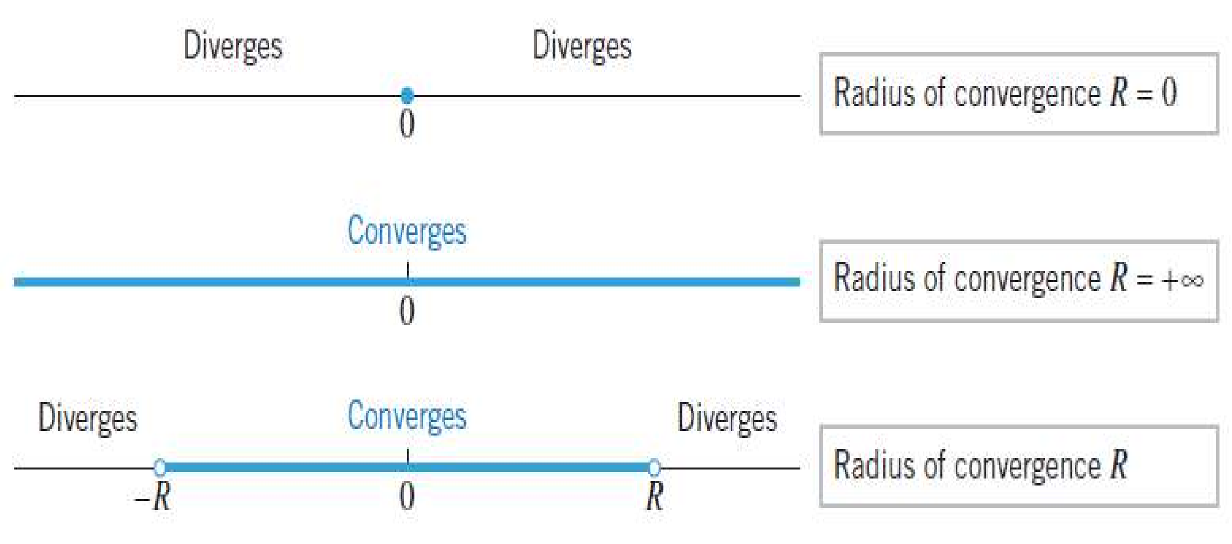
(b) *The series converges absolutely (and hence converges) for all real values of x.*

(c) *The series converges absolutely (and hence converges) for all x in some finite open interval $(-R, R)$ and diverges if $x < -R$ or $x > R$. At either of the values $x = R$ or $x = -R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.*

- This theorem states that the convergence set for a power series in x is always an interval centered at x=0.
- For this reason ("centered"), the convergence set of a power series in x is called the interval of convergence.

Radius of Convergence

- When the convergence set is only x=0, we say that the series has radius of convergence of 0.
- When the convergence set is $(-\infty, +\infty)$, we say that the series has radius of convergence $+\infty$.
- When the convergence set is between –R and R, we say that the series has radius of convergence R.



Finding the Interval of Convergence

- The usual procedure for finding the interval of convergence of a power series is to apply the ratio test for absolute convergence.
- Example: Find the interval of convergence and radius of convergence of $\sum_{=0}^{\infty} \frac{}{!}$.
- <u>Solution:</u>
- Apply the ratio test for absolute convergence

$$= \lim_{\to +\infty} \left| \frac{^{+1}}{} \right| = \lim_{\to +\infty} \left| \frac{^{+1}}{(\ +1)!} * \frac{!}{} \right| = \lim_{\to +\infty} \left| \frac{}{+1} \right| = 0 < 1$$

since zero is always less than one for all values of x, the series converges for all x, the interval of convergence is $(-\infty, +\infty)$ and the radius of convergence is R = $+\infty$.

Another Example of Finding the Interval and Radius of Convergence

- We are going to do the ratio test for absolute convergence on $\sum_{=0}^{\infty} \frac{(-1)}{3\ (\ +1)}$. Therefore, we should notice that $|(-1)\ | = |(-1)^{\ +1}| = 1$.
- This gives $= \lim_{\to +\infty} \left|\frac{\ +1}{}\right| = \lim_{\to +\infty} \left|\frac{\ +1}{3^{\ +1}(\ +2)} * \frac{3\ (\ +1)}{}\right|$

$$= \lim_{\to +\infty} \left|\frac{*\ 1}{3\ * 3^{1}(\ +2)} * \frac{3\ (\ +1)}{}\right| =$$

$$\lim_{\to +\infty} \left|\frac{}{3(\ +2)} * \frac{(\ +1)}{1}\right|$$

$$= \lim_{\to +\infty} \left[\frac{|\ |}{3}\right] * \frac{(\ +1)}{(\ +2)} = \frac{|\ |}{3} * \lim_{\to +\infty} \frac{1+\frac{1}{}}{1+\frac{2}{}} = \frac{|\ |}{3} * 1 =$$

Example continued

- The only thing left to find out is whether the series converges or diverges at x = 3 and x = -3, so we must test those endpoints:
  - When you substitute x = 3 into the series $\sum_{=0}^{\infty}\frac{(-1)}{3(\ \ +1)}$,
    $\sum_{=0}^{\infty}\frac{(-1)\ 3}{3(\ \ +1)}$ is the result which simplifies to $\sum_{=0}^{\infty}\frac{(-1)}{(\ \ +1)}$ and that equals 1 - $\frac{1}{2}$ + $\frac{1}{3}$ - $\frac{1}{4}$ + … which is the alternating harmonic series and it converges and so does the series when x = 3.
  - When you substitute x = -3 into the series $\sum_{=0}^{\infty}\frac{(-1)}{3(\ \ +1)}$,
    $\sum_{=0}^{\infty}\frac{(-1)\ (-3)}{3(\ \ +1)}$ is the result which simplifies to
    $\sum_{=0}^{\infty}\frac{(-1)\ \ *(-1)}{(\ \ +1)} = \sum_{=0}^{\infty}\frac{1}{(\ \ +1)}$ and that equals 1 + $\frac{1}{2}$ + $+\frac{1}{4}$ + … which is the harmonic series and it diverges and so does the series when x = -3.
  - Therefore, the interval of convergence is (-3,3] or -3<x≤3.
- Please do NOT forget to check the endpoints. It is a

Power Series in x - $_0$

- From the Maclaurin ***series*** we can write a "power series in x - $_0$" in Sigma notation:

$$\sum_{=0}^{\infty} \frac{(\ )(\ _0)}{!}(\ -\ _0)$$

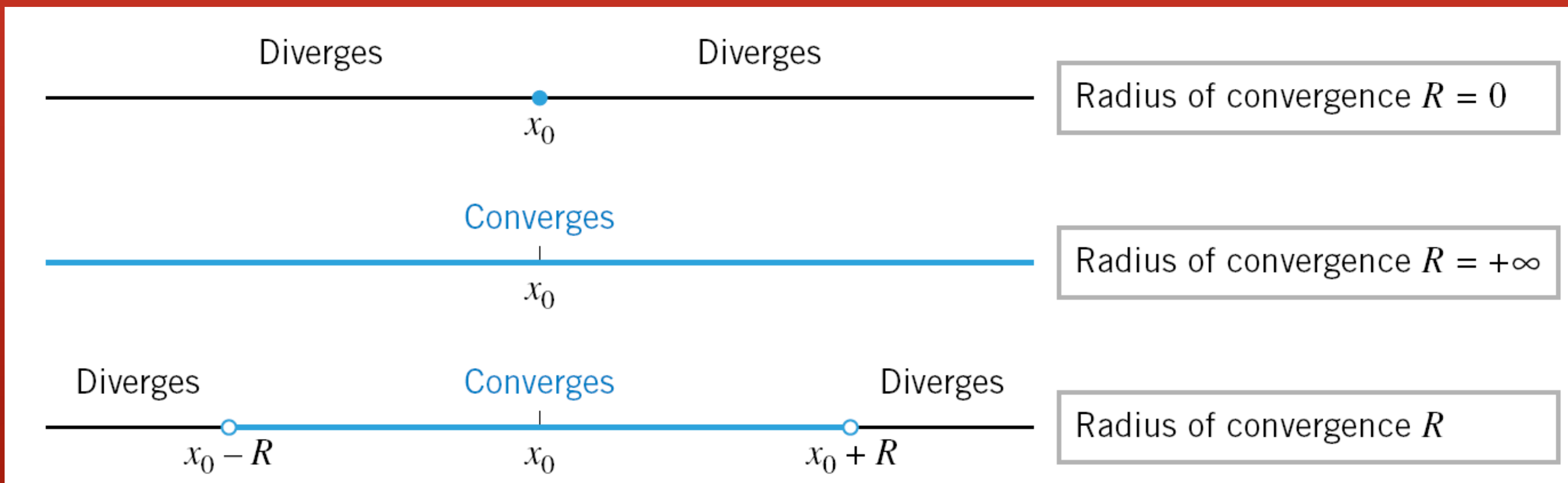
9.8.3 THEOREM *For a power series* $\sum c_k(x - x_0)^k$, *exactly one of the following statements is true:*

(a) *The series converges only for* $x = x_0$.

(b) *The series converges absolutely (and hence converges) for all real values of* x.

(c) *The series converges absolutely (and hence converges) for all* x *in some finite open interval* $(x_0 - R, x_0 + R)$ *and diverges if* $x < x_0 - R$ *or* $x > x_0 + R$. *At either of the values* $x = x_0 - R$ *or* $x = x_0 + R$, *the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.*

Visual Demonstration of Theorem 9.8.3



- These results are similar to those on slide #10 for Maclaurin series, but those are centered about 0 instead of 0

Interval of Convergence example for a Power Series in $- \ _0$

- Find the interval of convergence and radius of convergence of the series $\sum_{=1}^{\infty} \frac{(\ -5)}{2}$.
- Solution: apply the ratio test for absolute convergence.
- This gives $= \lim_{\to+\infty} \left| \frac{\ _{+1}}{\ } \right| = \lim_{\to+\infty} \left| \frac{(\ -5)^{+1}}{(\ +1)^2} * \frac{2}{(\ -5)} \right|$

$$= \lim_{\to+\infty} \left[\left| \ -5 \right| \left(\frac{\ }{\ +1} \right)^2 \right] = \left| \ -5 \right| \lim_{\to+\infty} \left(\frac{1}{1+\frac{1}{\ }} \right)^2$$

$$= \left| \ -5 \right| * 1$$

$$\left| \ -5 \right| < 1$$

-1<x-5<1

+5 +5 +5

4 < x < 6 Therefore, the radius of convergence is 1 since 4 and 6 are both 1 from 5 and we must test the endpoints to determine the interval of convergence.

Test Endpoints from example #4

- The only thing left to find out is whether the series converges or diverges at x = 4 and x = 6, so we must test those endpoints:
  - When you substitute x = 4 into the series $\sum_{=1}^{\infty} \frac{(\quad -5)}{2}, \sum_{=1}^{\infty} \frac{(-1)}{2}$ is the result which converges **<u>absolutely</u>** since it is a p-series with p=2 and 2>1 so the series converges when x = 4.
  - When you substitute x = 6 into the series $\sum_{=1}^{\infty} \frac{(\quad -5)}{2}, \sum_{=1}^{\infty} \frac{(1)}{2}$ is the result which converges since it is a p-series with p=2 and 2>1 so the series converges when x = 6.
  - Therefore, the interval of convergence is [4,6] or 4 ≤ x ≤6.
- Please do NOT forget to check the endpoints. It is a very common mistake on the AP exam.

Other

- You may read about functions defined by power series leading to Bessel functions in honor of the German mathematician and astronomer Friedrich Wilhelm Bessel (1784-1846) if you are interested, but they are not on the AP or IB exam.

View from our hotel in Ixtapa, Mexico