Section 9.5

Infinite Series: "The Comparison, Ratio, and Root Tests"

All graphics are attributed to:

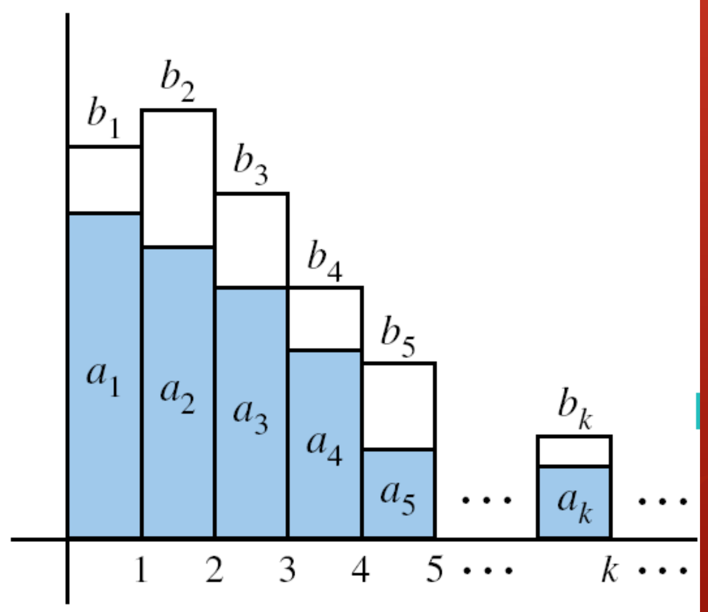
- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Introduction

- In this section we will develop some more basic convergence tests for series with nonnegative terms.
- Later, we will use some of these tests to study the convergence of Taylor series which have to do with polynomials.

The Comparison Test

- The comparison test uses the known convergence or divergence of a series to deduce the convergence or divergence of another series.
- If you visualize the terms in the series as areas of rectangles, the comparison test states that if the total area $\sum$ is finite, then the total area $\sum$ must also be finite.
- Also, if the total area $\sum$ is infinite, then the total area $\sum$ must also be infinite.



For each rectangle, b_k is the entire area and a_k is the area of the blue portion.

Formal Comparison Test Theorem

9.5.1 THEOREM (*The Comparison Test*) *Let* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *be series with nonnegative terms and suppose that*

$$a_1 \le b_1,\ a_2 \le b_2,\ a_3 \le b_3,\ \ldots,\ a_k \le b_k,\ \ldots$$

(*a*) *If the "bigger series"* Σb_k *converges, then the "smaller series"* Σa_k *also converges.*

(*b*) *If the "smaller series"* Σa_k *diverges, then the "bigger series"* Σb_k *also diverges.*

- Note: It is not essential that the condition ≤ hold for all k. The conclusions of the theorem remain true if this condition is "eventually true".

Steps for Using the Comparison Test

Step 1. Guess at whether the series $\sum u_k$ converges or diverges.

Step 2. Find a series that proves the guess to be correct. That is, if we guess that $\sum u_k$ diverges, we must find a divergent series whose terms are "smaller" than the corresponding terms of $\sum u_k$, and if we guess that $\sum u_k$ converges, we must find a convergent series whose terms are "bigger" than the corresponding terms of $\sum u_k$.

Suggestions to Help with Step 1

- It is somewhat unusual that the first step on the previous slide says that you need to guess something. Therefore, there are some informal principles to help with that process.

- In most cases, the series $\sum$ will have its general term expressed as a fraction.

  1. Constant terms in the denominator of can usually be deleted without affecting the convergence or divergence of the series.
  2. If a polynomial in k appears as a factor in the numerator or denominator of , all but the leading term in the polynomial can usually be discarded without affecting the convergence or divergence of the series.

Example Using the Comparison Test

- According to the first principle on the previous slide (Constant terms in the denominator of   can usually be deleted without affecting the convergence or divergence of the series), we should be able to drop the constant in the denominator of $\sum_{=1}^{\infty}\frac{1}{\sqrt{\ }-\frac{1}{2}}$ without affecting the convergence or divergence.

- That tells us that $\sum_{=1}^{\infty}\frac{1}{\sqrt{\ }-\frac{1}{2}}$ is likely to behave like the p-series $\sum_{=1}^{\infty}\frac{1}{\sqrt{\ }} = \sum_{=1}^{\infty}\frac{1}{^{\frac{1}{2}}}$ which is a divergent p-series since $0<1/2\leq 1$

- Also, $\sum_{=1}^{\infty}\frac{1}{\sqrt{\ }-\frac{1}{2}} > \sum_{=1}^{\infty}\frac{1}{^{\frac{1}{2}}}$ for k=1,2,3… so $\sum_{=1}^{\infty}\frac{1}{^{\frac{1}{2}}}$ is a divergent series that is "smaller" than the given series.

- Therefore, the given series also diverges.

Another Example Using the Comparison Test

- According to the second principle (if a polynomial in k appears as a factor in the numerator or denominator of , all but the leading term in the polynomial can usually be discarded without affecting the convergence or divergence of the series), we should be able to drop the other term in the denominator of $\sum_{=1}^{\infty} \frac{1}{2\ ^{2}+}$ without affecting the convergence or divergence.

- That tells us that $\sum_{=1}^{\infty} \frac{1}{2\ ^{2}+}$ → $\sum_{=1}^{\infty} \frac{1}{2\ ^{2}}$ is likely to behave like the p-series $\sum_{=1}^{\infty} \frac{1}{\ ^{2}}$ which is a convergent p-series since 2>1.

- Also, $\sum_{=1}^{\infty} \frac{1}{2\ ^{2}+} < \sum_{=1}^{\infty} \frac{1}{2\ ^{2}}$ for k=1,2,3… so $\sum_{=1}^{\infty} \frac{1}{2\ ^{2}}$ is a convergent series that is “bigger” than the given series.

- Therefore, the given series also converges.

The Limit Comparison Test

- In those two examples, we were able to guess about convergence or divergence using informal principles and our ability to find related series we could compare to.
- Unfortunately, it is not always easy to find a comparable series.
- Therefore, we need an alternative that can often be easier to apply.
- We still need to first guess at the convergence or divergence and find a series that supports our guess.

The Limit Comparison Test Theorem

9.5.4 THEOREM (*The Limit Comparison Test*) *Let* $\sum a_k$ *and* $\sum b_k$ *be series with positive terms and suppose that*

$$\rho = \lim_{k \to +\infty} \frac{a_k}{b_k}$$

If ρ *is finite and* $\rho > 0$*, then the series both converge or both diverge.*

- The proof is on page A39 in the appendices if you are interested.

Example Similar to First Example

- $\sum_{=1}^{\infty} \frac{1}{\sqrt{\ } - \frac{1}{2}}$ is the first example we looked at in this section.
- This time we will consider $\sum_{=1}^{\infty} \frac{1}{\sqrt{\ } + 1}$ which is similar and also likely to behave like $\sum_{=1}^{\infty} \frac{1}{\ ^{\frac{1}{2}}}$.
- Apply Limit Comparison Test:
- Let $= \frac{1}{\sqrt{\ } + 1}$ and $= \frac{1}{\sqrt{\ }}$

Another Limit Comparison Test Example

- $\sum_{=1}^{\infty} \frac{3\ ^{3} - 2\ ^{2} + 4}{7 - \ ^{3} + 2}$ is likely to behave like $\sum_{=1}^{\infty} \frac{3\ ^{3}}{7} = \sum_{=1}^{\infty} \frac{3}{4}$
- $\sum_{=1}^{\infty} \frac{3}{4}$ is a convergent p-series (4>1)
- Apply Limit Comparison Test:
- Let $= \frac{3\ ^{3} - 2\ ^{2} + 4}{7 - \ ^{3} + 2}$ and $= \frac{3}{4}$
- $= \lim_{\to +\infty} — = \lim_{\to +\infty} \frac{\frac{3\ ^{3} - 2\ ^{2} + 4}{7 - \ ^{3} + 2}}{\frac{3}{4}} = \lim_{\to +\infty} \frac{3\ ^{3} - 2\ ^{2} + 4}{7 - \ ^{3} + 2} * \frac{4}{3}$

  $= \lim_{\to +\infty} \frac{3\ ^{7} - 2\ ^{6} + 4\ ^{4}}{3\ ^{7} - 3\ ^{3} + 6}$
- $= \frac{3}{3} = 1$
- Since is finite and > 0, the series converges following the convergence of $\sum^{\infty} \frac{3}{4}$.

The Ratio Test

- The comparison test and the limit comparison test first require making a guess about convergence and then finding a series to compare to, both of which can be difficult.
- If you cannot find an appropriate series for comparison, you can often use the Ratio Test.
- The Ratio Test does not require an initial guess or a series for comparison.

The Ratio Test Theorem

9.5.5 THEOREM (*The Ratio Test*) *Let* $\sum u_k$ *be a series with positive terms and suppose that*

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k}$$

(a) *If* $\rho < 1$, *the series converges.*

(b) *If* $\rho > 1$ *or* $\rho = +\infty$, *the series diverges.*

(c) *If* $\rho = 1$, *the series may converge or diverge, so that another test must be tried.*

- The proof of the Ratio Test is on page A40 in the appendices if you are interested.

Ratio Test Example

- Determine whether the following series converges or diverges:
- b) $\sum_{=1}^{\infty} \frac{}{2}$
- $= \lim_{\to +\infty} \frac{+1}{} = \lim_{\to +\infty} \frac{\frac{+1}{2^{+1}}}{\frac{}{2}} = \lim_{\to +\infty} \frac{+1}{2\ 2^{1}} * \frac{2}{}$

  $\lim_{\to +\infty} \frac{+1}{2} / - = \frac{1}{2} < 1$
- Therefore the series converges.

Another Ratio Test Example

- Determine whether the following series converges or diverges:
- d) $\sum_{=1}^{\infty} \frac{(2\ )!}{4}$
- $= \lim_{\to +\infty} \frac{\quad {}^{+1}}{\quad} = \lim_{\to +\infty} \frac{\frac{[2(\ +1)]!}{4^{\ +1}}}{\frac{(2\ )!}{4}} = \lim_{\to +\infty} \frac{\frac{(2\ +2)!}{4\ * 4^{1}}}{\frac{(2\ )!}{4}}$

$$\lim_{\to +\infty} \frac{(2\ +2)!}{4\ * 4^{1}} * \frac{4}{(2\ )!} = \frac{1}{2} * \lim_{\to +\infty} (2\ +2)(2\ +1) = \infty > 1$$

- Therefore the series diverges.
- There are several more examples on page 635 that you can refer to while doing your homework.

The Root Test

- In cases where it is difficult or inconvenient to find the limit required for the ratio test, the next test is sometimes useful.
- Its proof is similar to the proof of the ratio test.

The Root Test

9.5.6 THEOREM (*The Root Test*) *Let* $\sum u_k$ *be a series with positive terms and suppose that*

$$\rho = \lim_{k \to +\infty} \sqrt[k]{u_k} = \lim_{k \to +\infty} (u_k)^{1/k}$$

(a) *If* $\rho < 1$, *the series converges.*

(b) *If* $\rho > 1$ *or* $\rho = +\infty$, *the series diverges.*

(c) *If* $\rho = 1$, *the series may converge or diverge, so that another test must be tried.*

Root Test Examples

- The root test is useful when the general term is raised to the k power.
- Example: determine whether the following series converge or diverge.
- a) $\sum_{=2}^{\infty}\left(\frac{4\quad-5}{2\quad+1}\right)$
- $= \lim_{\to+\infty}(\quad)^{\frac{1}{}} = \lim_{\to+\infty}\frac{4\quad-5}{2\quad+1}$ / – = 2 > 1 therefore diverges
- b) $\sum_{=1}^{\infty}\frac{1}{(\ln(\quad+1))}$
- $= \lim_{\to+\infty}(\quad)^{\frac{1}{}} = \lim_{\to+\infty}\frac{1}{\ln(\quad+1)} = \frac{1}{\infty}$ = 0 < 1 therefore converges

Running List of Ideas

- The Squeezing Theorem for Sequences
- Sums of Geometric Series
- Telescoping Sums
- Harmonic Series
- Convergence Tests
  - The Divergence Test
  - The Integral Test
  - Convergence of p-series
  - The Comparison Test
  - The Limit Comparison Test
  - The Ratio Test
  - The Root Test

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