SECTION 7.2

PRINCIPALS OF INTEGRAL EVALUATION: "INTEGRATION BY PARTS"

ALL GRAPHICS ARE ATTRIBUTED TO:

Calculus,10/E by Howard Anton, Irl Bivens, and Stephen Davis

INTRODUCTION

- In this section we will discuss an integration technique that is essentially an antiderivative of the formula for differentiating a product of two functions.

THE PRODUCT RULE AND INTEGRATION BY PARTS

- Since there is not a product rule for integration, we need to develop a general method for evaluating integrals of the form $\int f(x)g(x)\,dx$
- If we start with the product rule and work backwards (integration), it will help identify a pattern.

$$\frac{d}{dx}[f(x)G(x)] = f(x)G'(x) + f'(x)G(x) = f(x)g(x) + f'(x)G(x)$$

THE PRODUCT RULE AND INTEGRATION BY PARTS – CON'T

- This pattern produces the formula we use to make difficult integrations easier. We call the method integration by parts.

$$\int u\,dv = uv - \int v\,du$$

EXAMPLE #1

Use integration by parts to evaluate $\int x \cos x \, dx$.

- The first step is to pick part of the expression x cos x to be u and another part to be dv (we will talk about strategies on a later slide).
- For now, let u = x and let dv = cosx dx. Then (2^{nd} step)

$$du = dx \quad \text{and} \quad v = \int dv = \int \cos x \, dx = \sin x$$

- The 3^{rd} step is to apply the integration by parts formula $\int u \, dv = uv - \int v \, du$

$$\int \underbrace{x}_{u} \underbrace{\cos x \, dx}_{dv} = \underbrace{x}_{u} \underbrace{\sin x}_{v} - \int \underbrace{\sin x}_{v} \underbrace{dx}_{du}$$

$$= x \sin x - (-\cos x) + C = x \sin x + \cos x + C$$

SOME GUIDELINES FOR CHOOSING U AND DV

- The goal is to choose u and dv to obtain a new integral that is easier than the original.
- The more you practice, the easier it is to pick u and dv correctly on your first try.
- One suggestion is to pick a part of the expression to be u that gets "easier/simpler/smaller" when you take its derivative.
- Another suggestion is to pick a part of the expression to be dv that you know how to integrate or that is easy to integrate.

EXAMPLE #2

Evaluate $\int \ln x \, dx$.

- One choice is to let u=1 and dv = lnx dx .
- If we did that, we would have to take the integral of lnx dx which we do not know how to do.
- STEP 1: Therefore, we should let u=lnx since we know its derivative and let dv = dx.
- STEP 2: That gives us $du = \frac{1}{x} dx \qquad v = \int dx = x$
- STEP 3: Apply the formula $\int u \, dv = uv - \int v \, du$

$$\int \ln x \, dx = \int u \, dv = uv - \int v \, du = x \ln x - \int dx = x \ln x - x + C$$

LIATE METHOD

- LIATE is an acronym for Logarithmic, Inverse trigonometric (which we have not done this year), Algebraic, Trigonometric, Exponential.
- If you have to take the integral of the product of two functions from different categories in the list, you will often be more successful if you select u to be the function whose category occurs earlier in the list.
- This method does not always work, but it works often enough to be a good rule of thumb.

LIATE EXAMPLE

Evaluate $\int xe^x \, dx$.

- We could either make u = x (algebraic) or u = e^x (exponential). According to LIATE, since algebraic is earlier in the list, we should make u = x and dv = e^x dx. STEP 1
- STEP 2: That give us $du = dx \quad \text{and} \quad v = \int e^x \, dx = e^x$
- STEP 3: Apply the formula $\int u \, dv = uv - \int v \, du$

$$\int xe^x \, dx = \int u \, dv = uv - \int v \, du = xe^x - \int e^x \, dx = xe^x - e^x + C$$

REPEATED INTEGRATION BY PARTS

- It is sometimes necessary to use integration by parts more than once in the same problem:

Example 4 Evaluate $\int x^2 e^{-x}\,dx$.

Solution. Let

$$u = x^2, \quad dv = e^{-x}\,dx, \quad du = 2x\,dx, \quad v = \int e^{-x}\,dx = -e^{-x}$$

so that from (3)

$$\begin{aligned}\int x^2 e^{-x}\,dx &= \int u\,dv = uv - \int v\,du \\ &= x^2(-e^{-x}) - \int -e^{-x}(2x)\,dx \\ &= -x^2 e^{-x} + 2\int x e^{-x}\,dx \end{aligned} \tag{4}$$

The last integral is similar to the original except that we have replaced x^2 by x. Another integration by parts applied to $\int xe^{-x}\,dx$ will complete the problem. We let

$$u = x, \quad dv = e^{-x}\,dx, \quad du = dx, \quad v = \int e^{-x}\,dx = -e^{-x}$$

so that

$$\int xe^{-x}\,dx = x(-e^{-x}) - \int -e^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

Finally, substituting this into the last line of (4) yields

$$\begin{aligned}\int x^2 e^{-x}\,dx &= -x^2e^{-x} + 2\int xe^{-x}\,dx = -x^2e^{-x} + 2(-xe^{-x} - e^{-x}) + C \\ &= -(x^2 + 2x + 2)e^{-x} + C\end{aligned}$$

pg 494 in book

may be easier to read

IF YOU MUST DO INTEGRATION BY PARTS MORE THAN TWICE, YOU MAY WANT TO USE A TABLE

Tabular Integration by Parts

Step 1. Differentiate $p(x)$ repeatedly until you obtain 0, and list the results in the first column.

Step 2. Integrate $f(x)$ repeatedly and list the results in the second column.

Step 3. Draw an arrow from each entry in the first column to the entry that is one row down in the second column.

Step 4. Label the arrows with alternating $+$ and $-$ signs, starting with a $+$.

Step 5. For each arrow, form the product of the expressions at its tip and tail and then multiply that product by $+1$ or -1 in accordance with the sign on the arrow. Add the results to obtain the value of the integral.

SEE PG 496 FOR EXAMPLE

REPEATED DIFFERENTIATION		REPEATED INTEGRATION
$x^2 - x$	$+$	$\cos x$
$2x - 1$	$-$	$\sin x$
2	$+$	$-\cos x$
0		$-\sin x$

$$\int (x^2 - x)\cos x\,dx = (x^2 - x)\sin x + (2x - 1)\cos x - 2\sin x + C$$

$$= (x^2 - x - 2)\sin x + (2x - 1)\cos x + C$$

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