Section 6.5

Area of a Surface of Revolution

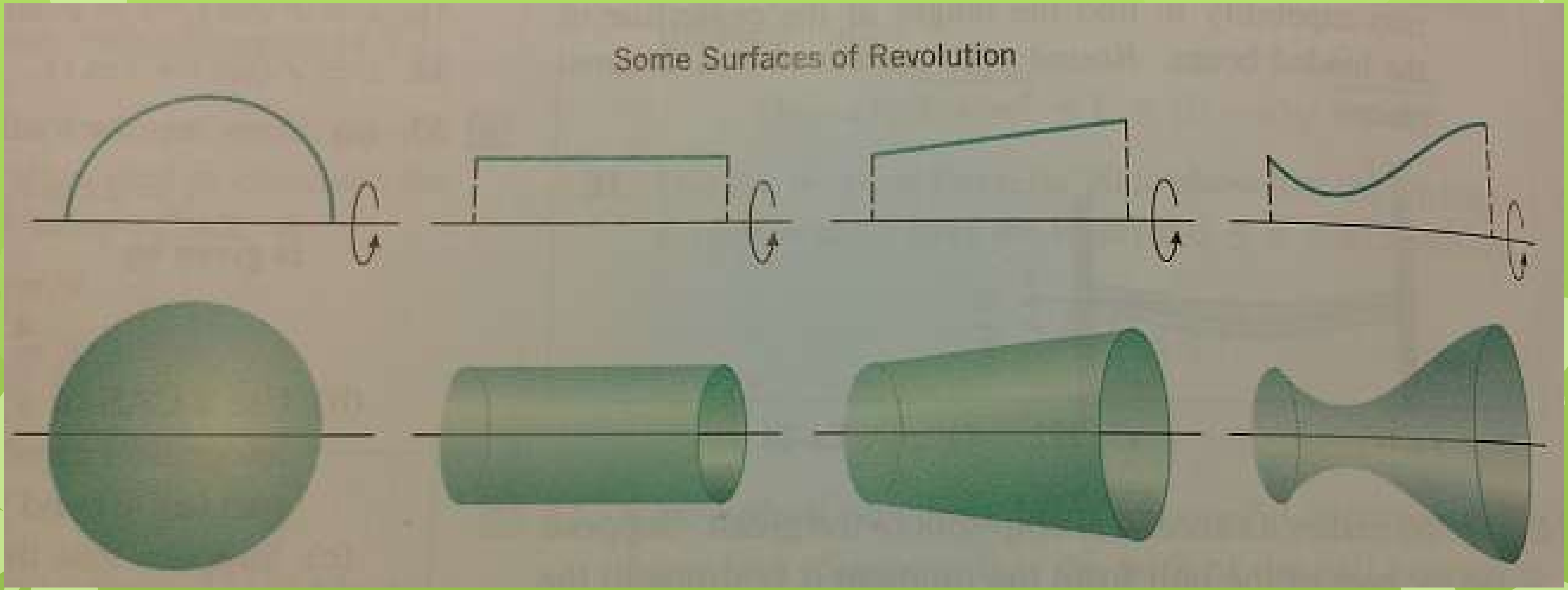
All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Introduction

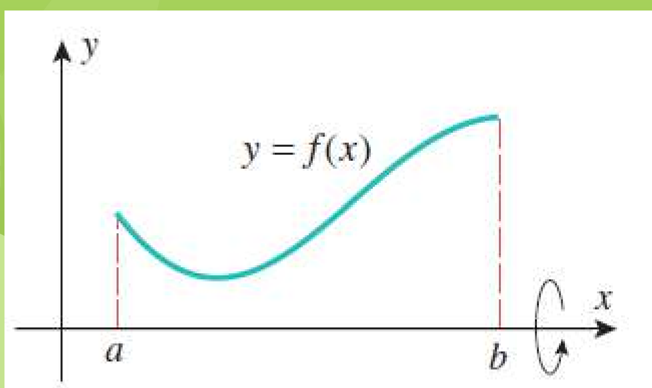
- In this section we will find the area of a surface that is generated by revolving a plane curve about a line.
- It is very similar to section 6.4, therefore, the equation is very similar to that of arc length.
- The difference is that we need to revolve it around a line like we did in sections 6.2 and 6.3.
- Since we are only rotating the outside curve (not the area between it and the axis or line), each small section will be approximated by the circumference of the circle infinitesimally narrow width.

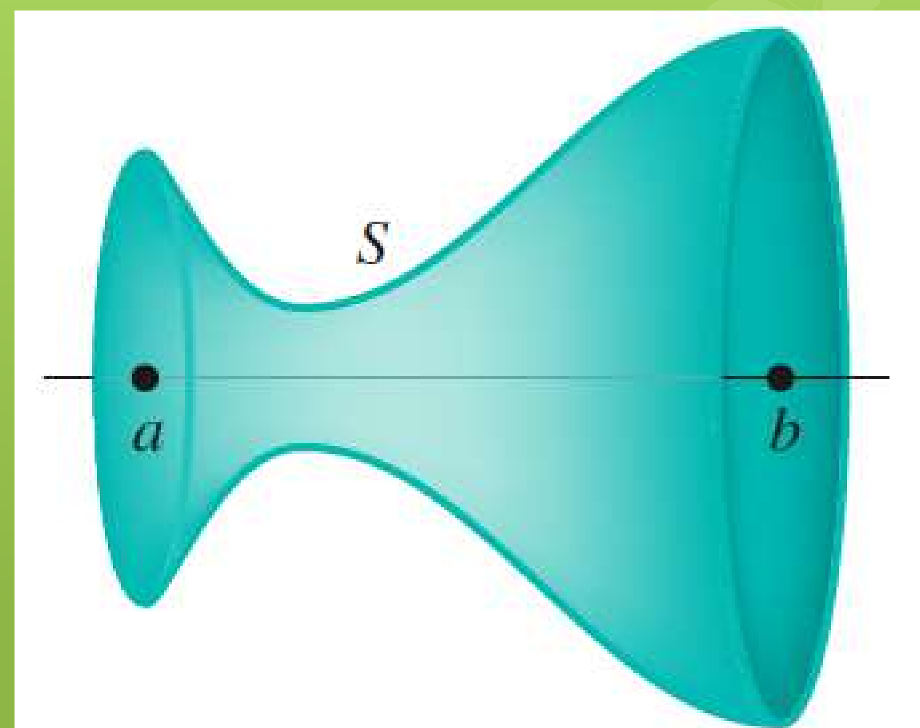
Examples Rotated about the x-axis



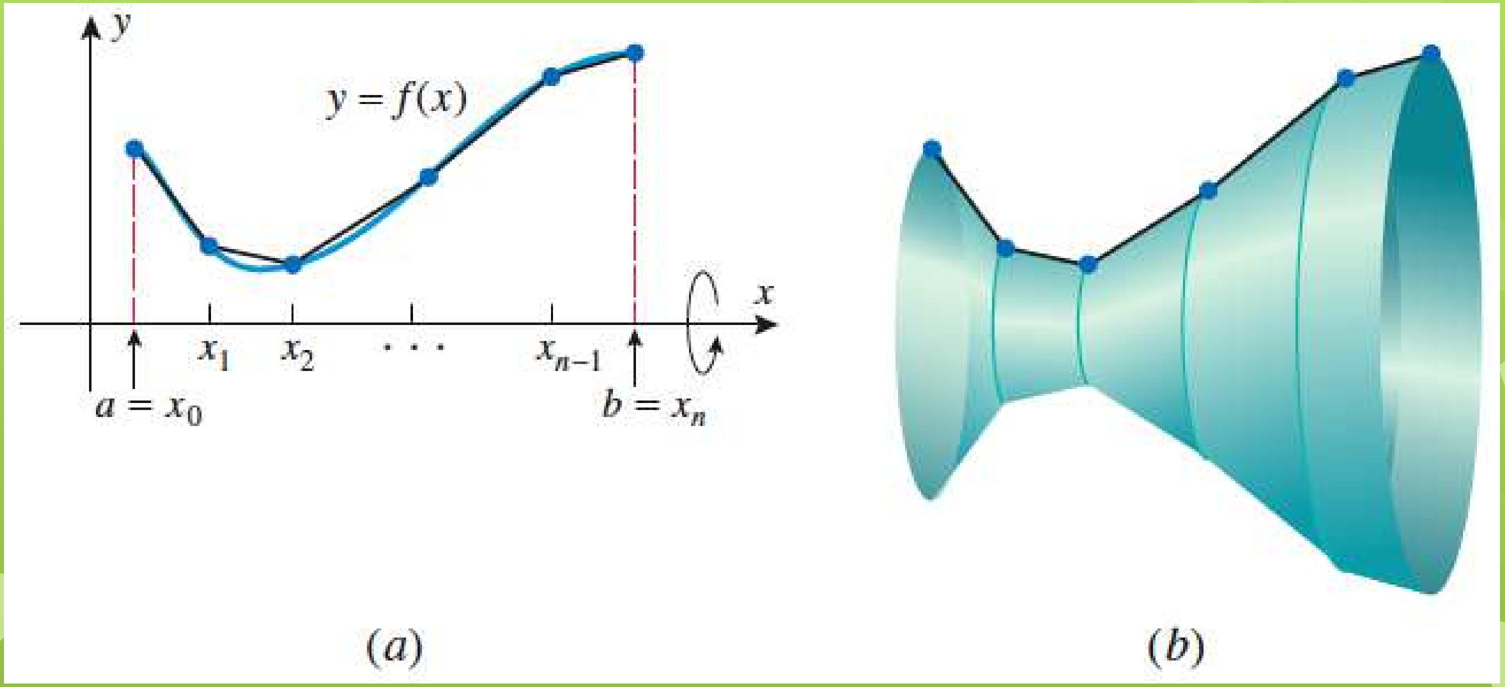
Around the x-axis

6.5.1 SURFACE AREA PROBLEM Suppose that f is a smooth, nonnegative function on $[a, b]$ and that a surface of revolution is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis (Figure 6.5.2). Define what is meant by the *area* S of the surface, and find a formula for computing it.





Break the Surface into Small Sections



Calculate Surface Area Using Riemann Sums

- Each of the subintervals into which we broke the surface area on the previous slide (figure b) is called a frustum which is a portion of a right circular cone.
- As we allow the number of subintervals to approach infinity, the width of each approaches zero.
- Each subinterval gets closer and closer to resembling a circle who's circumference is 2π r.
- We calculated the length of each subinterval in figure a on the previous slide last class using the distance formula.

Combine the Distance Formula and the Circumference

- The arc length (distance) formula from last class and the circumference which represents the rotation around a line combine together to generate the following Riemann sum-like expression:

$$S \approx \sum_{k=1}^{n} 2\pi f(x_k^{**})\sqrt{1+[f'(x_k^{*})]^2}\,\Delta x_k$$

- When we take the limit as the number of subintervals approaches infinity, this Riemann sum will give us the exact surface area.

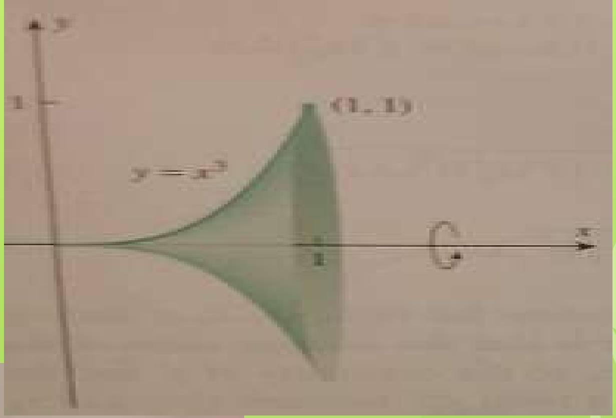
Integral – when about the x-axis

6.5.2 DEFINITION If f is a smooth, nonnegative function on $[a, b]$, then the surface area S of the surface of revolution that is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is defined as

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example about the x-axis



Example 1 Find the area of the surface that is generated by revolving the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis.

Solution. First sketch the curve; then imagine revolving it about the x-axis (Figure 6.5.6). Since $y = x^3$, we have $dy/dx = 3x^2$, and hence from (4) the surface area S is

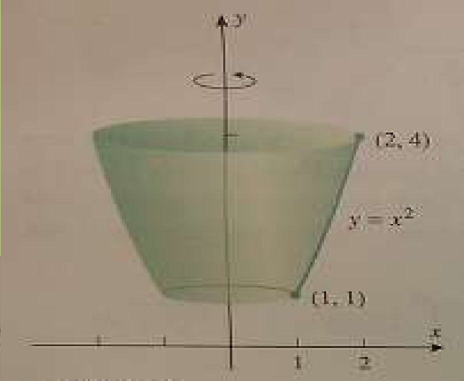
$$\begin{aligned}
S &= \int_0^1 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\
&= \int_0^1 2\pi x^3\sqrt{1 + (3x^2)^2}\,dx \\
&= 2\pi \int_0^1 x^3(1 + 9x^4)^{1/2}\,dx \\
&= \frac{2\pi}{36}\int_1^{10} u^{1/2}\,du \qquad \boxed{\begin{array}{c} u = 1 + 9x^4 \\ du = 36x^3\,dx \end{array}} \\
&= \frac{2\pi}{36}\cdot\frac{2}{3}u^{3/2}\Big]_{u=1}^{10} = \frac{\pi}{27}(10^{3/2} - 1) \approx 3.56
\end{aligned}$$

Integral – when about the y-axis

$$S = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\, dy = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example about the y-axis

Example 2 Find the area of the surface that is generated by revolving the portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ about the y-axis.



▲ Figure 6.5.7

Solution. First sketch the curve; then imagine revolving it about the y-axis (Figure 6.5.7). Because the curve is revolved about the y-axis we will apply Formula (5). Toward this end, we rewrite $y = x^2$ as $x = \sqrt{y}$ and observe that the y-values corresponding to $x = 1$ and $x = 2$ are $y = 1$ and $y = 4$. Since $x = \sqrt{y}$, we have $dx/dy = 1/(2\sqrt{y})$, and hence from (5) the surface area S is

$$S = \int_1^4 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_1^4 2\pi\sqrt{y}\sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\, dy$$

$$= \pi \int_1^4 \sqrt{4y+1}\, dy$$

$$= \frac{\pi}{4}\int_5^{17} u^{1/2}\, du \qquad \boxed{\begin{matrix} u = 4y+1 \\ du = 4\,dy \end{matrix}}$$

$$= \frac{\pi}{4}\cdot\frac{2}{3}u^{3/2}\Big]_{u=5}^{17} = \frac{\pi}{6}(17^{3/2} - 5^{3/2}) \approx 30.85$$

At my cousin's wedding