

5.5

Multiple–Angle and Product–to–Sum Formulas



What You Should Learn

- Use multiple–angle formulas to rewrite and evaluate trigonometric functions
- Use power–reducing formulas to rewrite and evaluate trigonometric functions
- Use half–angle formulas to rewrite and evaluate trigonometric functions
- Use product–to–sum and sum–to–product formulas to rewrite and evaluate trigonometric functions.



Multiple–Angle Formulas



Multiple – Angle Formulas

In this section, you will study four additional categories of trigonometric identities.

1. The first category involves *functions of multiple angles* such as $\sin ku$ and $\cos ku$.
2. The second category involves *squares of trigonometric functions* such as $\sin^2 u$.
3. The third category involves *functions of half-angles* such as $\sin \frac{u}{2}$.
4. The fourth category involves *products of trigonometric functions* such as $\sin u \cos v$.



Multiple – Angle Formulas

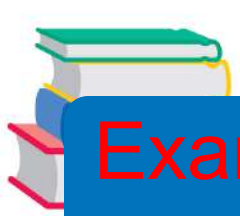
You should **memorize the double–angle formulas below** because they are used often in trigonometry and calculus.

Double-Angle Formulas

$$\sin 2u = 2 \sin u \cos u$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

$$\begin{aligned}\cos 2u &= \cos^2 u - \sin^2 u \\ &= 2 \cos^2 u - 1 \\ &= 1 - 2 \sin^2 u\end{aligned}$$



Example 1 – Solving a Multiple–Angle Equation

Solve $2 \cos x + \sin 2x = 0$.

Solution:

Begin by rewriting the equation so that it involves functions of x (rather than $2x$). Then factor and solve as usual.

$$2\cos x + \sin 2x = 0$$

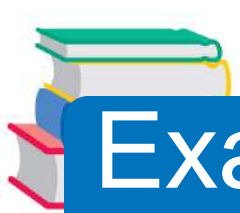
Write original equation.

$$2 \cos x + 2 \sin x \cos x = 0$$

Double–angle formula

$$2 \cos x(1 + \sin x) = 0$$

Factor.



Example 1 – Solution

cont'd

$$2 \cos x = 0$$

$$1 + \sin x = 0$$

Set factors equal to zero.

$$\cos x = 0$$

$$\sin x = -1$$

Isolate trigonometric functions.

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$x = \frac{3\pi}{2}$$

Solutions in $[0, 2\pi)$

So, the general solution is

$$x = \frac{\pi}{2} + 2n\pi \quad \text{and} \quad x = \frac{3\pi}{2} + 2n\pi \quad \text{General solution}$$

where n is an integer. Try verifying this solution graphically.



Power-Reducing Formulas



Power-Reducing Formulas

The double-angle formulas can be used to obtain the following **power-reducing formulas**.

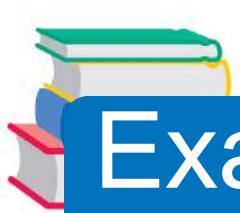
You need to know that these exist and how to find them, but you do not need to memorize them. 😊

Power-Reducing Formulas

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$



Example 4 – *Reducing a Power*

Rewrite $\sin^4 x$ as a sum of first powers of the cosines of multiple angles.

Solution:

$$\sin^4 x = (\sin^2 x)^2$$

Property of exponents

$$= \left(\frac{1 - \cos 2x}{2} \right)^2$$

Power-reducing formula

$$= \frac{1}{4} (1 - 2 \cos 2x + \cos^2 2x)$$

Expand binomial.

$$= \frac{1}{4} \left(1 - 2 \cos 2x + \frac{1 + \cos 4x}{2} \right)$$

Power-reducing formula



Example 4 – *Solution*

cont'd

$$= \frac{1}{4} - \frac{1}{2} \cos 2x + \frac{1}{8} + \frac{1}{8} \cos 4x$$

Distributive Property

$$= \frac{3}{8} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x$$

Simplify.

$$= \frac{1}{8}(3 - 4 \cos 2x + \cos 4x)$$

Factor.



Half–Angle Formulas



Half-Angle Formulas

You can derive some useful alternative forms of the power-reducing formulas by replacing u with $u/2$. The results are called **half-angle formulas**.

You need to know that these exist and how to find them, but you do not need to memorize them. 😊

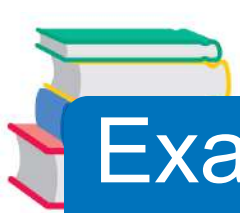
Half-Angle Formulas

$$\sin \frac{u}{2} = \pm \sqrt{\frac{1 - \cos u}{2}}$$

$$\cos \frac{u}{2} = \pm \sqrt{\frac{1 + \cos u}{2}}$$

$$\tan \frac{u}{2} = \frac{1 - \cos u}{\sin u} = \frac{\sin u}{1 + \cos u}$$

The signs of $\sin \frac{u}{2}$ and $\cos \frac{u}{2}$ depend on the quadrant in which $\frac{u}{2}$ lies.



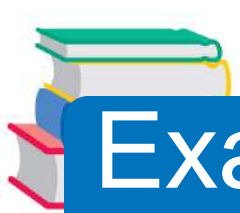
Example 5 – *Using a Half–Angle Formula*

Find the exact value of $\sin 105^\circ$.

Solution:

Begin by noting that 105° is half of 210° . Then, using the half–angle formula for $\sin(u/2)$ and the fact that 105° lies in Quadrant II, you have

$$\begin{aligned}\sin 105^\circ &= \sqrt{\frac{1 - \cos 210^\circ}{2}} \\ &= \sqrt{\frac{1 - (-\cos 30^\circ)}{2}}\end{aligned}$$

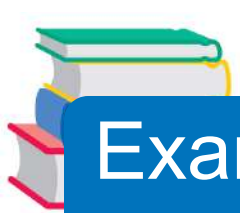


Example 5 – *Solution*

cont'd

$$= \sqrt{\frac{1 + (\sqrt{3}/2)}{2}}$$
$$= \frac{\sqrt{2 + \sqrt{3}}}{2}.$$

The positive square root is chosen because $\sin \theta$ is positive in Quadrant II.



Example 6 – Solving a Trigonometric Equation

Find all solutions of $1 + \cos^2 x = 2 \cos^2 \frac{x}{2}$ in the interval $[0, 2\pi)$.

Solution:

$$1 + \cos^2 x = 2 \cos^2 \frac{x}{2}$$

Write original equation.

$$1 + \cos^2 x = 2 \left(\pm \sqrt{\frac{1 + \cos x}{2}} \right)^2$$

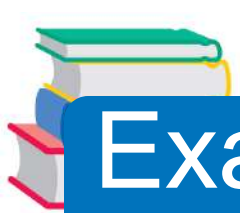
Half-angle formula

$$1 + \cos^2 x = 1 + \cos x$$

Simplify.

$$\cos^2 x - \cos x = 0$$

Simplify.



Example 6 – *Solution*

cont'd

$$\cos x(\cos x - 1) = 0$$

Factor.

By setting the factors $\cos x$ and $\cos x - 1$ equal to zero, you find the solutions in the interval $[0, 2\pi)$ are $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, and $x = 0$.



Product-to-Sum Formulas



Product-to-Sum Formulas

Each of the following **product-to-sum formulas** is easily verified using the sum and difference formulas.

You need to know that these exist and how to find them, but you do not need to memorize them. 😊

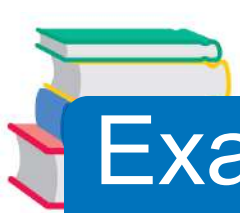
Product-to-Sum Formulas

$$\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$$

$$\sin u \cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \frac{1}{2}[\sin(u + v) - \sin(u - v)]$$



Example 7 – *Writing Products as Sums*

Rewrite the product as a sum or difference.

$$\cos 5x \sin 4x$$

Solution:

Using the appropriate product-to-sum formula, you obtain

$$\begin{aligned}\cos 5x \sin 4x &= \frac{1}{2} [\sin(5x + 4x) - \sin(5x - 4x)] \\ &= \frac{1}{2} \sin 9x - \frac{1}{2} \sin x.\end{aligned}$$



Sum-to-Product Formulas

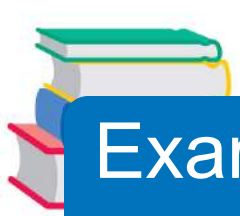
$$\sin u + \sin v = 2 \sin\left(\frac{u + v}{2}\right) \cos\left(\frac{u - v}{2}\right)$$

$$\sin u - \sin v = 2 \cos\left(\frac{u + v}{2}\right) \sin\left(\frac{u - v}{2}\right)$$

$$\cos u + \cos v = 2 \cos\left(\frac{u + v}{2}\right) \cos\left(\frac{u - v}{2}\right)$$

$$\cos u - \cos v = -2 \sin\left(\frac{u + v}{2}\right) \sin\left(\frac{u - v}{2}\right)$$

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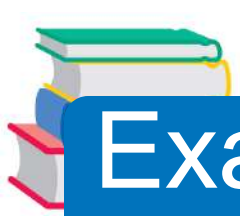
Example 8 – Using a Sum-to-Product Formula

Find the exact value of $\cos 195^\circ + \cos 125^\circ$.

Solution:

Using the appropriate sum-to-product formula, you obtain

$$\begin{aligned}\cos 195^\circ + \cos 105^\circ &= 2 \cos\left(\frac{195^\circ + 105^\circ}{2}\right) \cos\left(\frac{195^\circ - 105^\circ}{2}\right) \\ &= 2 \cos 150^\circ \cos 45^\circ \\ &= 2\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\ &= -\frac{\sqrt{6}}{2}\end{aligned}$$



Example 8 – *Solution*

cont'd

$$\begin{aligned} &= 2\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\ &= -\frac{\sqrt{6}}{2} \end{aligned}$$