5.5

Multiple–Angle and Product–to–Sum Formulas

What You Should Learn

- Use multiple–angle formulas to rewrite and evaluate trigonometric functions
- Use power–reducing formulas to rewrite and evaluate trigonometric functions
- Use half–angle formulas to rewrite and evaluate trigonometric functions
- Use product–to–sum and sum–to–product formulas to rewrite and evaluate trigonometric functions.

Multiple–Angle Formulas

Multiple – Angle Formulas

In this section, you will study four additional categories of trigonometric identities.

1. The first category involves *functions of multiple angles* such as sin *ku* and cos *ku*.

2. The second category involves *squares of trigonometric functions* such as $\sin^2 u$.

3. The third category involves *functions of half–angles* such as $\sin \frac{u}{2}$.

4. The fourth category involves *products of trigonometric functions* such as sin *u* cos *v*.

Multiple – Angle Formulas

You should memorize the **double–angle formulas** below because they are used often in trigonometry and calculus.

Double-Angle Formulas

$$\sin 2u = 2 \sin u \cos u$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

$$\begin{aligned} \cos 2u &= \cos^2 u - \sin^2 u \\ &= 2 \cos^2 u - 1 \\ &= 1 - 2 \sin^2 u \end{aligned}$$

Example 1 – *Solving a Multiple–Angle Equation*

Solve $2 \cos x + \sin 2x = 0$.

Solution:

Begin by rewriting the equation so that it involves functions of x (rather than $2x$). Then factor and solve as usual.

$$2\cos x + \sin 2x = 0$$ Write original equation.

$$2 \cos x + 2 \sin x \cos x = 0$$ Double–angle formula

$$2 \cos x(1 + \sin x) = 0$$ Factor.

Example 1 – *Solution*

cont'd

$2 \cos x = 0$ $\quad$ $1 + \sin x = 0$ $\quad$ Set factors equal to zero.

$\cos x = 0$ $\quad$ $\sin x = -1$ $\quad$ Isolate trigonometric functions.

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad$ $x = \frac{3\pi}{2}$ $\quad$ Solutions in $[0, 2\pi)$

So, the general solution is

$x = \frac{\pi}{2} + 2n\pi$ and $x = \frac{3\pi}{2} + 2n\pi$ $\quad$ General solution

where *n* is an integer. Try verifying this solution graphically.

Power–Reducing Formulas

Power–Reducing Formulas

The double–angle formulas can be used to obtain the following **power–reducing formulas**.

You need to know that these exist and how to find them, but you do not need to memorize them. ☺

Power-Reducing Formulas

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

Example 4 – *Reducing a Power*

Rewrite $\sin^4 x$ as a sum of first powers of the cosines of multiple angles.

Solution:

$$\sin^4 x = (\sin^2 x)^2$$ Property of exponents

$$= \left(\frac{1 - \cos 2x}{2}\right)^2$$ Power–reducing formula

$$= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$$ Expand binomial.

$$= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1 + \cos 4x}{2}\right)$$ Power–reducing formula

Example 4 – *Solution*

cont'd

$$= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x$$

Distributive Property

$$= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

Simplify.

$$= \frac{1}{8}(3 - 4\cos 2x + \cos 4x)$$

Factor.

Half–Angle Formulas

Half–Angle Formulas

You can derive some useful alternative forms of the power–reducing formulas by replacing *u* with *u*/2. The results are called **half-angle formulas**.

You need to know that these exist and how to find them, but you do not need to memorize them. ☺

Half-Angle Formulas

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

The signs of $\sin\frac{u}{2}$ and $\cos\frac{u}{2}$ depend on the quadrant in which $\frac{u}{2}$ lies.

Example 5 – *Using a Half–Angle Formula*

Find the exact value of sin 105°.

Solution:

Begin by noting that 105° is half of 210°. Then, using the half–angle formula for sin(*u*/2) and the fact that 105° lies in Quadrant II, you have

$$\sin 105^\circ = \sqrt{\frac{1 - \cos 210^\circ}{2}}$$

$$= \sqrt{\frac{1 - (-\cos 30^\circ)}{2}}$$

Example 5 – *Solution*

cont'd

$$= \sqrt{\frac{1 + \left(\sqrt{3}/2\right)}{2}}$$

$$= \frac{\sqrt{2 + \sqrt{3}}}{2}.$$

The positive square root is chosen because sin θ is positive in Quadrant II.

Example 6 – *Solving a Trigonometric Equation*

Find all solutions of $1 + \cos^2 x = 2\cos^2 \frac{x}{2}$ in the interval $[0, 2\pi)$.

Solution:

$$1 + \cos^2 x = 2\cos^2 \frac{x}{2}$$ Write original equation.

$$1 + \cos^2 x = 2\left(\pm\sqrt{\frac{1 + \cos x}{2}}\right)^2$$ Half-angle formula

$$1 + \cos^2 x = 1 + \cos x$$ Simplify.

$$\cos^2 x - \cos x = 0$$ Simplify.

Example 6 – *Solution*

cont'd

$$\cos x(\cos x - 1) = 0 \qquad \text{Factor.}$$

By setting the factors cos *x* and cos *x* – 1 equal to zero, you find the solutions in the interval $[0,2\pi)$ are $x = \pi/2$, $x = 3\pi/2$, and $x = 0$.

Product–to–Sum Formulas

Product–to–Sum Formulas

Each of the following **product–to–sum formulas** is easily verified using the sum and difference formulas.

You need to know that these exist and how to find them, but you do not need to memorize them. ☺

Product-to-Sum Formulas

$$\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$$

$$\sin u \cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \frac{1}{2}[\sin(u + v) - \sin(u - v)]$$

Example 7 – *Writing Products as Sums*

Rewrite the product as a sum or difference.

$\cos 5x \sin 4x$

Solution:

Using the appropriate product-to-sum formula, you obtain

$$\cos 5x \sin 4x = \frac{1}{2}[\sin(5x + 4x) - \sin(5x - 4x)]$$

$$= \frac{1}{2}\sin 9x - \frac{1}{2}\sin x.$$

Sum-to-Product Formulas

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

You need to know that these exist and how to find them, but you do not need to memorize them. ☺

Example 8 – *Using a Sum–to–Product Formula*

Find the exact value of cos 195° + cos 125°.

Solution:

Using the appropriate sum-to-product formula, you obtain

$$\cos 195° + \cos 105° = 2\cos\left(\frac{195° + 105°}{2}\right)\cos\left(\frac{195° - 105°}{2}\right)$$

$$= 2 \cos 150° \cos 45°$$

$$= 2\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$= -\frac{\sqrt{6}}{2}$$

Example 8 – *Solution*

cont'd

$$= 2\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$= -\frac{\sqrt{6}}{2}$$