Section 5.2

Integration: "The Indefinite Integral"

All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Introduction

- In the last section we saw how antidifferentiation could be used to find exact areas. In this section we will develop some fundamental results about antidifferentiation.

5.2.1 DEFINITION A function F is called an *antiderivative* of a function f on a given open interval if $F'(x) = f(x)$ for all x in the interval.

- For example, $F(x) = 1/3\ x^3$ is an antiderivative "F" of $f(x) = x^2$ because the derivative of F(x) is $F'(x) = x^2 = f(x)$.
- The problem is that $F(x) = 1/3\ x^3$ is not the ONLY antiderivative of f(x).

The Problem with the Previous Example

- If we add any constant C to $1/3\ x^3$, then the function $G(x) = 1/3\ x^3 + C$ is also an antiderivative of f(x) since the derivative of G(x) is $G'(x) = x^2 + 0 = f(x)$
- For example, take the derivatives of the following: $F(x) = 1/3\ x^3 - 5$, $F(x) = 1/3\ x^3 + 1/4$, $F(x) = 1/3\ x^3 + 1$, $F(x) = 1/3\ x^3 + 7$, etc.

5.2.2 THEOREM *If $F(x)$ is any antiderivative of $f(x)$ on an open interval, then for any constant C the function $F(x) + C$ is also an antiderivative on that interval. Moreover, each antiderivative of $f(x)$ on the interval can be expressed in the form $F(x) + C$ by choosing the constant C appropriately.*

The Indefinite Integral

- The process of finding antiderivatives is called antidifferentiation or integration.
- Since F’(x) = f(x), when we go backwards

$$\int f(x)dx = F(x) + C$$

means the same thing.

- For a more specific example:

$$\int x^2\,dx = \frac{1}{3}x^3 + C \text{ is equivalent to } \frac{d}{dx}\left[\frac{1}{3}x^3\right] = x^2$$

- The integral of f(x) with respect to x is equal to F(x) plus a constant.

Integration Formulas

- Here are some examples of derivative formulas and their equivalent integration formulas:
- The integration formula is just "backwards" when compared to the derivative formula you know.

DIFFERENTIATION FORMULA	INTEGRATION FORMULA
1. $\frac{d}{dx}[x] = 1$	$\int dx = x + C$
2. $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r \quad (r \neq -1)$	$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$
3. $\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$
4. $\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x\,dx = -\cos x + C$
5. $\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
6. $\frac{d}{dx}[-\cot x] = \csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
7. $\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$

Integration Formulas

- Here are more examples of derivative formulas and their equivalent integration formulas:
- When you need to refer back to these (you probably will need to quite often), you will find the list on pg 324. You might want to mark it with a post-it.

DIFFERENTIATION FORMULA	INTEGRATION FORMULA
8. $\frac{d}{dx}[-\csc x] = \csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + C$
9. $\frac{d}{dx}[e^x] = e^x$	$\int e^x \, dx = e^x + C$
10. $\frac{d}{dx}\left[\frac{b^x}{\ln b}\right] = b^x \quad (0 < b, b \neq 1)$	$\int b^x \, dx = \frac{b^x}{\ln b} + C \quad (0 < b, b \neq 1)$
11. $\frac{d}{dx}[\ln \lvert x \rvert] = \frac{1}{x}$	$\int \frac{1}{x} \, dx = \ln \lvert x \rvert + C$

Examples

- Here are some common examples to follow when integrating x raised to a power other than -1.
- Remember: <u>To integrate a power of x, add 1 to the exponent and divide by the new exponent.</u>

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \boxed{r = 2}$$

$$\int x^3\,dx = \frac{x^4}{4} + C \qquad \boxed{r = 3}$$

$$\int \frac{1}{x^5}\,dx = \int x^{-5}\,dx = \frac{x^{-5+1}}{-5+1} + C = -\frac{1}{4x^4} + C \qquad \boxed{r = -5}$$

$$\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \tfrac{2}{3}x^{\frac{3}{2}} + C = \tfrac{2}{3}(\sqrt{x})^3 + C \qquad \boxed{r = \tfrac{1}{2}}$$

Properties of the Indefinite Integral

- Our first properties of antiderivatives (integrals) follow directly from the simple constant factor, sum, and difference rules for limits and derivatives:
- 1. $$\int cf(x)\,dx = c\int f(x)\,dx$$
- 2. $$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$
- 3. $$\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

Combining Those Properties

- We can combine those three properties for combinations of sums, differences, and/or multiples of constants.
- Example:

$$\int (3x^6 - 2x^2 + 7x + 1)\,dx = 3\int x^6\,dx - 2\int x^2\,dx + 7\int x\,dx + \int 1\,dx$$
$$= \frac{3x^7}{7} - \frac{2x^3}{3} + \frac{7x^2}{2} + x + C$$

- General Rule:

$$\int [c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)]\,dx$$
$$= c_1 \int f_1(x)\,dx + c_2 \int f_2(x)\,dx + \cdots + c_n \int f_n(x)\,dx$$

Consider Simplifying First

- Sometimes it is useful to rewrite an integrand (the thing you are taking the integral of) in a different form before performing the integration.
- Examples:

▶ **Example 4** Evaluate

(a) $\int \frac{\cos x}{\sin^2 x}\,dx$ (b) $\int \frac{t^2 - 2t^4}{t^4}\,dt$ (c) $\int \frac{x^2}{x^2+1}\,dx$

Solution (a).

$$\int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin x}\frac{\cos x}{\sin x}\,dx = \int \csc x \cot x\,dx = -\csc x + C$$

Formula 8 in Table 5.2.1

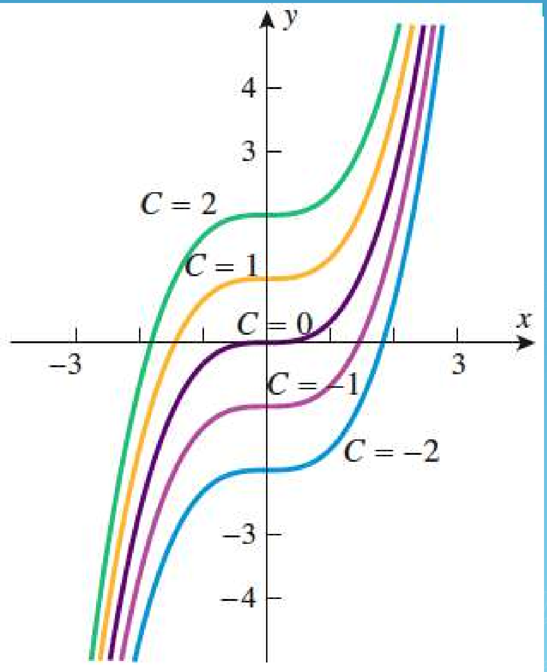
Solution (b).

$$\int \frac{t^2 - 2t^4}{t^4}\,dt = \int \left(\frac{1}{t^2} - 2\right)dt = \int (t^{-2} - 2)\,dt$$

$$= \frac{t^{-1}}{-1} - 2t + C = -\frac{1}{t} - 2t + C$$

Integral Curves

- As we discussed on slide #4, any of these curves could be the integral (antiderivative) of f(x) = x^2 because we do not know what the value of C is.



$y = \frac{1}{3}x^3 + C$

Initial Condition

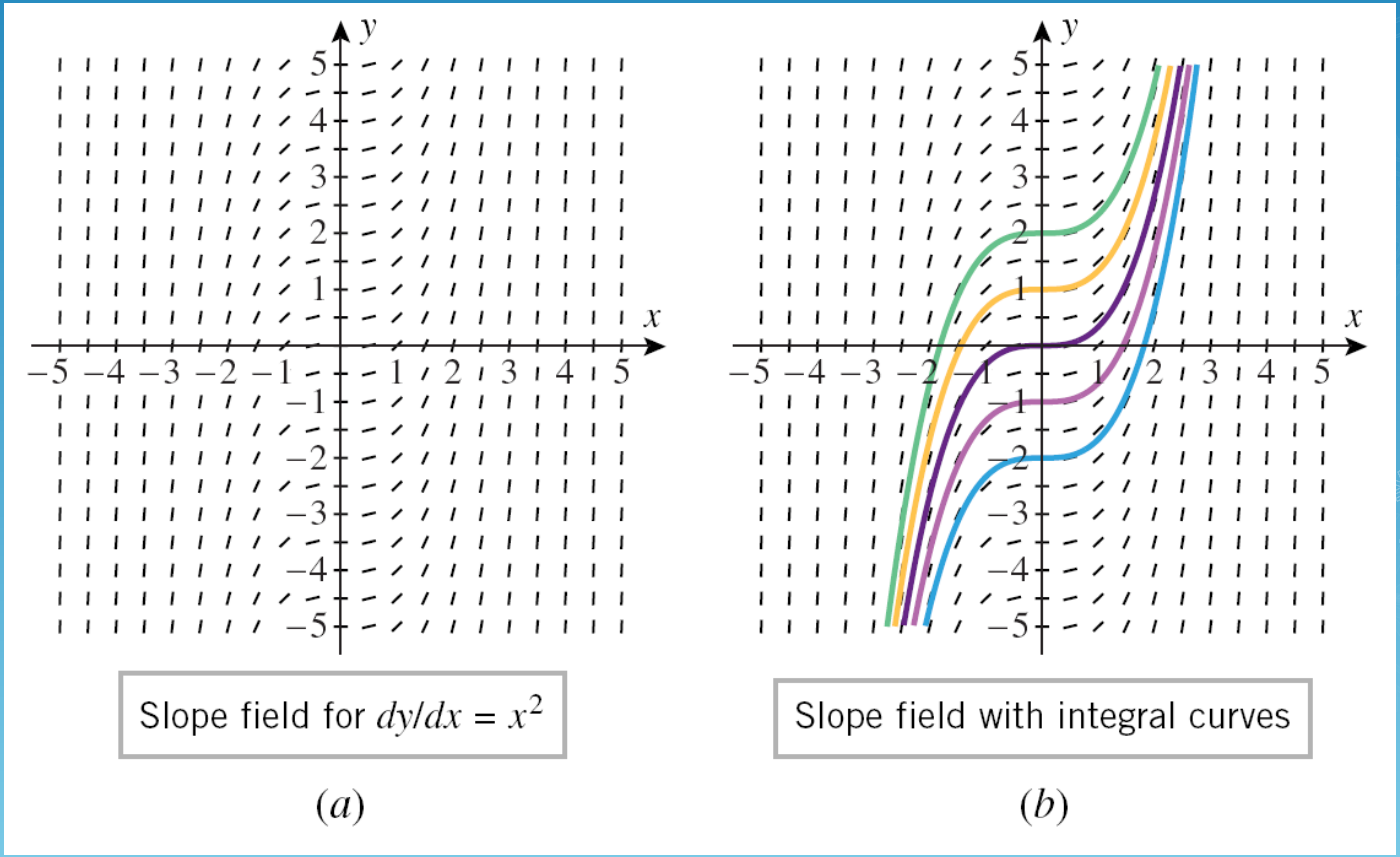
- When an "initial condition" is introduced, such as a requirement that the graph pass through a certain point, we may then solve for C and find the specific integral.
- Example: Instead of finding that the integral of $f(x) = x^2$ is $F(x) = 1/3\ x^3 + C$, if we were given that the graph passes through the point (2,1) we could solve for C: $1 = 1/3\ (2)^3 + C$
- This gives us $1=8/3 + C$ so $C=-5/3$ and our anti-derivative is more specific:

$$F(x)=1/3\ x^3 -5/3$$

Slope Fields

- If we interpret dy/dx as the slope of a tangent line, then at a point (x,y) on an integral curve of the equation $dy/dx=f(x)$, the slope of the tangent line is $f(x)$.
- We can find the slopes of the tangent lines by doing repeated substitution and drawing small portions of the tangent lines through those points.
- The resulting picture is called a slope field and it shows the "direction" of the integral curves at the gridpoints.
- We can do it by hand, but it is a lot of work and it is more commonly done by computer.

Example



y
x
5
4
3
2
1
−1
−2
−3
−4
−5
−5 −4 −3 −2 −1 1 2 3 4 5
Slope field for $dy/dx = x^2$
(a)
Slope field with integral curves
(b)

I love the Eiffel Tower