4.4

Trigonometric Functions of Any Angle

What You Should Learn

- Evaluate trigonometric functions of any angle
- Find reference angles
- Evaluate trigonometric functions of real numbers

Introduction

Introduction

Following is the definition of trigonometric functions of Any Angle. This applies when the radius is not one (not a unit circle).

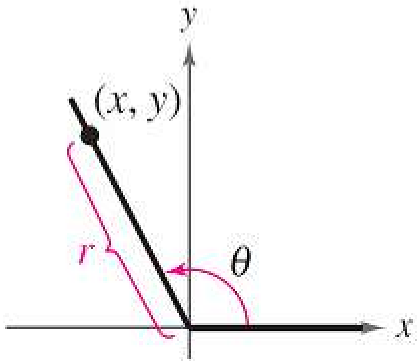
Definition of Trigonometric Functions of Any Angle

Let θ be an angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$.

$\sin\theta = \frac{y}{r}$ $\qquad$ $\cos\theta = \frac{x}{r}$

$\tan\theta = \frac{y}{x}, \quad x \neq 0$ $\qquad$ $\cot\theta = \frac{x}{y}, \quad y \neq 0$

$\sec\theta = \frac{r}{x}, \quad x \neq 0$ $\qquad$ $\csc\theta = \frac{r}{y}, \quad y \neq 0$



Introduction

Note: when $x = 0$, the tangent and secant of θ are undefined.

For example, the tangent of 90° is undefined since the sine of 90° is 1 and the cosine of 90° is 0. 1/0 is undefined.

Similarly, when $y = 0$, the cotangent and cosecant of θ are undefined.

Example 1 – *Evaluating Trigonometric Functions*

Let (–3, 4) be a point on the terminal side of θ (see Figure 4.34).

Find the sine, cosine, and tangent of θ.

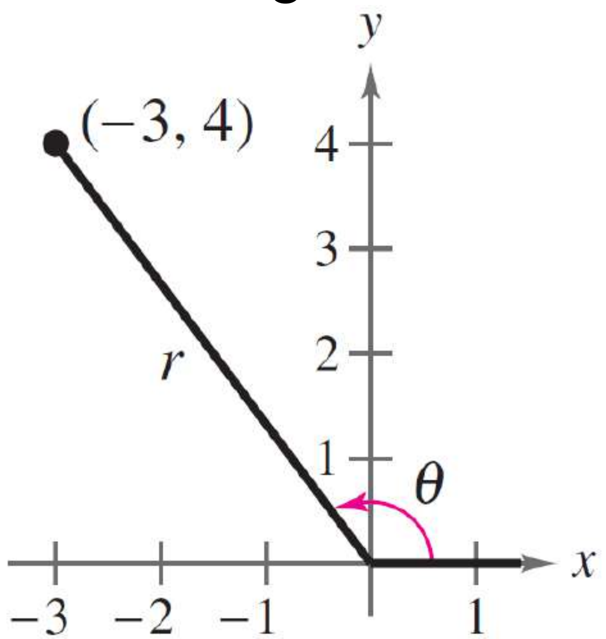


Figure 4.34

Example 1 – *Solution*

Referring to Figure 4.34, you can see by using the Pythagorean Theorem and the given point that $x = -3$, $y = 4$, and

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
&= \sqrt{(-3)^2 + 4^2} \\
&= \sqrt{25} \\
&= 5.
\end{aligned}
$$

Example 1 – *Solution*

cont'd

So, you have

$$\sin\theta = \frac{y}{r} = \frac{4}{5},$$

$$\cos\theta = \frac{x}{r} = -\frac{3}{5},$$

and

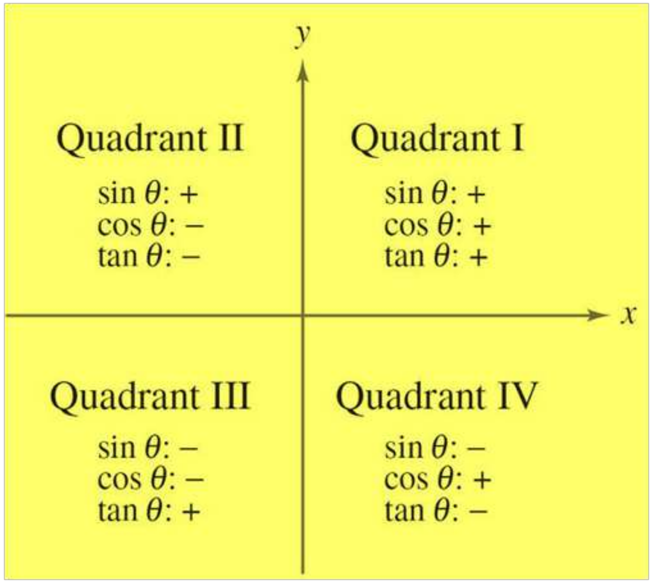
$$\tan\theta = \frac{y}{x} = -\frac{4}{3}.$$

Introduction

The *signs* of the trigonometric functions in the four quadrants can be determined easily from the definitions of the functions. For instance, because

$$\cos \theta = \frac{x}{r}$$

it follows that $\cos \theta$ is positive wherever $x > 0$, which is in Quadrants I and IV.



We will discuss “All Students Take Calculus” in class as a way to help us remember this.

Reference Angles

Reference Angles

The values of the trigonometric functions of angles greater than 90° (or less than 0°) can be determined from their values at corresponding acute angles called **reference angles.**

Definition of Reference Angle

Let θ be an angle in standard position. Its **reference angle** is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

Reference Angles

Figure 4.37 shows the reference angles for θ in Quadrants II, III, and IV.

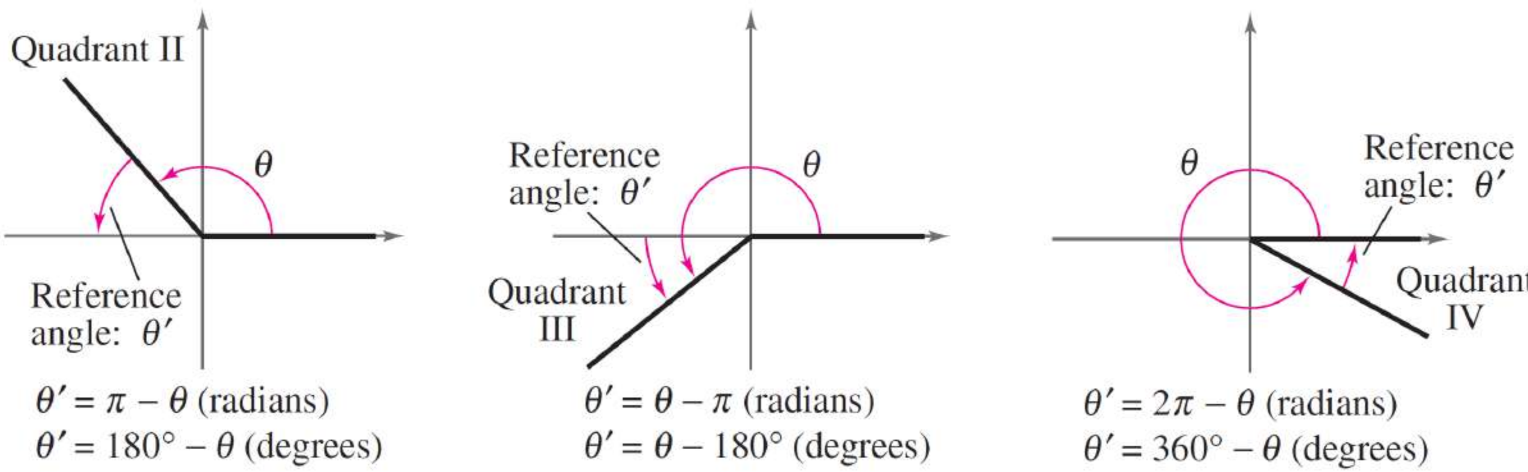


Figure 4.37

Example 4 – *Finding Reference Angles*

Find the reference angle θ'.

a. $\theta = 300°$ **b.** $\theta = 2.3$ **c.** $\theta = -135°$

Solution:

a. Because 300° lies in Quadrant IV, the angle it makes with the *x*-axis is

$$\theta' = 360° - 300°$$

$$= 60°.$$

Degrees

Example 4 – *Solution*

cont'd

b. Because 2.3 lies between $\pi/2 \approx 1.5708$ and $\pi \approx 3.1416$, it follows that it is in Quadrant II and its reference angle is

$$\theta' = \pi - 2.3$$

Radians

$$\approx 0.8416.$$

c. First, determine that $-135°$ is coterminal with $225°$, which lies in Quadrant III. So, the reference angle is

$$\theta' = 225° - 180°$$

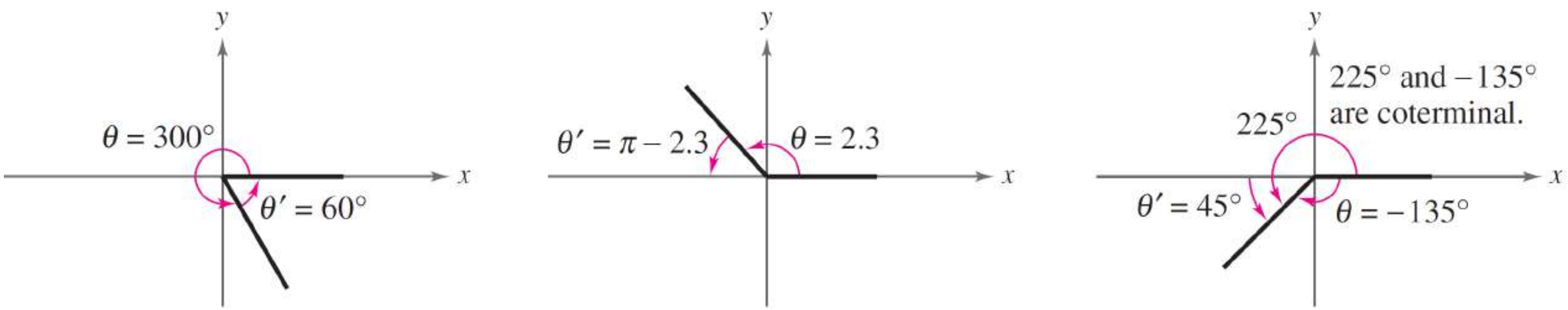
Degrees

$$= 45°.$$

Example 4 – *Solution*

cont'd

Figure 4.38 shows each angle θ and its reference angle θ'.



(a) (b) (c)

Figure 4.38

Trigonometric Functions of Real Numbers

Trigonometric Functions of Real Numbers

To see how a reference angle is used to evaluate a trigonometric function, consider the point (x, y) on the terminal side of θ, as shown in Figure 4.39.

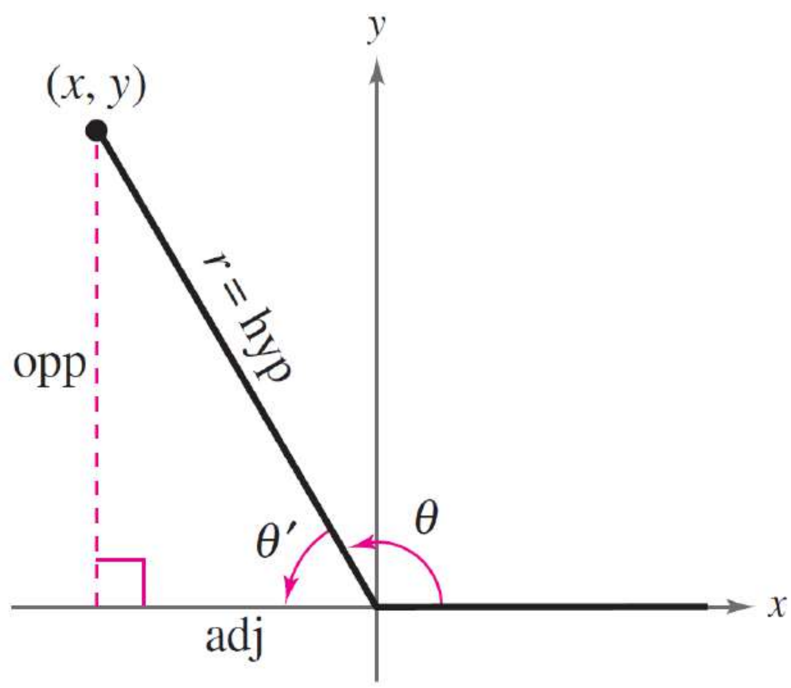


Figure 4.39

Trigonometric Functions of Real Numbers

By definition, you know that

$$\sin \theta = \frac{y}{r}$$

and

$$\tan \theta = \frac{y}{x}.$$

For the right triangle with acute angle θ' and sides of lengths $|x|$ and $|y|$, you have

and $$\sin \theta' = \frac{\text{opp}}{\text{hyp}} = \frac{|y|}{r} \qquad \tan \theta' = \frac{\text{opp}}{\text{adj}} = \frac{|y|}{|x|}.$$

Trigonometric Functions of Real Numbers

So, it follows that sin θ and sin θ' are equal, *except possibly in sign*. The same is true for tan θ and tan θ' and for the other four trigonometric functions. In all cases, the sign of the function value can be determined by the quadrant in which θ lies.

Evaluating Trigonometric Functions of Any Angle

To find the value of a trigonometric function of any angle θ:

1. Determine the function value of the associated reference angle θ'.
2. Depending on the quadrant in which θ lies, affix the appropriate sign to the function value.

Example 5 – *Trigonometric Functions of Nonacute Angles*

Evaluate each trigonometric function.

a. $\cos \frac{4\pi}{3}$ **b.** tan = (–210°) **c.** $\csc \frac{11\pi}{4}$

Solution:

a. Because $\theta = 4\pi/3$ lies in Quadrant III, the reference angle is $\theta' = (4\pi/3) - \pi = \pi/3$, as shown in Figure 4.40.

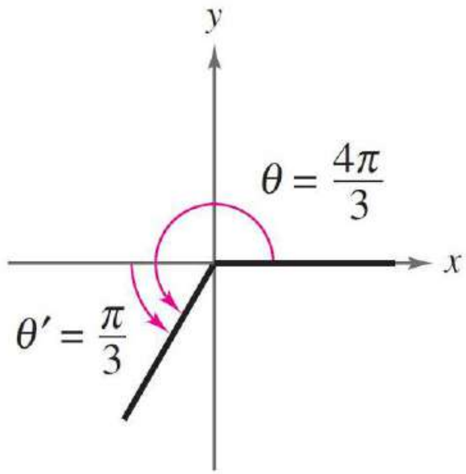


Figure 4.40

Moreover, the cosine is negative in Quadrant III, so

$$\cos \frac{4\pi}{3} = (-)\cos \frac{\pi}{3}$$
$$= -\frac{1}{2}.$$

Example 5 – *Solution*

cont'd

b. Because –210° + 360° = 150°, it follows that –210° is coterminal with the second-quadrant angle 150°. Therefore, the reference angle is θ' = 180° – 150° = 30°, as shown in Figure 4.41.

Finally, because the tangent is negative in Quadrant II, you have.

$$\tan(-210^\circ) = (-)\tan 30^\circ$$
$$= -\frac{\sqrt{3}}{3}.$$

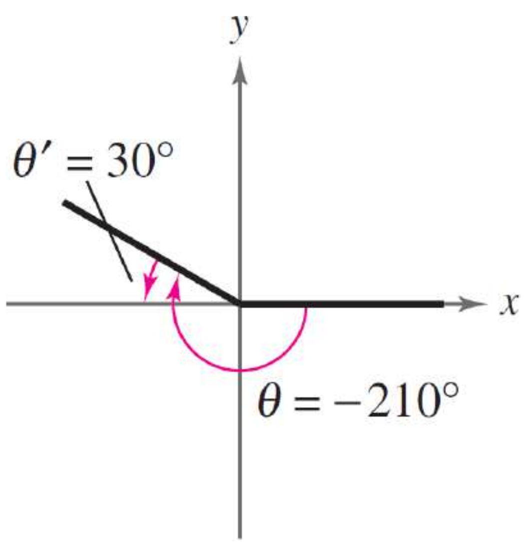


Figure 4.41

Example 5 – *Solution*

cont'd

c. Because $(11\pi/4) - 2\pi = 3\pi/4$, it follows that $11\pi/4$ is coterminal with the second-quadrant angle $3\pi/4$. Therefore, the reference angle is $\theta' = \pi - (3\pi/4) = \pi/4$, as shown in Figure 4.42.

Because the cosecant is positive in Quadrant II, you have

$$\csc\frac{11\pi}{4} = (+)\csc\frac{\pi}{4}$$
$$= \frac{1}{\sin(\pi/4)}$$
$$= \sqrt{2}.$$

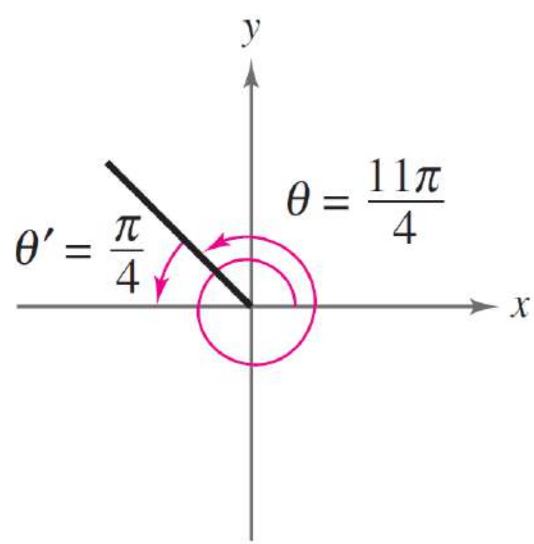


Figure 4.42