Section 4.2

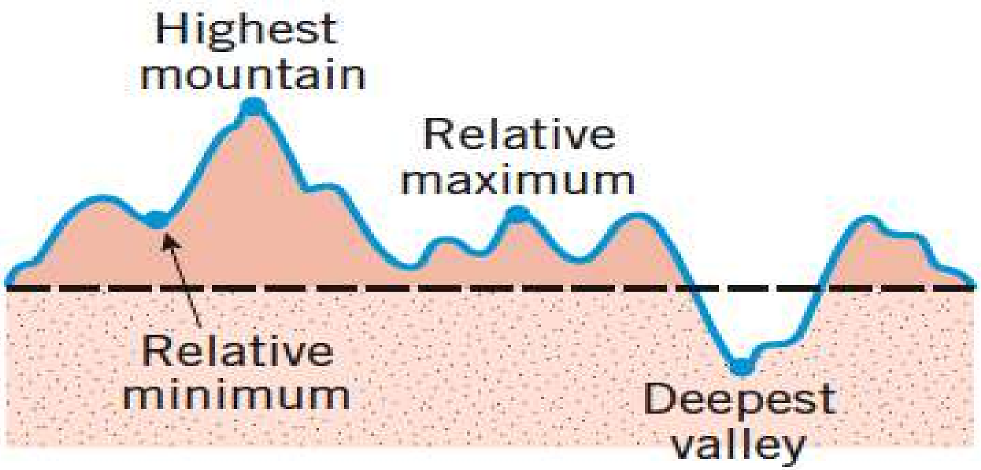
The Derivative in Graphing and Applications- "Analysis of Functions II: Relative Extrema; Graphing Polynomials"

All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Relative Maxima and Minima

- In this section we will develop methods for finding the high and low points on the graph of a functions .



- We will also spend some time discussing multiplicity and types of roots of polynomials.

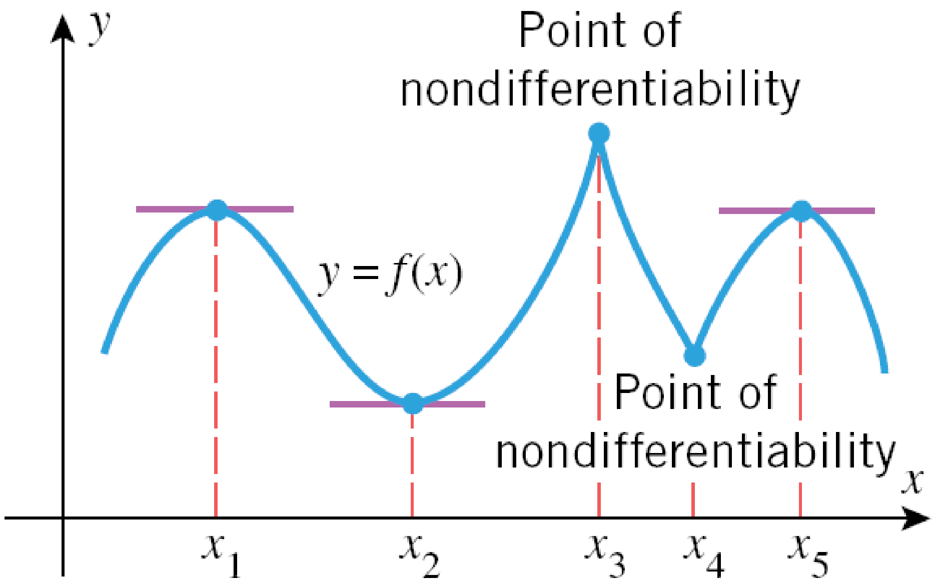
Relative Maxima and Minima con't

- The relative maxima (plural for maximum) are the high points in their *immediate vicinity* (the top of each little mountain on the previous slide).
- The relative minima (plural for minimum) are the low points in their *immediate vicinity* (the bottom of each little valley on the previous slide).

4.2.1 DEFINITION A function f is said to have a ***relative maximum*** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the largest value, that is, $f(x_0) \geq f(x)$ for all x in the interval. Similarly, f is said to have a ***relative minimum*** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the smallest value, that is, $f(x_0) \leq f(x)$ for all x in the interval. If f has either a relative maximum or a relative minimum at x_0, then f is said to have a ***relative extremum*** at x_0.

Relative Extrema

- Relative extrema means the extreme points which are the relative maxima and minima.
- They are called ***critical points*** and they occur where the graph of the function has:
  - horizontal tangent lines (slope zero) called ***stationary points***
  - and/or where the function is not differentiable



- The points x_1, x_2, x_3, x_4, and x_5 above are critical points.
- Of these, x_1, x_2, and x_5 are stationary points.

Maximum # of Critical Points in a Polynomial Function

- The maximum # of critical points that a polynomial of degree n can have is n-1 because the derivative is always one degree less than the original function due to the power rule.

Example

▶ **Example 2** Find all critical points of $f(x) = x^3 - 3x + 1$.

Solution. The function f, being a polynomial, is differentiable everywhere, so its critical points are all stationary points. To find these points we must solve the equation $f'(x) = 0$. Since

$$f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$$

we conclude that the critical points occur at $x = -1$ and $x = 1$. This is consistent with the graph of f in Figure 4.2.4. ◀

- They took the derivative, factored, and set that equal to zero.
- Then they did the zero product property to find the stationary points which are critical numbers.
- There are no points where f(x) is not differentiable since it is a polynomial which is differentiable everywhere.

Results of last Example

- We found the stationary points on the last slide were x = 1 and x = -1 which is where the slope of the tangent lines are zero.
- You can see on the graph to the right that that is where the relative extrema lie.
- The relative maximum is at x = -1 and the relative minimum is at x=1.
- You can tell which is which without graphing by using the first derivative test or the second derivative test (see later slide).

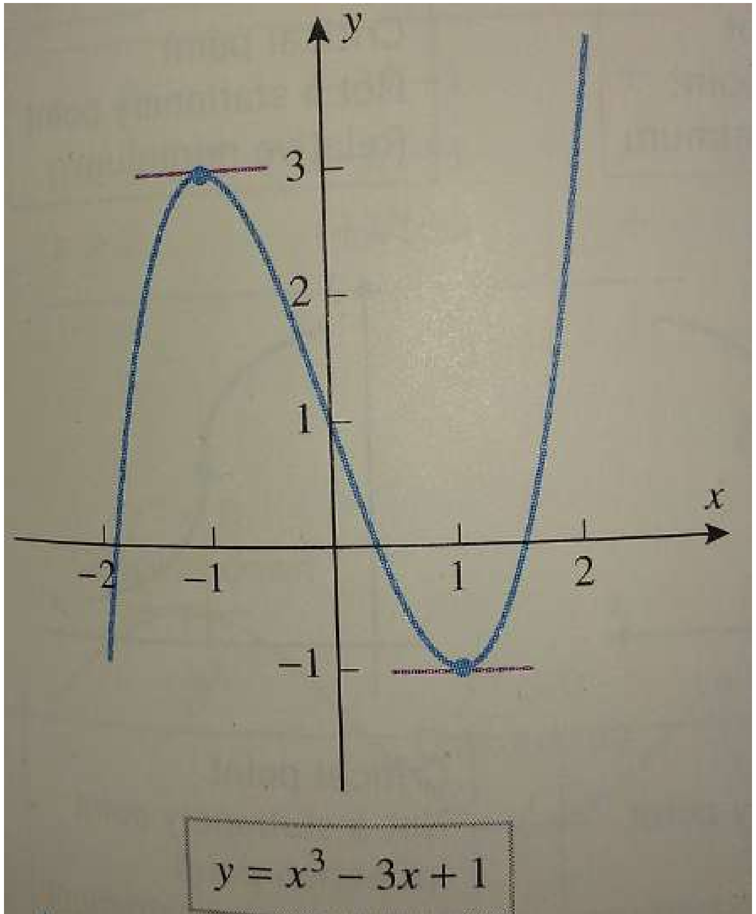


Figure 4.2.4

Example with non-differentiable points

▶ **Example 3** Find all critical points of $f(x) = 3x^{5/3} - 15x^{2/3}$.

Solution. The function f is continuous everywhere and its derivative is

$$f'(x) = 5x^{2/3} - 10x^{-1/3} = 5x^{-1/3}(x-2) = \frac{5(x-2)}{x^{1/3}}$$

- First they did the power rule and got a common denominator.
- Next we need to find the critical points, including the stationary points (where f'(x) = 0).

Example with non-differentiable points con't

We see from this that $f'(x) = 0$ if $x = 2$ and $f'(x)$ is undefined if $x = 0$. Thus $x = 0$ and $x = 2$ are critical points and $x = 2$ is a stationary point. This is consistent with the graph of f shown in Figure 4.2.5. ◀

- It is undefined if x=0 because you cannot divide by zero.

- The results match the graph
  because there is a relative min
  at x=2 and a relative max at x=0.

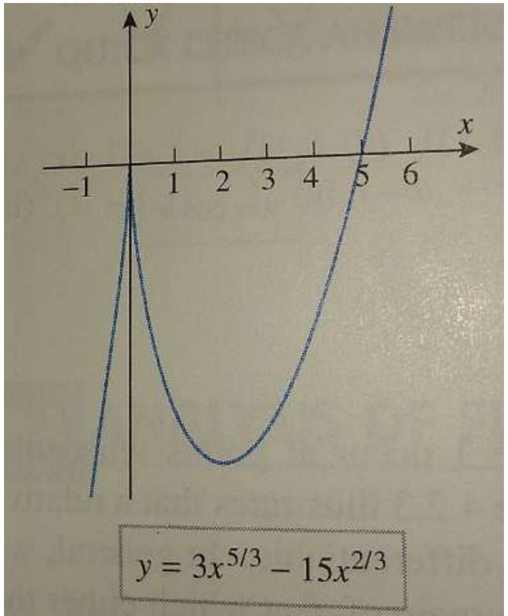
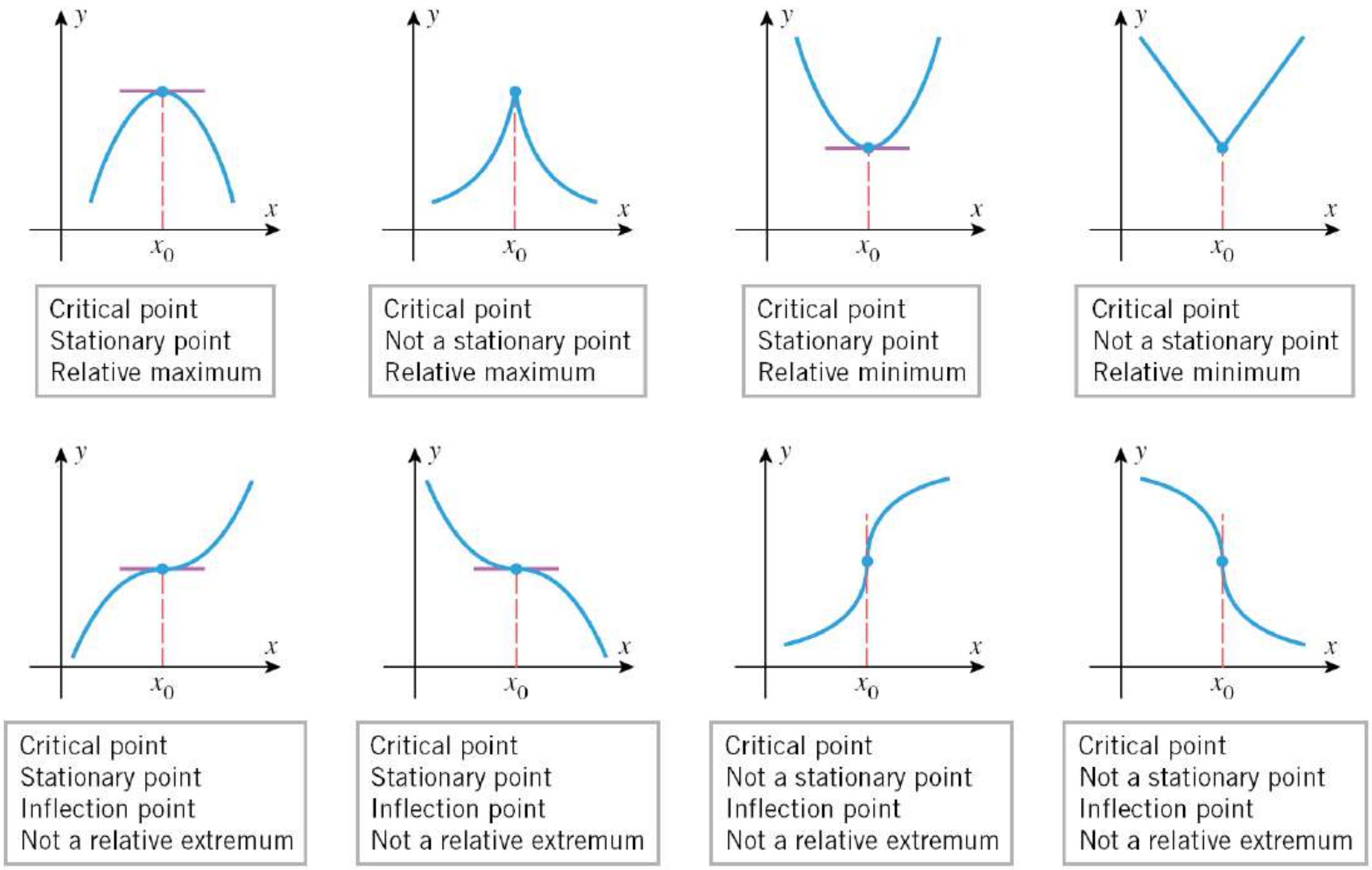


Figure 4.2.5

First Derivative Test

- The relative extrema of a function must occur at critical points, but they do not occur at every critical point.
- Relative extrema only occur at the critical points where f'(x) changes sign.
- See examples on next slide.
- None of the points in the bottom row are relative extrema because the derivative does not change sign at those values of x.

All are critical points, not all are relative extrema



Interpreting your f'(x) Interval Table

- When you finish your f'(x) interval table
  - where your graph changes from increasing to decreasing, there is a relative maximum
  - where your graph changes from decreasing to increasing, there is a relative minimum

4.2.3 THEOREM ***(First Derivative Test)*** *Suppose that f is continuous at a critical point x_0.*

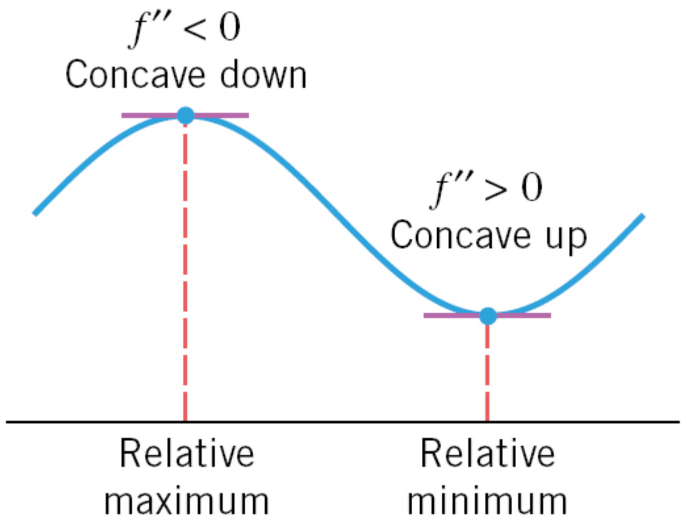
(a) *If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0, then f has a relative maximum at x_0.*

(b) *If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0, then f has a relative minimum at x_0.*

(c) *If $f'(x)$ has the same sign on an open interval extending left from x_0 as it does on an open interval extending right from x_0, then f does not have a relative extremum at x_0.*

Second Derivative Test

- There is another test for relative extrema that is based on concavity.
  - If the function is concave up at a critical point, then that x value is a relative minimum.
  - If the function is concave down at a critical point, then that x value is a relative maximum.



When to use the Second Derivative Test

- I prefer the second derivative test, but I only use it for functions where the second derivative is easy for me to do.

4.2.4 THEOREM (*Second Derivative Test*) *Suppose that f is twice differentiable at the point x_0.*

(a) *If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a relative minimum at x_0.*

(b) *If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a relative maximum at x_0.*

(c) *If $f'(x_0) = 0$ and $f''(x_0) = 0$, then the test is inconclusive; that is, f may have a relative maximum, a relative minimum, or neither at x_0.*

Example: First Derivative Test

- If you think back to the function from Section 4.1 notes, $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ the stationary points were x=0, -2, 1.
- The interval table showed:

Interval	Test #	Test in f '(x)	Result	Effect
(-infinity, -2)	-3	$12(-3)^3+12(-3)^2-24(-3)$	-144	Decreasing ↘
(-2, 0)	-1	$12(-1)^3+12(-1)^2-24(-1)$	+24	Increasing ↗
(0, 1)	.5	$12(.5)^3+12(.5)^2-24(.5)$	-7.5	Decreasing ↘
(1, +infinity)	2	$12(2)^3+12(2)^2-24(2)$	+96	Increasing ↗

- The slope changes from increasing to decreasing at x=0, so the first derivative test says that will be a relative maximum.
- The slope changes from decreasing to increasing at x=-2 and x=1, so the first derivative test says that those will be relative minimums.

Same Example Using Second Derivative Test

- Using the same function $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ and stationary points x=0, -2, 1, we can substitute them into the second derivative instead which was $f''(x) = 36x^2 + 24x - 24$.
  - $f''(0) = 36(0)^2 + 24(0) - 24 = -24$: since the second derivative is negative, the graph is concave down at x=0 so there is a relative maximum at that value.
  - $f''(-2) = 36(-2)^2 + 24(-2) - 24 = +72$

    $f''(1) = 36(1)^2 + 24(1) - 24 = +36$: since the second derivative is positive, the graph is concave up at x=-2 and x = 1 so there are relative minimums at those values.
- If you compare the results of the first and second derivative tests on this slide and the last, you will see that they are the same.

Multiplicity and Degree of Polynomial Functions

- Please look quickly at pages 249-251 for a quick review of the topics.
- Multiplicity (page 249):
  - After factoring the polynomial, if the degree of a factor is 1, there will be a single root at the x value that comes from that factor.
  - If the degree of a factor is 2, there will be a double root (bounce) at the x value that comes from that factor.
  - If the degree of a factor is 3, there will be a triple root (similar to a cubic) at the x value that comes from that factor.
- Analysis of Polynomials (page 250-251):
  - When a polynomial is in standard form, if the degree is even then the end behavior is the same on both ends (up, up if a is positive and down, down if a is negative).
  - If the degree is odd, the end behavior will not be the same on both ends (down, up if a is positive and up, down if a is negative).

Sea Lions at Pier 39 in San Francisco