Topics in Differentiation: "Derivatives of Exponential Functions"

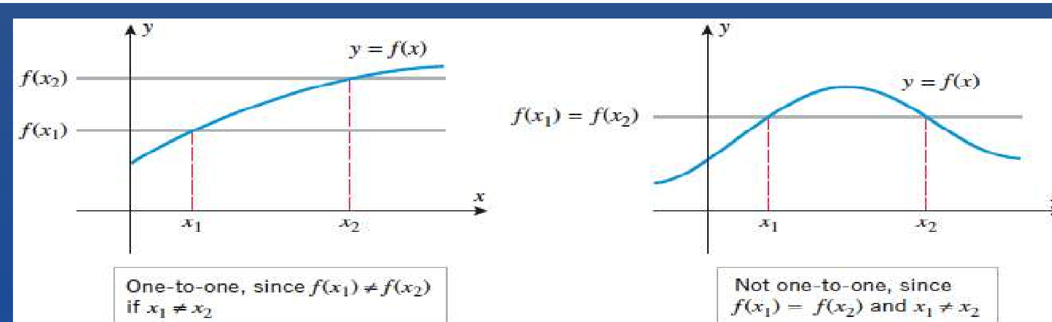
SECTION 3.3

All graphics are attributed to:

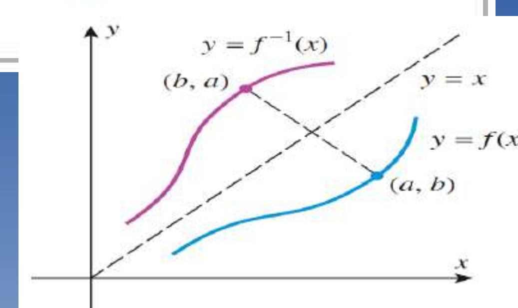
- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

One-to-One and Inverse Functions (reminders from Section 0.4)

0.4.4 THEOREM *(The Horizontal Line Test)* *A function has an inverse function if and only if its graph is cut at most once by any horizontal line.*



0.4.5 THEOREM *If f has an inverse, then the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about the line $y = x$; that is, each graph is the mirror image of the other with respect to that line.*



One-to-One and Inverse Functions (continued)

- Note: Sometimes, it is necessary to restrict the domain of an inverse $f^{-1}(x)=x$ or of an original f(x) in order to obtain a function (see examples on page 44).

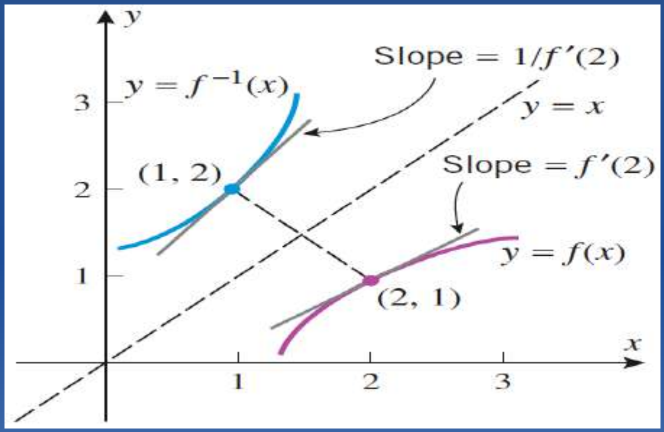
- A function f(x) has an inverse iff it is one-to-one (invertible), each x has one y and each y has one x (must pass vertical and horizontal line tests).

Application to Section 3.3

- In this section we will show how the derivative of a one-to-one function can be used to obtain the derivative of its inverse function.
- This will provide the tools we need to obtain derivative formulas for exponential functions from the derivative formulas for logarithmic functions.
- You could also use this method to obtain derivative formulas for inverse trig. functions, but we are going to skip that this year.

Slope of the tangent line to the inverse

- The slope of the tangent line to y = $f^{-1}(x)$ (which is the inverse) at the point (x,y) is the reciprocal of the slope of the tangent line to y = f(x) at (y,x).



- There is a more detailed explanation on page 197 if you are interested.

Result of that relationship?

In general, if f is a differentiable and one-to-one function, then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(1))}$$

provided $f'(f^{-1}(x)) \neq 0$.

- It may seem more obvious to you to use different notation. $\frac{dy}{dx} = \frac{1}{dx/dy}$
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Increasing or Decreasing Functions are One-to-One

- If the graph of a function is always increasing ($f'(x)>0$) or always decreasing ($f'(x)<0$), then it will pass the horizontal and vertical line tests which means that the function is one-to-one.
- There is an example of a one-to-one function problem on the next page that might help you see when these rules might help.

Two ways of calculating the derivative of f^{-1} if f is 1 to 1

Consider the function $f(x) = x^5+x+1$.

a) Show that f is one-to-one on the interval $(-\infty, +\infty)$.

b) Find a formula for the derivative of f^{-1}.

c) Compute $(f^{-1})'(1)$.

Solution (a). Since

$$f'(x) = 5x^4 + 1 > 0$$

for all real values of x, it follows from Theorem 3.3.1 that f is one-to-one on t
$(-\infty, +\infty)$.

I found the derivative using the power rule. No matter which value of x you plug in, it will always give a positive answer (>0) since you must raise the number to the 4th power.

Solution (b). Let $y = f^{-1}(x)$. Differentiating $x = f(y) = y^5 + y + 1$ imp
respect to x yields

$$\frac{d}{dx}[x] = \frac{d}{dx}[y^5 + y + 1]$$

$$1 = (5y^4 + 1)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{5y^4 + 1}$$

I took the derivative using implicit differentiation. (If you prefer, this would be the perfect time to use the previous rule about the reciprocal of the derivative of the original after interchanging x and y).

Solution (c):

To compute $(f^{-1})'(1)$, we need to find the y value when X = 1 by using the inverse equation $x = y^5+y+1$ and substituting 1 in for x: $1 = y^5+y+1$. When you solve this, you get y=0.

$(f^{-1})'(1) = \frac{dy}{dx}$ *evaluated when* $(x = 1) = \frac{dy}{dx}$ *evaluated when* $(y = 0) = \frac{1}{5*0^4+1} = 1$

Derivative of Exponential Functions

- After the last few slides, you probably do not want to see a proof here. ☺ If you would like to read the proof(s) of these derivatives, please see pages 198-199.

Base b (not e)	Base e (only)
The derivative when you have any base other than e to just the x power. $$\frac{d}{dx}[b^x] = b^x \ln b$$	The derivative when you have only base e to just the x power. $$\frac{d}{dx}[e^x] = e^x$$
The derivative when you have any base other than e to something other than just the x power (u). $$\frac{d}{dx}[b^u] = b^u \ln b \cdot \frac{du}{dx}$$	The derivative when you have only base e to to something other than just the x power (u). $$\frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx}$$

Short examples

$$\frac{d}{dx}[2^x] = 2^x \ln 2$$

$$\frac{d}{dx}[e^{-2x}] = e^{-2x} \cdot \frac{d}{dx}[-2x] = -2e^{-2x}$$

$$\frac{d}{dx}[e^{x^3}] = e^{x^3} \cdot \frac{d}{dx}[x^3] = 3x^2 e^{x^3}$$

$$\frac{d}{dx}[e^{\cos x}] = e^{\cos x} \cdot \frac{d}{dx}[\cos x] = -(\sin x)e^{\cos x}$$

Logarithmic Differentiation Example

$y = (x^2 + 1)^{\sin x}$

$ln\, y = \ln(x^2 + 1)^{\sin x}$

Take the natural logarithm of both sides

$ln\, y = (\sin x) * \ln\,(x^2 + 1)$

Apply the power property of logarithms.

$\frac{1}{y}\frac{dy}{dx} = (\cos x) * \ln(x^2 + 1) + \frac{1}{x^2 + 1}(2X) * \ sinx$

Take the derivative of both sides using the product rule and $\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$.

$y * \frac{1}{y}\frac{dy}{dx} = ((\cos x) * \ln(x^2 + 1) + \frac{2x * sinx}{x^2 + 1}) * y$

To get $\frac{dy}{dx}$ alone, multiply both sides by y. I also simplified the right side.

$\frac{dy}{dx} = ((\cos x) * \ln(x^2 + 1) + \frac{2xsinx}{x^2 + 1}) * (x^2 + 1)^{\sin x}$

Substitute the original function in for y.

My friend made us wear wigs for her birthday: