Topics in Differentiation: "Derivative of Logarithmic Functions"

SECTION 3.2

All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Derivatives of Logarithmic Functions

- In this section, we will obtain derivative formulas for logarithmic functions, and we will explain why the natural logarithm function is preferred over logarithms with other bases in Calculus.
- There is a proof using the definition of the derivative of ln x on page 192.

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$$

Natural Log vs. Common Log

Natural Logarithm	Common Logarithm
The derivative when you are taking the natural log of just x. $$\frac{d}{dx}[\ln x] = \frac{1}{x}, \qquad x > 0$$	The derivative when you are taking the logarithm with a base other than e of just x. $$\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}, \qquad x > 0$$
The derivative when you are taking the natural log of something other than x (u). You must use the Chain Rule. $$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$$	The derivative when you are taking the log with a base other than e of something other than x (u). You must use the Chain Rule. $$\frac{d}{dx}[\log_b u] = \frac{1}{u \ln b} \cdot \frac{du}{dx}$$

Why is the Natural Logarithm preferred?

- Among all possible bases, base e produces the simplest formula for the derivative of $\log_b x$. This is the main reason (for now ☺) that the natural logarithm function is preferred over other logarithms in Calculus.

- Look at the previous slide and notice how much smaller the formulas are in the first column than those in the second column.

Examples

Find $\frac{d}{dx}[\ln(x^2+1)]$.

$$\frac{d}{dx}[\ln(x^2+1)]$$

Apply $\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$ and the Chain Rule

$$= \frac{1}{x^2+1} * the\ derivative\ of\ the\ inside$$

$$= \frac{1}{x^2+1} * 2x \quad = \frac{2x}{x^2+1}$$

Find $\frac{d}{dx}[\ln(\cos x)]$.

$$\frac{d}{dx}[\ln(\cos x)]$$

Apply $\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$ and the Chain Rule

$$= \frac{1}{\cos x} * the\ derivative\ of\ the\ inside$$

$$= \frac{1}{\cos x} * -\sin x \quad = \frac{-\sin x}{\cos x} \quad = -\tan x$$

Logarithmic Differentiation

- Some derivatives are very long and messy to calculate directly, especially when they are composed of products, quotients, and powers all in one function like $y = \frac{x^2 \sqrt[5]{7x-14}}{(1+x^2)^4}$
- You would have to do the product and chain rules within the quotient rule and it would take quite awhile.
- Therefore, we are going to first take the natural logarithm of both sides and use log properties to simplify before we even start to find the derivative.

Logarithmic Differentiation Example

$$y = \frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}$$

$$ln\, y = \ln\left(\frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}\right)$$

Take the natural logarithm of both sides

$$ln\, y = \ln x^2\sqrt[3]{7x-14} - \; ln(1+x^2)^4$$

Apply the quotient property of logarithms.

$$ln\, y = \ln x^2 + ln(7x-14)^{\frac{1}{3}} - \; ln(1+x^2)^4$$

Apply the product property of logarithms.

$$ln\, y = 2\ln x + \frac{1}{3}\ln(7x-14) - 4\ln(1+x^2)$$

Apply the power rule of logarithms.

$$\frac{1}{y}\frac{dy}{dx} = 2 * \frac{1}{x} * 1 + \frac{1}{3} * \frac{1}{7x-14} * 7 - 4 * \frac{1}{1+x^2} * 2x$$

Take the derivative of both sides and use

$$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$$

$$y * \frac{1}{y}\frac{dy}{dx} = \left(\frac{2}{x} + \frac{7}{3(7x-14)} - \frac{8x}{1+x^2}\right) * y$$

To get $\frac{dy}{dx}$ alone, multiply both sides by y. I also simplified the right side.

$$\frac{dy}{dx} = \left(\frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2}\right) * \frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}$$

Substitute the original function in for y.

Logarithmic Differentiation Steps

1. Take the natural log of both sides (whatever you do to one side, you must do to the other) since base e is easier than any other base (see slide #5).
2. Apply ALL of the possible properties of logarithms.
3. Take the derivative of both sides.
4. Multiply both sides by y.
5. Substitute the original in for y and simplify.

What I will be doing next week.