11.4 Limits at Infinity and Limits of Sequences

What You Should Learn

- Evaluate limits of functions at infinity.
- Find limits of sequences.

Limits at Infinity and Horizontal Asymptotes

Limits at Infinity and Horizontal Asymptotes

There are two basic problems in calculus: finding **tangent lines** and finding the **area** of a region.

We have seen earlier how limits can be used to solve the tangent line problem. In this section, you will see how a different type of limit, a *limit at infinity*, can be used to solve the area problem. To get an idea of what is meant by a limit at infinity, consider the function

$f(x) = (x + 1)/(2x)$.

Limits at Infinity and Horizontal Asymptotes

The graph of *f* is shown in Figure 11.29. From earlier work, you know that $y = \frac{1}{2}$ is a horizontal asymptote of the graph of this function.

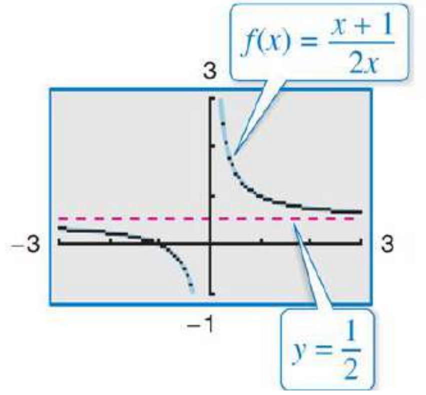

Figure 11.29

Using limit notation, this can be written as follows.

$$\lim_{x \to -\infty} f(x) = \frac{1}{2}$$ Horizontal asymptote to the left

$$\lim_{x \to \infty} f(x) = \frac{1}{2}$$ Horizontal asymptote to the right

Limits at Infinity and Horizontal Asymptotes

These limits mean that the value of $f(x)$ gets arbitrarily close to $\frac{1}{2}$ as x decreases or increases without bound.

Definition of Limits at Infinity

If f is a function and L_1 and L_2 are real numbers, then the statements

$$\lim_{x \to -\infty} f(x) = L_1 \qquad \text{Limit as } x \text{ approaches } -\infty$$

and

$$\lim_{x \to \infty} f(x) = L_2 \qquad \text{Limit as } x \text{ approaches } \infty$$

denote the **limits at infinity.** The first statement is read "*the limit of $f(x)$ as x approaches $-\infty$ is L_1*," and the second is read "*the limit of $f(x)$ as x approaches ∞ is L_2*."

Limits at Infinity and Horizontal Asymptotes

Limits at Infinity

If r is a positive real number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0. \qquad \text{Limit toward the right}$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0. \qquad \text{Limit toward the left}$$

Example 1 – *Evaluating a Limit at Infinity*

Find the limit.

$$\lim_{x\to\infty}\left(4-\frac{3}{x^2}\right)$$

Solution:

Use the properties of limits.

$$\lim_{x\to\infty}\left(4-\frac{3}{x^2}\right)=\lim_{x\to\infty}4-\lim_{x\to\infty}\frac{3}{x^2}$$

$$=\lim_{x\to\infty}4-3\left(\lim_{x\to\infty}\frac{1}{x^2}\right)$$

$$= 4 - 3(0)$$

$$= 4$$

Example 1 – *Solution*

cont'd

So, the limit of

$$f(x) = 4 - \frac{3}{x^2}$$

as *x* approaches ∞ is 4.

Limits at Infinity and Horizontal Asymptotes

Limits at Infinity for Rational Functions

Consider the rational function

$$f(x) = \frac{N(x)}{D(x)}$$

where

$$N(x) = a_n x^n + \cdot \cdot \cdot + a_0 \qquad \text{and} \qquad D(x) = b_m x^m + \cdot \cdot \cdot + b_0.$$

The limit of $f(x)$ as x approaches positive or negative infinity is as follows.

$$\lim_{x \to \pm\infty} f(x) = \begin{cases} 0, & n < m \\ \dfrac{a_n}{b_m}, & n = m \end{cases}$$

If $n > m$, then the limit does not exist.

Limits of Sequences

Limits of Sequences

Limits of sequences have many of the same properties as limits of functions. For instance, consider the sequence whose nth term is $a_n = 1/2^n$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

As n increases without bound, the terms of this sequence get closer and closer to 0, and the sequence is said to **converge** to 0. Using limit notation, you can write

$$\lim_{n \to \infty} \frac{1}{2^n} = 0.$$

Limits of Sequences

The following relationship shows how limits of functions of x can be used to evaluate the limit of a sequence.

Limit of a Sequence

Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x \to \infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that

$$f(n) = a_n$$

for every positive integer n, then

$$\lim_{n \to \infty} a_n = L.$$

Limits of Sequences

A sequence that does not converge is said to **diverge.** For instance, the sequence

$$1, -1, 1, -1, 1, \ldots$$

diverges because it does not approach a unique number.

Example 4 – *Finding the Limit of a Sequence*

Find the limit of each sequence. (Assume *n* begins with 1.)

a. $a_n = \dfrac{2n + 1}{n + 4}$

b. $b_n = \dfrac{2n + 1}{n^2 + 4}$

c. $c_n = \dfrac{2n^2 + 1}{4n^2}$

Solution:

a. $\lim\limits_{n \to \infty} \dfrac{2n + 1}{n + 4} = 2$ $\quad \dfrac{3}{5}, \dfrac{5}{6}, \dfrac{7}{7}, \dfrac{9}{8}, \dfrac{11}{9}, \dfrac{13}{10}, \ldots \to 2$

Example 4 – *Solution*

cont'd

b. $\lim_{n\to\infty} \dfrac{2n+1}{n^2+4} = 0$ $\qquad \dfrac{3}{5}, \dfrac{5}{8}, \dfrac{7}{13}, \dfrac{9}{20}, \dfrac{11}{29}, \dfrac{13}{40}, \ldots \to 0$

c. $\lim_{n\to\infty} \dfrac{2n^2+1}{4n^2} = \dfrac{1}{2}$ $\qquad \dfrac{3}{4}, \dfrac{9}{16}, \dfrac{19}{36}, \dfrac{33}{64}, \dfrac{51}{100}, \dfrac{73}{144}, \ldots \to \dfrac{1}{2}$