11 Limits and an Introduction to Calculus

11.3 The Tangent Line Problem

What You Should Learn

- Understand the tangent line problem.
- Use a tangent line to approximate the slope of a graph at a point.
- Use the limit definition of slope to find exact slopes of graphs.
- Find derivatives of functions and use derivatives to find slopes of graphs.

Tangent Line to a Graph

Tangent Line to a Graph

We learned how the slope of a line indicates the rate at which a line rises or falls. For a line, this rate (or slope) is the same at every point on the line.

For graphs other than lines, the rate at which the graph rises or falls changes from point to point.

Tangent Line to a Graph

For instance, in Figure 11.18, the parabola is rising more quickly at the point (x_1, y_1) than it is at the point (x_2, y_2). At the vertex (x_3, y_3), the graph levels off, and at the point (x_4, y_4) the graph is falling.

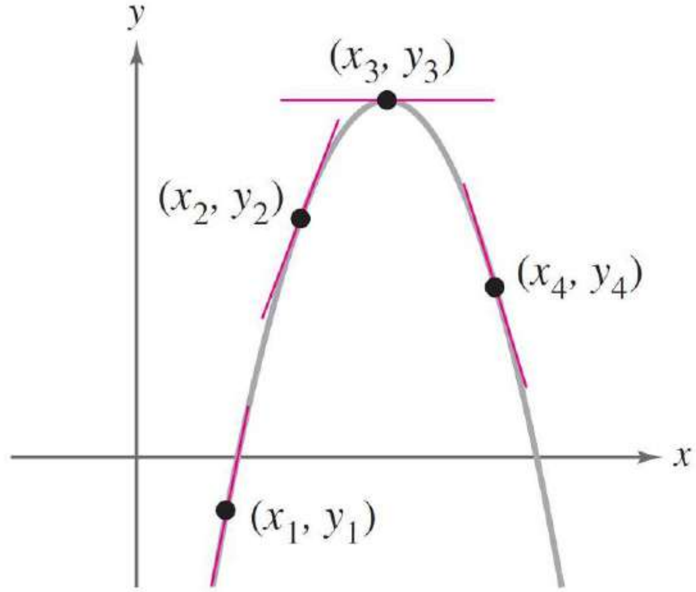


Figure 11.18

Tangent Line to a Graph

To determine the rate at which a graph rises or falls at a *single point*, you can find the slope of the tangent line at that point. In simple terms, the **tangent line** to the graph of a function f at a point $P(x_1, y_1)$ is the line that best approximates the slope of the graph at the point. Figure 11.19 shows other examples of tangent lines.

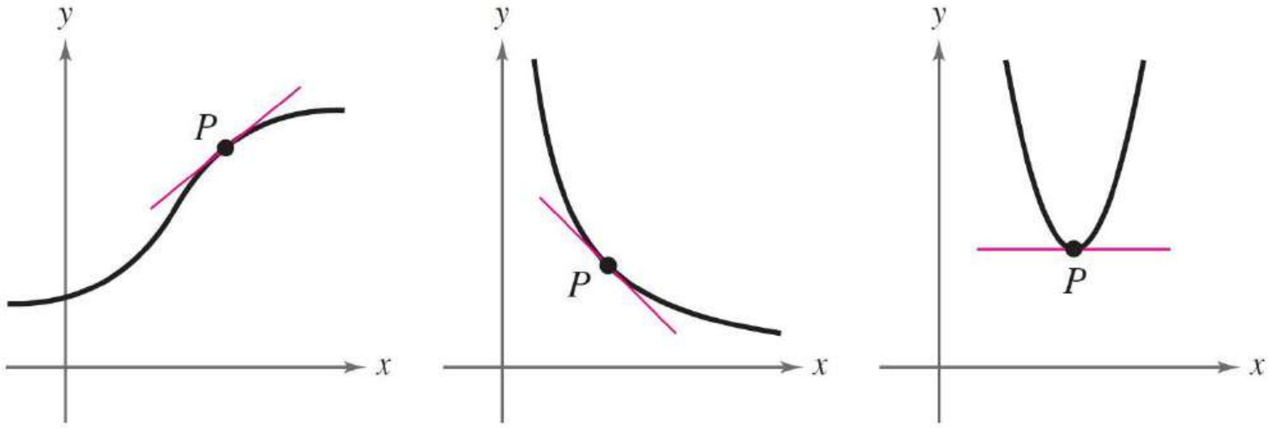


Figure 11.19

Tangent Line to a Graph

From geometry, you know that a line is tangent to a circle when the line intersects the circle at only one point (see Figure 11.20). Tangent lines to noncircular graphs, however, can intersect the graph at more than one point.

For instance, in the first graph in Figure 11.19, if the tangent line were extended, then it would intersect the graph at a point other than the point of tangency.

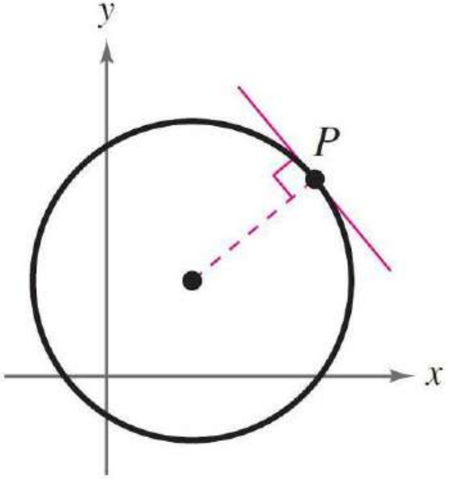


Figure 11.20

Slope of a Graph

Slope of a Graph

Because a tangent line approximates the slope of a graph at a point, the problem of finding the slope of a graph at a point is the same as finding the slope of the tangent line at the point.

Example 1 – *Visually Approximating the Slope of a Graph*

Use the graph in Figure 11.21 to approximate the slope of the graph of $f(x) = x^2$ at the point (1, 1).

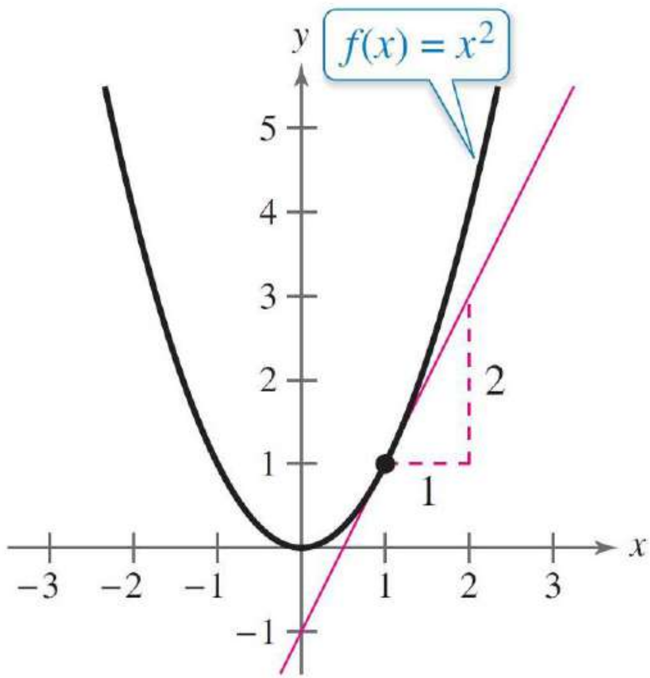


Figure 11.21

Example 1 – *Solution*

From the graph of $f(x) = x^2$, you can see that the tangent line at (1, 1) rises approximately two units for each unit change in x.

So, you can estimate the slope of the tangent line at (1, 1) to be

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x}$$

$$\approx \frac{2}{1}$$

$= 2$

Example 1 – *Solution*

cont'd

Because the tangent line at the point (1, 1) has a slope of about 2, you can conclude that the graph of *f* has a slope of about 2 at the point (1, 1).

Slope and the Limit Process

Slope and the Limit Process

In Example 1, you approximated the slope of a graph at a point by creating a graph and then “eyeballing” the tangent line at the point of tangency.

A more systematic method of approximating tangent lines makes use of a **secant line** through the point of tangency and a second point on the graph, as shown in Figure 11.23.

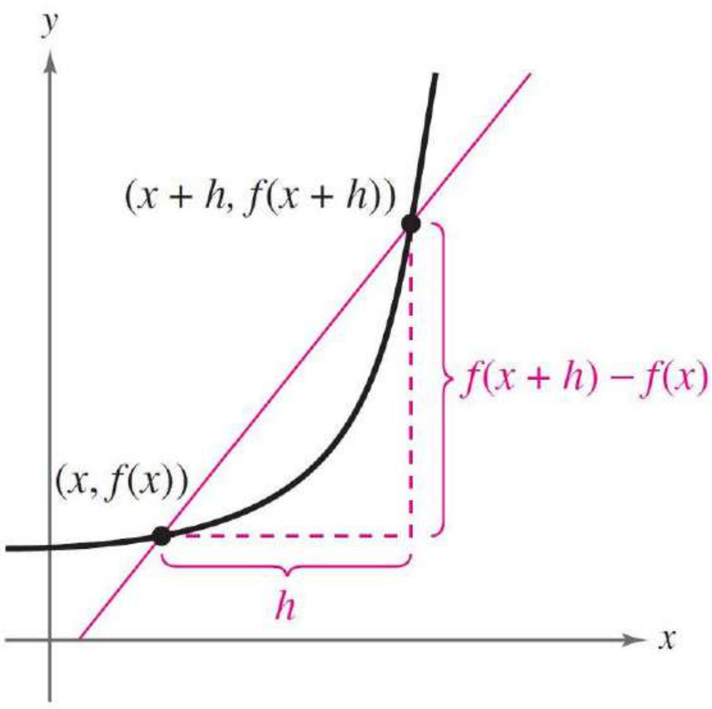


Figure 11.23

Slope and the Limit Process

If (x, $f(x)$) is the point of tangency and ($x + h$, $f(x + h)$) is a second point on the graph of f, then the slope of the secant line through the two points is given by

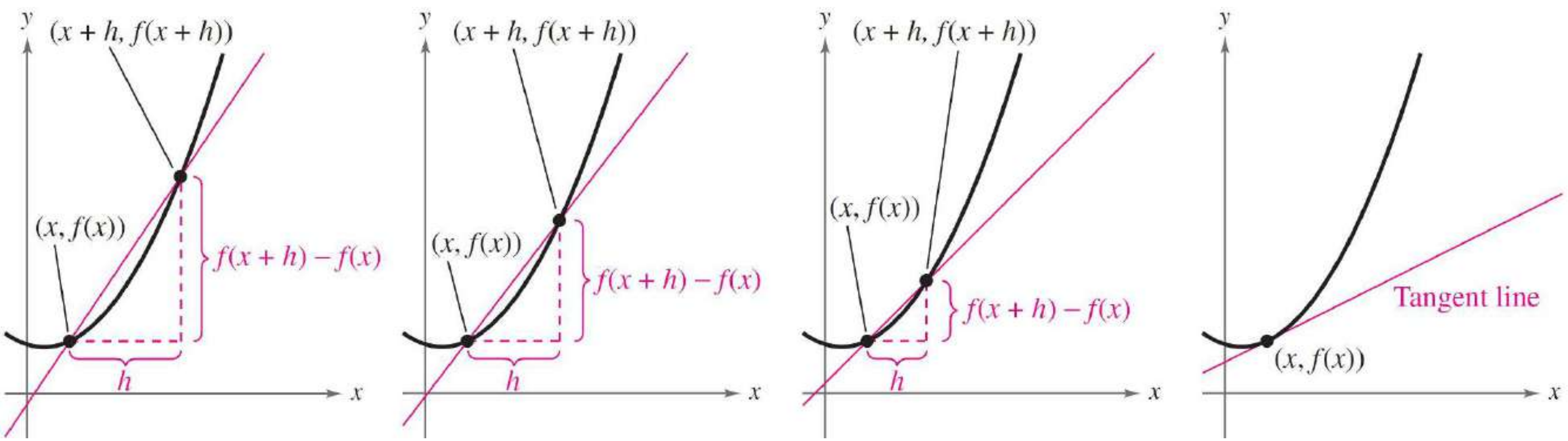
$$m_{\text{sec}} = \frac{f(x+h) - f(x)}{h}.$$

Slope of secant line

The right side of this equation is called the **difference quotient.** The denominator h is the *change in x*, and the numerator is the *change in y*.

Slope and the Limit Process

The beauty of this procedure is that you obtain more and more accurate approximations of the slope of the tangent line by choosing points closer and closer to the point of tangency, as shown in Figure 11.24.



As *h* approaches 0, the secant line approaches the tangent line.

Figure 11.24

Slope and the Limit Process

Definition of the Slope of a Graph

The **slope** m of the graph of f at the point $(x, f(x))$ is equal to the slope of its tangent line at $(x, f(x))$, and is given by

$$m = \lim_{h \to 0} m_{\text{sec}}$$

$$= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

provided this limit exists.

Example 3 – *Finding the Slope of a Graph*

Find the slope of the graph of $f(x) = x^2$ at the point (–2, 4).

Solution:

Find an expression that represents the slope of a secant line at (–2, 4).

$$m_{\text{sec}} = \frac{f(-2+h) - f(-2)}{h}$$

Set up difference quotient.

$$= \frac{(-2+h)^2 - (-2)^2}{h}$$

Substitute into $f(x) = x^2$.

$$= \frac{4 - 4h + h^2 - 4}{h}$$

Expand terms.

Example 3 – *Solution*

cont'd

$$= \frac{-4h + h^2}{h}$$ Simplify.

$$= \frac{\cancel{h}(-4 + h)}{\cancel{h}}$$ Factor and divide out.

$$= -4 + h, \quad h \neq 0$$ Simplify.

Next, take the limit of m_{sec} as approaches 0.

$$m = \lim_{h \to 0} m_{\text{sec}}$$

$$= \lim_{h \to 0} (-4 + h)$$

Example 3 – *Solution*

cont'd

$= -4 + 0$

$= -4$

The graph has a slope of –4 at the point (–2, 4), as shown in Figure 11.25.

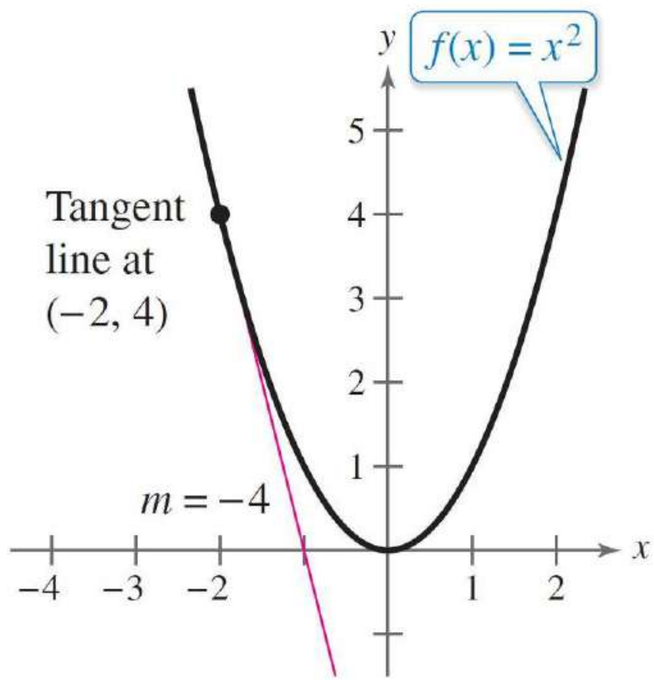


Figure 11.25

The Derivative of a Function

The Derivative of a Function

Let us consider the function $f(x) = x^2 + 1$ and use the limit process to derive another function $m = 2x$, that represents the slope of the graph of f at the point $(x, f(x))$. This derived function is called the **derivative** of f at x. It is denoted by $f'(x)$, which is read as "f prime of x".

Definition of the Derivative

The **derivative** of f at x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

provided this limit exists.

The Derivative of a Function

Remember that the derivative $f'(x)$ is a formula for the slope of the tangent line to the graph of f at the point $(x, f(x))$.

Example 6 – *Finding a Derivative*

Find the derivative of

$$f(x) = 3x^2 - 2x.$$

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[3(x+h)^2 - 2(x+h)] - (3x^2 - 2x)}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 3x^2 + 2x}{h}$$

Example 6 – *Solution*

cont'd

$$= \lim_{h \to 0} \frac{6xh + 3h^2 - 2h}{h}$$

$$= \lim_{h \to 0} \frac{h(6x + 3h - 2)}{h}$$

$$= \lim_{h \to 0} (6x + 3h - 2)$$

$= 6x + 3(0) - 2$

$= 6x - 2$

So, the derivative of $f(x) = 3x^2 - 2x$ is

$f\,(x) = 6x - 2$ Derivative of f at x