Parametric and Polar Curves; Conic Sections
“Parametric Equations; Tangent Lines and Arc Length for Parametric Curves”

Section 10.1

All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Chapter Introduction

- In this chapter, we will study alternative ways of expressing curves in the plane.
  - Curves described in component form (parametric curves)
  - Polar curves
  - Conic sections
- You have seen all of these in previous years, and we will now find tangent lines, arc length, etc. associated with them.

Parametric Equations

- Parametric equations express motion in such a way that the x- and y-coordinates, as functions of time, are:
  - $x = f(t)$
  - $y = f(t)$
- The resulting curve (C) shows the trajectory of the particle described by the equations.

- To graph a parametric equation, you may want to start by eliminating the parameter, or by making a table.

- NOTE: t = time is the parameter for many, but it can be any independent variable that varies over some interval of real numbers.

Example 1

- Sketch the trajectory over the time interval [0,10] of the particle whose parametric equations of motion are:
  - $x = t - 3\sin t$ and
  - $y = 4 - 3\cos t$.

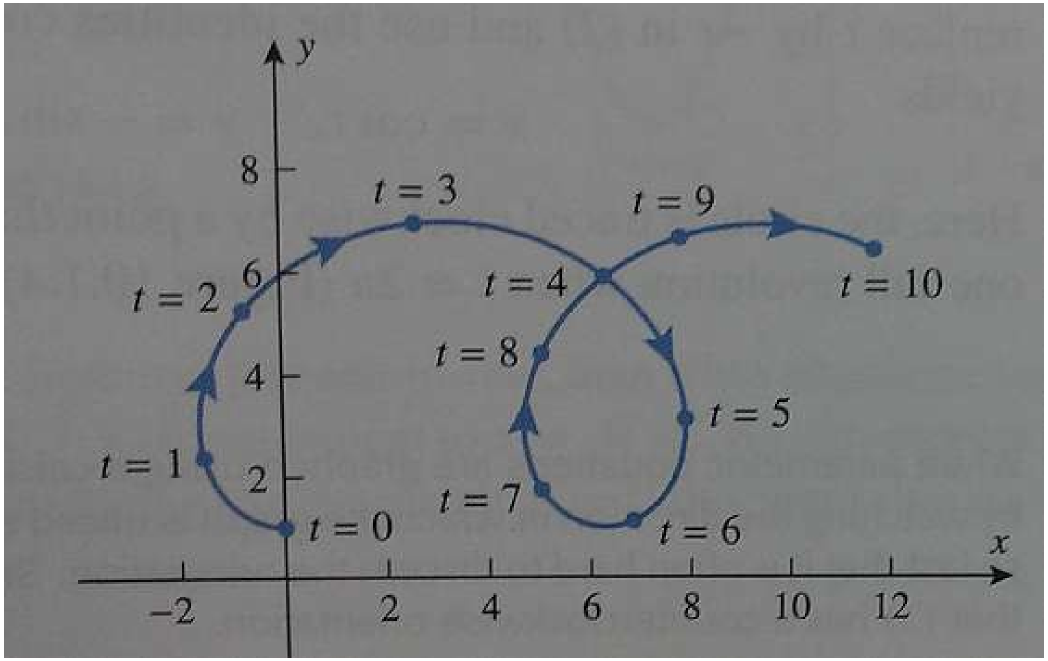
- Solution: Start by making a table and plotting points.
  - $x = 0 - 3*\sin(0) = 0$
  - $y = 4 - 3*\cos(0) = 1$

Table 10.1.1

t	x	y
0	0.0	1.0
1	−1.5	2.4
2	−0.7	5.2
3	2.6	7.0
4	6.3	6.0
5	7.9	3.1
6	6.8	1.1
7	5.0	1.7
8	5.0	4.4
9	7.8	6.7
10	11.6	6.5

Example 1 con't

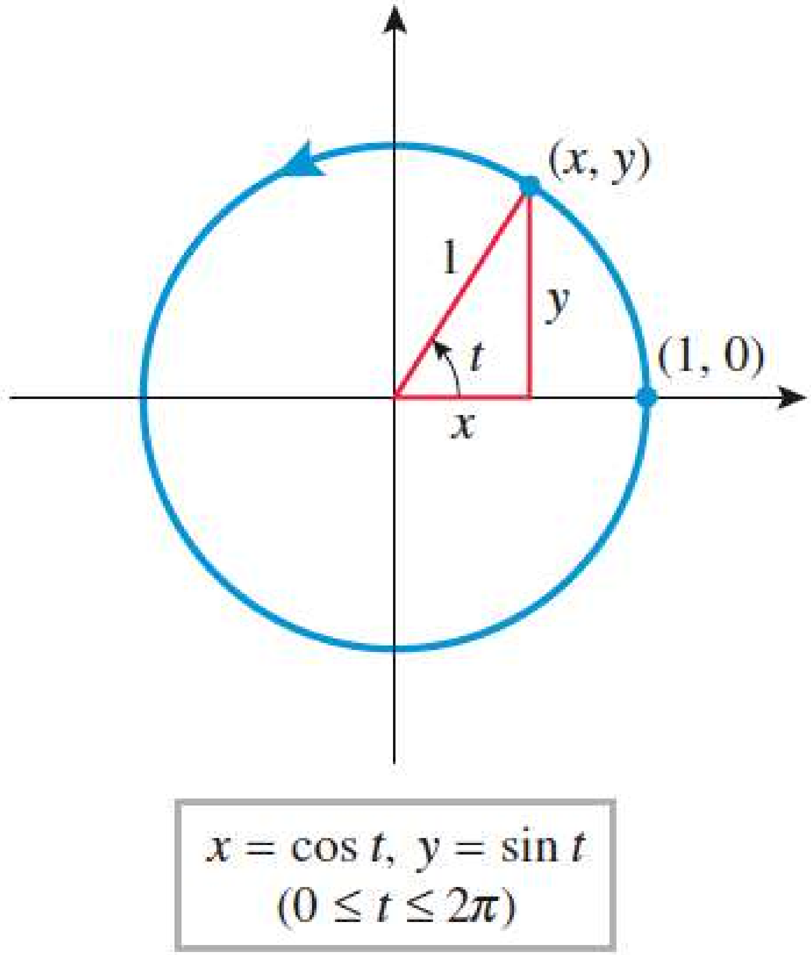
- After completing the table, connect the points with a smooth curve, indicate direction of travel, and label the points with their associated value(s) of t.
- There is no t-axis in the picture.



Example 2

- Find the graph of the parametric equations
  - x = cos t, y = sin t on the interval [o,2].

- Solution: It is often faster to graph by eliminating the parameter, t. For this example, we will use the Pythagorean identity ^{2}t + ^{2}t = 1 and substitution which gives ^{2} + ^{2} = 1 which is a circle.



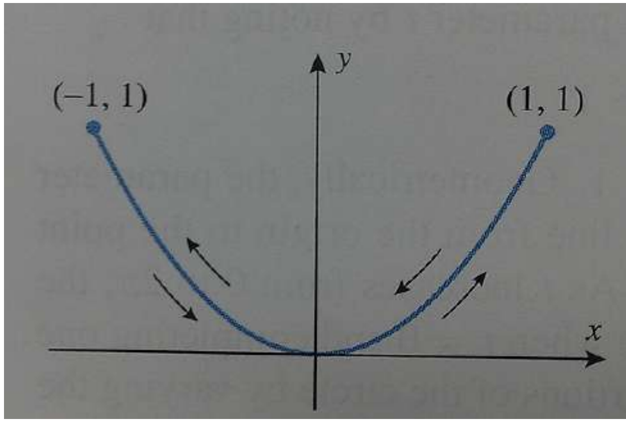
Orientation

- Orientation is the direction in which the graph is traced as the parameter increases.
- Indicating this direction of travel on the graph is the difference between a curve and a parametric curve.
- The orientation in example 2 was counterclockwise (see the arrow).
- To reverse the orientation, we could replace t with –t in example 2.
- This does not work for every parametric

NOTE

- Not all parametric equations produce curves with definite orientations; if the equations are badly behaved, then the point tracing the curve may leap around sporadically or move back and forth, failing to determine a definite direction.
- Example: $x = \sin t$, $y = \sin^2 t$

  This has an orientation that moves periodically back and forth along the parabola.



Tangent Lines to Parametric Curves

- When we have a curve given by parametric equations with continuous first derivatives with respect to t, we can find — using the chain rule.
- — = — without eliminating the parameter.

Example

- Find the slope of the tangent line to the unit circle x = cos t, y = sin t at the point where t = $\frac{\;}{6}$.

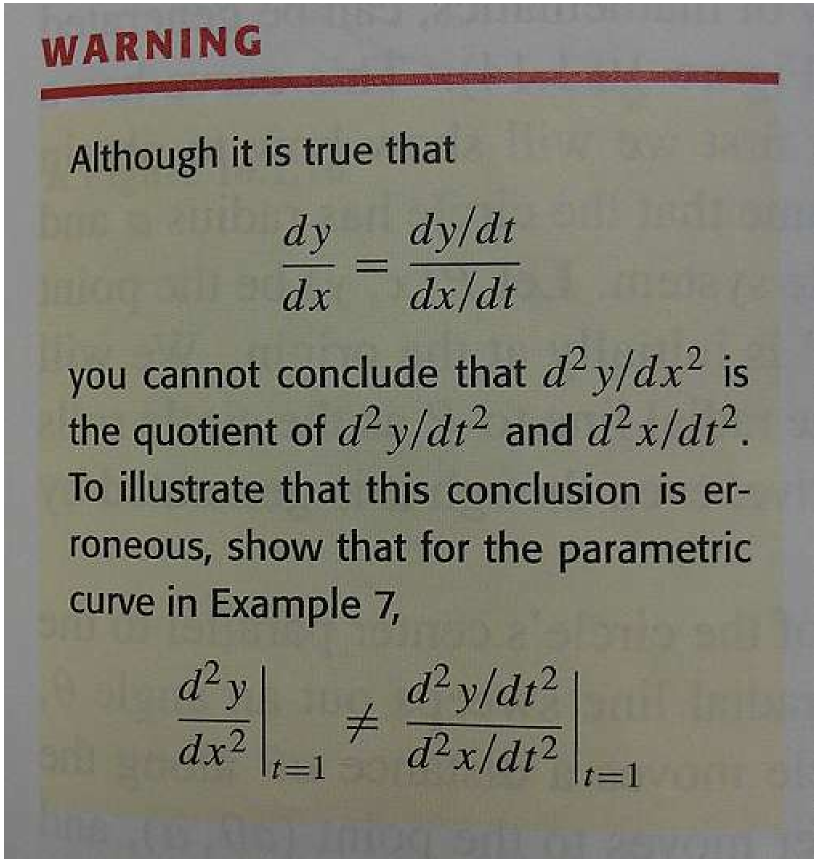
- <u>Solution</u>: $\frac{\;}{\;}$ = cos t and $\frac{\;}{\;}$ = -sin t which gives

us $\frac{\;}{\;} = \frac{\;}{\;} = \frac{\cos}{-\sin} = -\cot$

$\frac{\;}{\;}\Big|_{t=\frac{\;}{6}} = -\cot\frac{\;}{6} = -\sqrt{3}$

Finding Without Eliminating the Parameter

- = =
- I will demonstrate what the warning means in class.



WARNING

Although it is true that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

you cannot conclude that d^2y/dx^2 is the quotient of d^2y/dt^2 and d^2x/dt^2. To illustrate that this conclusion is erroneous, show that for the parametric curve in Example 7,

$$\left.\frac{d^2y}{dx^2}\right|_{t=1} \neq \left.\frac{d^2y/dt^2}{d^2x/dt^2}\right|_{t=1}$$

Example of

- Without eliminating the parameter, find a) —— and

  b) —— at (1,1) on the semicubical parabola (fancy

  name for the curve) when $x = t^2$, $y = t^3$.

- Solution: a) —— = $3\,t^2$ and —— = 2t which gives us

$$—— = \frac{——}{——} = \frac{3\,t^2}{2t} = \frac{3}{2}t$$

Example of con't

b) $\frac{\quad}{\quad}'$ = the derivative of $\frac{\quad}{\quad}$ with respect to t = $\frac{3}{2}$

which gives us $\frac{\quad}{\quad} = \frac{\quad}{\quad}' = \frac{\frac{\quad}{\quad}'}{\frac{\quad}{\quad}} = \frac{\frac{3}{2}}{2} = \frac{3}{4}$

and since(1,1) occurs when t=1,

$3 \qquad 3$

Arc Length of Parametric Curves

- This arc length formula also comes from the distance formula, as we discussed last week.

10.1.1 ARC LENGTH FORMULA FOR PARAMETRIC CURVES If no segment of the curve represented by the parametric equations

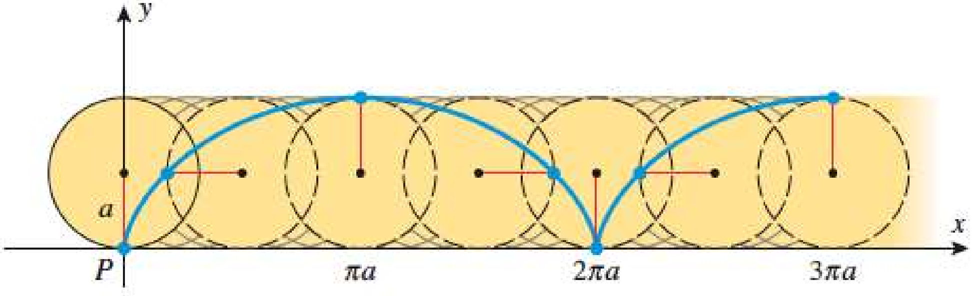
$$x = x(t), \quad y = y(t) \qquad (a \le t \le b)$$

is traced more than once as t increases from a to b, and if dx/dt and dy/dt are continuous functions for $a \le t \le b$, then the arc length L of the curve is given by

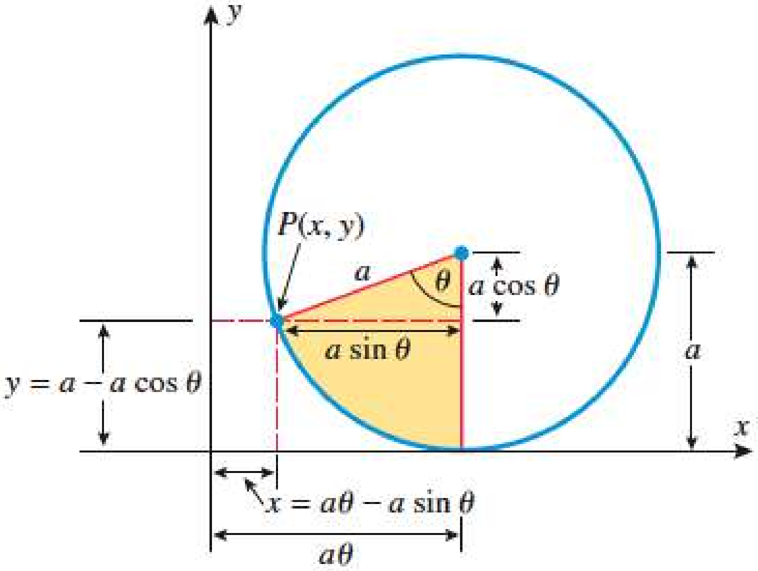
$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (9)$$

The Cycloid (The Apple of Discord)

- There is some reading on pages 698-699 that might interest you regarding application of our work in section 10.1 that relates to the early study of differentiation and integration during the 1600's and:



A cycloid



Oh, the memories… ☺