Section 0.5

"Before Calculus": Exponential and Logarithmic Functions

All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Remember from last year?

- Rewrite logarithms with different bases.
- Use properties of logarithms to evaluate or rewrite logarithmic expressions.
- Use properties of logarithms to expand or condense logarithmic expressions.
- Use logarithmic functions to model and solve real-life problems.

Change of Base

Most calculators have only two types of log keys, one for common logarithms (base 10) and one for natural logarithms (base *e*).

Although common logs and natural logs are the most frequently used, you may occasionally need to evaluate logarithms to other bases. To do this, you can use the following **change-of-base formula.**

Change of Base

Change-of-Base Formula

Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base using any of the following formulas.

Base b	*Base 10*	*Base e*
$\log_a x = \dfrac{\log_b x}{\log_b a}$	$\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$	$\log_a x = \dfrac{\ln x}{\ln a}$

Example 1 – *Changing Bases Using Common Logarithms*

a. $\text{Log}_4 25 = \dfrac{\log_{10} 25}{\log_{10} 4}$

$\approx \dfrac{1.39794}{0.60206}$

≈ 2.23

$\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$

Use a Calculator.

Simplify.

b. $\text{Log}_2 12$

$= \dfrac{\log_{10} 12}{\log_{10} 2} \approx \dfrac{1.07918}{0.30103}$

≈ 3.58

Properties of Logarithms

Properties of Logarithms

Let a be a positive real number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, then the following properties are true.

	Logarithm with Base a	*Natural Logarithm*
1. Product Property:	$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$
2. Quotient Property:	$\log_a \frac{u}{v} = \log_a u - \log_a v$	$\ln \frac{u}{v} = \ln u - \ln v$
3. Power Property:	$\log_a u^n = n \log_a u$	$\ln u^n = n \ln u$

Example 3 – *Using Properties of Logarithms*

Write each logarithm in terms of ln 2 and ln 3.

a. $\ln 6$ **b.** $\ln \frac{2}{27}$

Solution:

a. $\ln 6 = \ln(2 \cdot 3)$

Rewrite 6 as 2 • 3.

$= \ln 2 + \ln 3$

b. $\ln \frac{2}{27} = \ln 2 - \ln 27$

Product Property

$= \ln 2 - \ln 3^3$

Quotient Property

$= \ln 2 - 3 \ln 3$

Rewrite 27 as 3^3

Power Property

Rewriting Logarithmic Expressions

The properties of logarithms are useful for rewriting logarithmic expressions in forms that simplify the operations of algebra. This is true because they convert complicated products, quotients, and exponential forms into simpler sums, differences, and products, respectively.

Example 5 – *Expanding Logarithmic Expressions*

Use the properties of logarithms to expand each expression.

a. $\log_4 5x^3y$

b. $\ln \frac{\sqrt{3x-5}}{7}$

Solution

a. $\log_4 5x^3y = \log_4 5 + \log_4 x^3 + \log_4 y$ Product Property

$= \log_4 5 + 3\log_4 x + \log_4 y$ Power Property

Example 5 – *Solution*

b. $\ln \frac{\sqrt{3x - 5}}{7} = \ln \frac{(3x - 5)^{1/2}}{7}$ — Rewrite radical using rational exponent.

$= \ln(3x - 5)^{1/2} - \ln 7$ — Quotient Property

$= \frac{1}{2} \ln(3x - 5) - \ln 7$ — Power Property

Rewriting Logarithmic Expressions

In Example 5, the properties of logarithms were used to *expand* logarithmic expressions.

In Example 6, this procedure is reversed and the properties of logarithms are used to *condense* logarithmic expressions.

Example 6 – *Condensing Logarithmic Expressions*

Use the properties of logarithms to condense each expression.

a. $\frac{1}{2} \log_{10} x + 3 \log_{10}(x + 1)$

b. $2\ln(x + 2) - \ln x$

c. $\frac{1}{3} [\log_2 x + \log_2(x - 4)]$

Example 6 – *Solution*

a. $\frac{1}{2}\log_{10}x + 3\log_{10}(x+1)$

$=\log_{10}x^{1/2} + \log_{10}(x+1)^3$ Power Property

$= \log_{10}\left[\sqrt{x}(x+1)^3\right]$ Product Property

b. $2\ln(x+2) - \ln x$ Power Property

$= \ln(x+2)^2 - \ln x$ Quotient Property

$= \ln\frac{(x+2)^2}{x}$

Example 6 – *Solution*

c. $\frac{1}{3}\,[\log_2 x + \log_2(x - 4)]$

$= \frac{1}{3}\log_2[x\,(x - 4)]$ Product Property

$= \log_2[x(x - 4)]^{1/3}$ Power Property

$= \log_2 \sqrt[3]{x(x - 4)}$ Rewrite with a radical.

Also from last year…

- Solve simple exponential and logarithmic equations.
- Solve more complicated exponential equations.
- Solve more complicated logarithmic equations.
- Use exponential and logarithmic equations to model and solve real-life problems.

Helpful Properties

- *One-to-One Properties*
- $a^x = a^y$ if and only if $x = y$.
- $\log_a x = \log_a y$ if and only if $x = y$.
- *Inverse Properties*
- $a^{\log_a x} = x$
- $\log_a a^x = x$

Example 1 – *Solving Simple Exponential and Logarithmic Exponential*

Original Equation	*Rewritten Equation*	*Solution*	*Property*
a. $2^x = 32$	$2^x = 2^5$	$x = 5$	One-to-One
b. $\log_4 x - \log_4 8 = 0$	$\log_4 x = \log_4 8$	$x = 8$	One-to-One
c. $\ln x - \ln 3 = 0$	$\ln x = \ln 3$	$x = 3$	One-to-One
d. $\left(\frac{1}{3}\right)^x = 9$	$3^{-x} = 3^2$	$x = -2$	One-to-One
e. $e^x = 7$	$\ln e^x = \ln 7$	$x = \ln 7$	Inverse
f. $\ln x = -3$	$e^{\ln x} = e^{-3}$	$x = e^{-3}$	Inverse
g. $\log_{10} x = -1$	$10^{\log_{10} x} = 10^{-1}$	$x = 10^{-1} = \frac{1}{10}$	Inverse
h. $\log_3 x = 4$	$3^{\log_3 x} = 3^4$	$x = 81$	Inverse

Strategies:

Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One Properties of exponential or logarithmic functions.
2. Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.
3. Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.

Solving Logarithmic Equations

- To solve a logarithmic equation, you can write it in exponential form. Many of you liked the "swoop" method.

- $$\ln x = 3$$

  $$x = e^3$$

- This procedure is applied after the logarithmic expression has been isolated.

Example 6 – *Solving Logarithmic Equations*

- Solve each logarithmic equation.
- **a.** $\ln 3x = 2$

- **b.** $\log_3(5x - 1) = \log_3(x + 7)$

Example 6 – *Solution*

Solution:

a.

$$\ln 3x = 2$$

$$3x = e^2$$

$$x = \frac{1}{3}e^2$$

The solution is

$$x \approx 2.46$$

- Check this in the original equation.

Example 6 – *Solution*

- **b.** $\log_3(5x - 1) = \log_3(x + 7)$ Write original equation.
- $5x - 1 = x + 7$ One-to-One Property
- $x = 2$ Solve for *x*
- The solution $x = 2$. Check this in the original equation.

Solving Logarithmic Equations

- Because the domain of a logarithmic function generally does not include all real numbers, you should be sure to check for extraneous solutions of logarithmic equations.

pH Value

- In case you forget to read directions sometimes, like I do ☺, the formula for pH value is on pg 62.

$$pH = -\log [H+]$$

(where [H+] denotes the concentrations of hydrogen ions measured in moles per liter.)