Precalculus Chapter

of the

Mathematics Framework

for California Public Schools:
Kindergarten Through Grade Twelve

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Precalculus

Precalculus

Precalculus combines concepts of trigonometry, geometry, and algebra that are needed to prepare students for the study of calculus. The course strengthens students' conceptual understanding of problems and mathematical reasoning in solving problems. Facility with these topics is especially important for students who intend to study calculus, physics, other sciences, and engineering in college. The main topics in the Precalculus course are complex numbers, rational functions, trigonometric functions and their inverses, inverse functions, vectors and matrices, and parametric and polar curves. Because the standards that comprise this course are mostly (+) standards, students who enroll in Precalculus should have met the college- and career-ready standards of the previous courses in the Integrated Pathway or Traditional Pathway. It is recommended that students complete Precalculus before taking an Advanced Placement calculus course.

What Students Learn in Precalculus

Students in Precalculus extend their work with complex numbers, which started in Mathematics III or Algebra II, to see that complex numbers can be represented in the Cartesian plane and that operations with complex numbers have a geometric interpretation. They connect their understanding of trigonometry and the geometry of the plane to express complex numbers in polar form.

Students begin working with vectors, representing them geometrically and performing operations with them. They connect the notion of vectors to complex numbers. Students also work with matrices and their operations, experiencing for the first time an algebraic system in which multiplication is not commutative. Additionally, they see the connection between matrices and transformations of the plane—namely, that a vector in the plane can be multiplied by a 2×2 matrix to produce another vector—and they work with matrices from the perspective of transformations. They also find inverse matrices and use matrices to represent and solve linear systems.

Students extend their work with trigonometric functions, investigating the reciprocal functions *secant*, *cosecant*, and *cotangent* and the graphs and properties associated with those functions. Students find inverse trigonometric functions by appropriately restricting the domains of the standard trigonometric functions and use them to solve problems that arise in modeling contexts.

Although students in Precalculus have worked previously with parabolas and circles, they now work with ellipses and hyperbolas. They also work with polar coordinates and curves defined parametrically and connect these to their other work with trigonometry and complex numbers.

Finally, students work with rational functions that are more complicated, graphing them and determining zeros, y-intercepts, symmetry, asymptotes, intervals for which the function is increasing or decreasing, and maximum or minimum points.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (**MP**) apply throughout each course and, together with the Standards for Mathematical Content, prescribe that students experience mathematics as a coherent, useful, and logical subject. The Standards for Mathematical Practice represent a picture of what it looks like for students to *do mathematics* and, to the extent possible, content instruction should include attention to appropriate practice standards. Table P-1 presents examples of how students can engage with each MP standard in Precalculus.

Table P-1. Standards for Mathematical Practice—Explanation and Examples for Precalculus

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	Students expand their repertoire of expressions and functions that can be used to solve problems. They grapple with understanding the connection between complex numbers, polar coordinates, and vectors.
MP.2 Reason abstractly and quantitatively.	Students understand the connection between transformations and matrices, seeing a matrix as an algebraic representation of a transformation of the plane.
MP.3 Construct viable arguments and critique the reasoning of others. **Students build proofs by induction and proofs by contradiction. CA 3.1** (for higher mathematics only).	Students continue to reason through the solution of an equation and justify their reasoning to their peers. They defend their choice of a function to model a real-world situation.
MP.4 Model with mathematics.	Students apply their new mathematical understanding to real-world problems. They also discover mathematics through experimentation and by examining patterns in data from real-world contexts.
MP.5 Use appropriate tools strategically.	Students continue to use graphing technology to deepen their understanding of the behavior of polynomial, rational, square root, and trigonometric functions.
MP.6 Attend to precision.	Students make note of the precise definition of *complex number*, understanding that real numbers are a subset of complex numbers. They pay attention to units in real-world problems and use unit analysis as a method for verifying answers.
MP.7 Look for and make use of structure.	Students understand that matrices form an algebraic system in which the order of multiplication matters, especially when solving linear systems using matrices. They see that complex numbers can be represented by polar coordinates and that the structure of the plane yields a geometric interpretation of complex multiplication.
MP.8 Look for and express regularity in repeated reasoning.	Students multiply several vectors by matrices and observe that some matrices produce rotations or reflections. They compute with complex numbers and generalize the results to understand the geometric nature of their operations.

Standard **MP.4** holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a prominent place in instruction. Some standards are marked with a star symbol (★) to indicate that they are *modeling standards*—that is, they may be applied to real-world modeling situations more so than other standards. Note that this does not preclude other standards from being taught with or through mathematical modeling.

In places where specific MP standards may be implemented with the Precalculus standards, the MP standards are noted in parentheses.

Precalculus Content Standards, by Conceptual Category

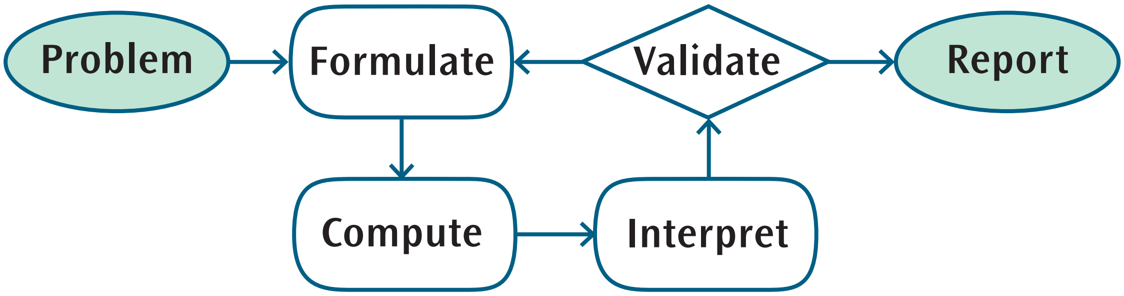
The Precalculus course is organized by conceptual category, domains, clusters, and then standards. The overall purpose and progression of the standards included in Precalculus are described below, according to each conceptual category. Note that the standards are *not* listed in an order in which they should be taught.

Conceptual Category: Modeling

Throughout the California Common Core State Standards for Mathematics (CA CCSSM), specific standards for higher mathematics are marked with a ★ symbol to indicate they are modeling standards. Modeling at the higher mathematics level goes beyond the simple application of previously constructed mathematics and includes real-world problems. True modeling begins with students asking a question about the world around them, and the mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise (e.g., Which of the quantities present in this situation are known and which are unknown?). Students need to decide on a solution path that may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They try to use previously derived models (e.g., linear functions), but may find that a new equation or function will apply. Additionally, students may see when trying to answer their question that solving an equation arises as a necessity and that the equation often involves the specific instance of knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, and the like), compute an answer or rewrite their expression to reveal new information, interpret and validate the results, and report out; see figure P-1. This is a new approach for many teachers and may be challenging to implement, but the effort should show students that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

Figure P-1. The Modeling Cycle



Readers are encouraged to consult appendix B (Mathematical Modeling) for further discussion of the modeling cycle and how it is integrated into the higher mathematics curriculum.

Conceptual Category: Functions

The standards of the Functions conceptual category can set the stage for students to learn other standards in Precalculus. At this level, expressions are often viewed as defining outputs for functions, and algebraic manipulations are then performed meaningfully with an eye toward what can be revealed about the function.

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context.

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★

Analyze functions using different representations.

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★

    d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior. ★

    e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★

10. **(+) Demonstrate an understanding of functions and equations defined parametrically and graph them. CA** ★

11. **(+) Graph polar coordinates and curves. Convert between polar and rectangular coordinate systems. CA**

Building Functions F-BF

Build new functions from existing functions.

3. Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

4. Find inverse functions.

   b. (+) Verify by composition that one function is the inverse of another.

   c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

   d. (+) Produce an invertible function from a non-invertible function by restricting the domain.

Although many of the standards in the Interpreting Functions and Building Functions domains appeared in previous courses, students now apply them in cases of polynomial functions of degree greater than two, more complicated rational functions, reciprocal trigonometric functions, and inverse trigonometric functions. Students examine end behavior of polynomial and rational functions and learn how to find asymptotes.

Students further their understanding of inverse functions. Previously, students found inverse functions only in simple cases (e.g., solving for x, when $f(x)=c$, finding inverses of linear functions); in Precalculus they explore the relationship between two functions that are inverses of each other (i.e., that f and g are inverses if $(f\circ g)(x)=x$ and $(g\circ f)(x)=x$). They may also begin to use inverse function notation, expressing g as $g=f^{-1}$ (**MP.2, MP.6**). Students in Precalculus construct inverse functions by appropriately restricting the domain of a given function and use inverses in different contexts. They understand how a function and its domain and range are related to its inverse function. They realize that finding an inverse function is more than simply "switching variables" and solving an equation. They can even find simpler inverses mentally, such as when they reverse the "steps" for the equation $f(x)=x^3-1$ to realize that the inverse of f must be $f^{-1}(x)=\sqrt[3]{x+1}$ (**MP.7**).

Students in Precalculus study parametric functions, understanding that a curve in the plane that might describe the path of a moving object can be represented with such functions. Students also work with polar coordinates and graph polar curves. Connections should be made between polar coordinates and the polar representation of complex numbers (**N-CN.4–5**). Students also discover the important role that trigonometric functions play in working with polar coordinates. These standards are new for a typical Precalculus curriculum. Students can investigate these new concepts in modeling situations, such as by recording points on the curve along which a tossed ball travels, graphing the points as vectors and deriving equations for $x(t)$ and $y(t)$ (**MP.4**). They can also investigate the relationship between the graphs of the sine and cosine as functions of θ, as well as the graph of the curve defined by $x(\theta)=\cos\theta$, $y(\theta)=\sin\theta$, drawing connections between these graphs.

Trigonometric Functions F-TF

Extend the domain of trigonometric functions using the unit circle.

4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Model periodic phenomena with trigonometric functions.

6. (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.
7. (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context. ★

Prove and apply trigonometric identities.

9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.
10. **(+) Prove the half angle and double angle identities for sine and cosine and use them to solve problems. CA**

These standards call for students to expand their understanding of the trigonometric functions by connecting properties of the functions to the unit circle. For example, students understand that since traveling 2π radians around the unit circle returns one to the same point on the circle, this must be reflected in the graphs of sine and cosine (**MP.8**). Students extend their knowledge of finding inverses to trigonometric functions and use these inverses in a wide range of application problems.

Students in Precalculus derive the addition and subtraction formulas for sine, cosine, and tangent, as well as the half angle and double angle identities for sine and cosine, and make connections among these. For example, students can derive from the addition formula for cosine $(\cos(x+y) = \cos x \cos y - \sin x \sin y)$ the double angle formula for cosine:

$$\cos 2x = \cos(x+x) = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x$$

Another opportunity for connections arises here, as students can investigate the relationship between these formulas and complex multiplication.

Conceptual Category: Number and Quantity

The Number and Quantity standards in Precalculus represent a culmination of students' understanding of number systems. Students investigate the geometry of complex numbers more fully and connect it to operations with complex numbers. Additionally, students develop the notion of a vector and connect operations with vectors and matrices to transformations of the plane.

The Complex Number System — N-CN

Perform arithmetic operations with complex numbers.

3. (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

Represent complex numbers and their operations on the complex plane.

4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.
5. (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example,* $\left(-1+\sqrt{3}i\right)^3 = 8$ *because* $\left(-1+\sqrt{3}i\right)$ *has modulus 2 and argument 120°.*
6. (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

As mentioned previously, complex numbers, polar coordinates, and vectors should be taught with an emphasis on connections between them. For instance, students connect the addition of complex numbers to the addition of vectors; they also investigate the geometric interpretation of multiplying complex numbers and connect that interpretation to polar coordinates by using the polar representation.

Vector and Matrix Quantities — N-VM

Represent and model with vector quantities.

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\boldsymbol{v}$, $|\boldsymbol{v}|$, $||\boldsymbol{v}||$, v).
2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.
3. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

Perform operations on vectors.

4. (+) Add and subtract vectors.
   a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
   b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
   c. Understand vector subtraction $\boldsymbol{v}-\boldsymbol{w}$ as $\boldsymbol{v}+(-\boldsymbol{w})$, where $-\boldsymbol{w}$ is the additive inverse of $\boldsymbol{w}$, with the same magnitude as $\boldsymbol{w}$ and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.
5. (+) Multiply a vector by a scalar.
   a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c\left(v_x, v_y\right) = \left(cv_x, cv_y\right)$.
   b. Compute the magnitude of a scalar multiple $c\boldsymbol{v}$ using $||c\boldsymbol{v}|| = |c|\boldsymbol{v}$. Compute the direction of $c\boldsymbol{v}$ knowing that when $|c|\boldsymbol{v} \neq 0$, the direction of $c\boldsymbol{v}$ is either along $\boldsymbol{v}$ (for $c > 0$) or against $\boldsymbol{v}$ (for $c < 0$).

Vector and Matrix Quantities *(continued)* N-VM

Perform operations on matrices and use matrices in applications.

6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
7. (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
8. (+) Add, subtract, and multiply matrices of appropriate dimensions.
9. (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.
10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is non-zero if and only if the matrix has a multiplicative inverse.
11. (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.
12. (+) Work with 2×2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Students investigate vectors as geometric objects in the plane that can be represented by ordered pairs, and they investigate matrices as objects that act on vectors. By working with vectors and matrices both geometrically and quantitatively, students discover that vector addition and subtraction behave according to certain properties, while matrices and matrix operations observe their own set of rules. Attending to structure, students discover with matrices a new set of mathematical objects and operations involving multiplication that is not commutative. They find inverse matrices by hand in 2×2 cases and use technology in other cases. Work with vectors and matrices sets the stage for solving systems of equations in the Algebra conceptual category.

Conceptual Category: Algebra

In the Algebra conceptual category, Precalculus students work with higher-degree polynomials and rational functions that are more complicated. As always, they attend to the meaning of the expressions they work with, and the expressions they encounter often arise in the context of functions. As in all other higher mathematics courses, students in Precalculus work with creating and solving equations and do so in contexts connected to real-world situations through modeling.

Seeing Structure in Expressions A-SSE

Interpret the structure of expressions.

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.* ★
2. Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

Arithmetic with Polynomials and Rational Expressions — A-APR

Rewrite rational expressions.

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x)+r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a non-zero rational expression; add, subtract, multiply, and divide rational expressions.

By the time students take Precalculus, they should have a well-developed understanding of the concept of a function. To make work with rational expressions more meaningful, students should be given opportunities to connect rational expressions to rational *functions* (whose outputs are defined by the expressions). For example, a traditional exercise with rational expressions might have the following form:

$$\text{Simplify } \frac{200}{x}+\frac{100}{x-10}$$

The intention here is that students will find a common denominator and transform the expression into $\frac{300x-2000}{x(x-10)}$. In contrast, students could view the two expressions as defining the outputs of two functions—f and g, respectively—where $f(x)=\frac{200}{x}$ and $g(x)=\frac{100}{x-10}$ (**MP.2**). In this case, f could be the function that represents the time it takes for a car to travel 200 miles at an average speed of x miles per hour, and g could be the function that represents the time it takes for the car to travel 100 miles at an average speed that is 10 miles per hour slower (**MP.4**). Students can be asked to consider the domains of the two functions, the domain on which the sum of the two functions defined by $(f+g)(x)=f(x)+g(x)$ makes sense, and what the sum denotes (total time to travel the 300 miles altogether). Furthermore, students can calculate tables of outputs for the two functions using a spreadsheet, add the outputs on the spreadsheet, and then graph the resulting outputs, discovering that the data fit the graph of the equation $y=\frac{300x-2000}{x(x-10)}$. Finally, if these expressions arise in a modeling context, students can interpret the results of studying these functions and their sum in the real-world context.

Creating Equations A-CED

Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA** ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.* ★

Reasoning with Equations and Inequalities A-REI

Solve systems of equations.

8. (+) Represent a system of linear equations as a single matrix equation in a vector variable.
9. (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3×3 or greater).

Standards **A-CED.1–4** appear in most other higher mathematics courses, as they represent skills that are commonly employed when students work with equations. Students in Precalculus expand these skills into several areas: trigonometric functions, by setting up and solving equations such as $\sin 2\theta = \frac{1}{2}$; parametric functions, by making sense of the equations $x = 2t$; $y = 3t + 1$, and $0 \le t \le 10$; and rational expressions, by sketching a rough graph of equations such as $y = \frac{300x - 2000}{x(x-10)}$ (**MP.7, MP.8**).

Students use matrix multiplication to connect their newfound knowledge of matrices to the representation of systems of linear equations. They can do this in modeling situations (e.g., those involving economic quantities or geometric elements).

Conceptual Category: Geometry

The standards of the Geometry conceptual category also connect to several other standards in the Precalculus curriculum. For example, students continue to work with conic sections (started in Mathematics III or Algebra II), and they view conic sections as examples of parametric functions (**F-IF.10**).

Similarity, Right Triangles, and Trigonometry G-SRT

Apply trigonometry to general triangles.

9. (+) Derive the formula $A = \frac{1}{2}ab\ \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Expressing Geometric Properties with Equations G-GPE

Translate between the geometric description and the equation for a conic section.

3. (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

3.1 Given a quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$, use the method for completing the square to put the equation into standard form; identify whether the graph of the equation is a circle, ellipse, parabola, or hyperbola and graph the equation. CA

Students in Precalculus continue to study trigonometric functions by discovering that these functions can also be used with general (non-right) triangles through the use of appropriate auxiliary lines. The Laws of Sines and Cosines can be derived once these auxiliary lines are introduced into general triangles. Students can then use these laws to solve problems, and they connect the relationships described by the laws to the geometry of vectors.

Precalculus Overview

Number and Quantity

The Complex Number System

- Perform arithmetic operations with complex numbers.
- Represent complex numbers and their operations on the complex plane.

Vector and Matrix Quantities

- Represent and model with vector quantities.
- Perform operations on vectors.
- Perform operations on matrices and use matrices in applications.

Algebra

Seeing Structure in Expressions

- Interpret the structure of expressions.

Arithmetic with Polynomials and Rational Expressions

- Rewrite rational expressions.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Solve systems of equations.

Functions

Interpreting Functions

- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build new functions from existing functions.

Trigonometric Functions

- Extend the domain of trigonometric functions using the unit circle.
- Model periodic phenomena with trigonometric functions.
- Prove and apply trigonometric identities.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Precalculus Overview *(continued)*

Geometry

Similarity, Right Triangles, and Trigonometry

- Apply trigonometry to general triangles.

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.

Number and Quantity

The Complex Number System — N-CN

Perform arithmetic operations with complex numbers.

3. (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

Represent complex numbers and their operations on the complex plane.

4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.
5. (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example, $\left(-1+\sqrt{3}i\right)^3 = 8$ because $\left(-1+\sqrt{3}i\right)$ has modulus 2 and argument 120°.*
6. (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

Vector and Matrix Quantities — N-VM

Represent and model with vector quantities.

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\boldsymbol{v}$, $|\boldsymbol{v}|$, $||\boldsymbol{v}||$, v).
2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.
3. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

Perform operations on vectors.

4. (+) Add and subtract vectors.
   a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
   b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
   c. Understand vector subtraction $\boldsymbol{v}-\boldsymbol{w}$ as $\boldsymbol{v}+(-\boldsymbol{w})$, where $-\boldsymbol{w}$ is the additive inverse of $\boldsymbol{w}$, with the same magnitude as $\boldsymbol{w}$ and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.
5. (+) Multiply a vector by a scalar.
   a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c\left(v_x, v_y\right) = \left(cv_x, cv_y\right)$.
   b. Compute the magnitude of a scalar multiple $c\boldsymbol{v}$ using $||c\boldsymbol{v}|| = |c|\boldsymbol{v}$. Compute the direction of $c\boldsymbol{v}$ knowing that when $|c|\boldsymbol{v} \neq 0$, the direction of $c\boldsymbol{v}$ is either along $\boldsymbol{v}$ (for $c > 0$) or against $\boldsymbol{v}$ (for $c < 0$).

P Precalculus

Perform operations on matrices and use matrices in applications.

6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
7. (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
8. (+) Add, subtract, and multiply matrices of appropriate dimensions.
9. (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.
10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is non-zero if and only if the matrix has a multiplicative inverse.
11. (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.
12. (+) Work with 2×2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Algebra

Seeing Structure in Expressions — A-SSE

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   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.* ★
2. Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

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6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a non-zero rational expression; add, subtract, multiply, and divide rational expressions.

Creating Equations A-CED

Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA** ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.* ★

Reasoning with Equations and Inequalities A-REI

Solve systems of equations.

8. (+) Represent a system of linear equations as a single matrix equation in a vector variable.
9. (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3×3 or greater).

Functions

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context.

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★

Analyze functions using different representations.

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   - d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior. ★
   - e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★

10. (+) Demonstrate an understanding of functions and equations defined parametrically and graph them. CA ★

11. (+) Graph polar coordinates and curves. Convert between polar and rectangular coordinate systems. CA

Building Functions F-BF

Build new functions from existing functions.

3. Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

4. Find inverse functions.

   b. (+) Verify by composition that one function is the inverse of another.

   c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

   d. (+) Produce an invertible function from a non-invertible function by restricting the domain.

Trigonometric Functions F-TF

Extend the domain of trigonometric functions using the unit circle.

4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Model periodic phenomena with trigonometric functions.

6. (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

7. (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context. ★

Prove and apply trigonometric identities.

9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

10. (+) Prove the half angle and double angle identities for sine and cosine and use them to solve problems. CA

Geometry

Similarity, Right Triangles, and Trigonometry G-SRT

Apply trigonometry to general triangles.

9. (+) Derive the formula $A = \frac{1}{2}ab\sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.

11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Expressing Geometric Properties with Equations G-GPE

Translate between the geometric description and the equation for a conic section.

3. (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

3.1 Given a quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$, use the method for completing the square to put the equation into standard form; identify whether the graph of the equation is a circle, ellipse, parabola, or hyperbola and graph the equation. CA