1 Functions and Their Graphs

1.2 Functions

What You Should Learn

- Decide whether a relation between two variables represents a function
- Use function notation and evaluate functions
- Find the domains of functions
- Use functions to model and solve real-life problems
- Evaluate difference quotients

Introduction to Functions

When there is some relation that matches each item from one set with exactly one item from a different set. Such a relation is called a **function.**

Definition of a Function

A **function** f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the **domain** (or set of inputs) of the function f, and the set B contains the **range** (or set of outputs).

Introduction to Functions

To help understand this definition, look at the function that relates the time of day to the temperature in Figure 1.12.

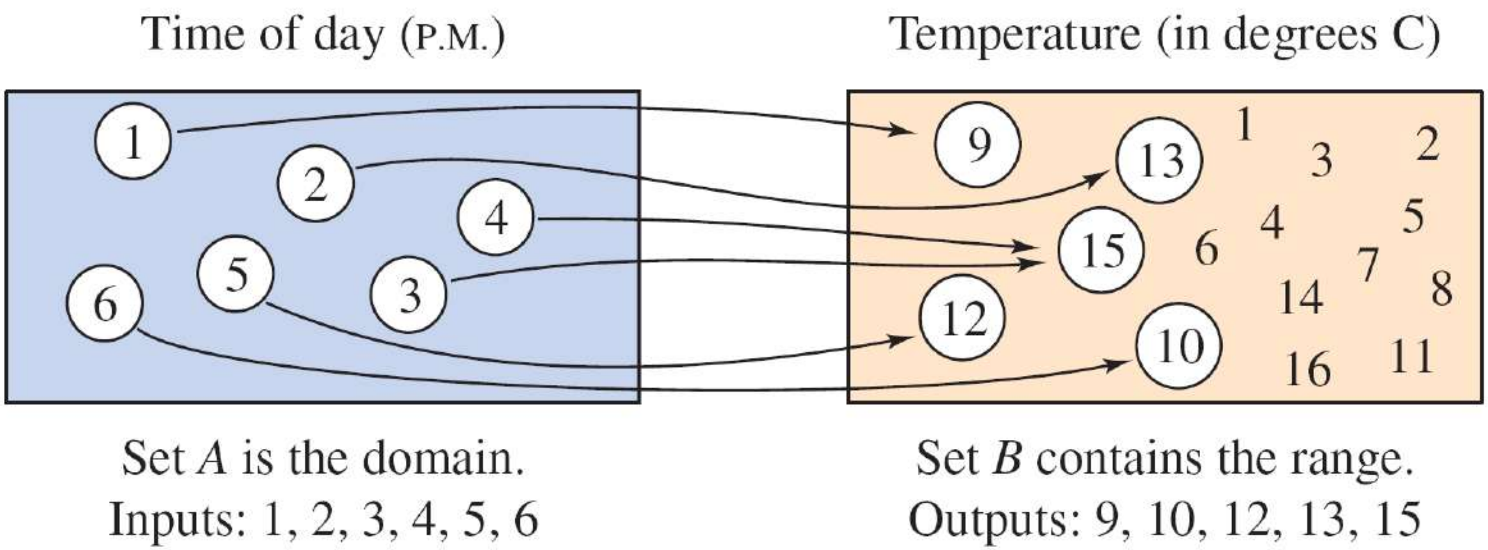


Figure 1.12

Introduction to Functions

This function can be represented by the ordered pairs

$$\{(1, 9°), (2, 13°), (3, 15°), (4, 15°), (5, 12°), (6, 10°)\}.$$

In each ordered pair, the first coordinate (x-value) is the **input** and the second coordinate (y-value) is the **output.**

Characteristics of a Function from Set A to Set B

1. Each element of A must be matched with an element of B.
2. Some elements of B may not be matched with any element of A.
3. Two or more elements of A may be matched with the same element of B.
4. An element of A (the domain) cannot be matched with two different elements of B.

Introduction to Functions

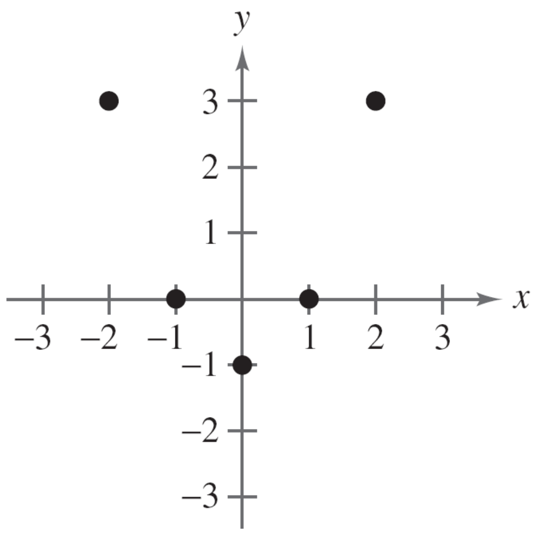
To determine whether or not a relation is a function, you must decide whether each input value is matched with exactly one output value. When any input value is matched with two or more output values, the relation is not a function.

Example 1 – *Testing for Functions*

Decide whether the relation represents *y* as a function of *x*.

Input, x	2	2	3	4	5
Output, y	11	10	8	5	1

(a)



(b)

Figure 1.13

Example 1 – *Solution*

a. This table *does not* describe y as a function of x. The input value 2 is matched with two different y-values.

b. The graph in Figure 1.13 *does* describe y as a function of x. Each input value is matched with exactly one output value.

Introduction to Functions

In algebra, it is common to represent functions by equations or formulas involving two variables. For instance, the equation $y = x^2$ represents the variable y as a function of x. In this equation, x is the **independent variable** and y is the **dependent variable.**

The domain of the function is the set of all values taken on by the independent variable x, and the range of the function is the set of all values taken on by the dependent variable y.

Function Notation

The symbol $f(x)$ is read as the *value of f at x* or simply *f of x.* The symbol $f(x)$ corresponds to the y-value for a given x. So, you can write $y = f(x)$.

Keep in mind that f is the *name* of the function, whereas $f(x)$ is the *output value* of the function at the *input value x.*

In function notation, the *input* is the independent variable and the *output* is the dependent variable. For instance, the function $f(x) = 3 – 2x$ has *function values* denoted by $f(–1)$, $f(0)$, and so on. To find these values, substitute the specified input values into the given equation.

Function Notation

For $x = -1$, $f(-1) = 3 - 2(-1) = 3 + 2 = 5$.

For $x = 0$, $f(0) = 3 - 2(0) = 3 - 0 = 3$.

Although f is often used as a convenient function name and x is often used as the independent variable, you can use other letters. For instance,

$$f(x) = x^2 - 4x + 7,$$

$$f(t) = t^2 - 4t + 7$$

and

$$g(s) = s^2 - 4s + 7$$

all define the same function.

Function Notation

In fact, the role of the independent variable is that of a "placeholder."

Consequently, the function could be written as

$f(\quad) = (\quad)^2 - 4(\quad) + 7.$

Example 3 – *Evaluating a Function*

Let $g(x) = -x^2 + 4x + 1$. Find each value of the function.

a. $g(2)$

b. $g(t)$

c. $g(x + 2)$

Solution:

a. Replacing x with 2 in $g(x) = -x^2 + 4x + 1$ yields the following.

$$g(2) = -(2)^2 + 4(2) + 1$$

$$= -4 + 8 + 1$$

$$= 5$$

Example 3 – *Solution*

cont'd

b. Replacing x with t yields the following.

$$g(t) = -(t)^2 + 4(t) + 1$$

$$= -t^2 + 4t + 1$$

c. Replacing x with $x + 2$ yields the following.

$$g(x + 2) = -(x + 2)^2 + 4(x + 2) + 1$$ Substitute $x + 2$ for x.

$$= -(x^2 + 4x + 4) + 4x + 8 + 1$$ Multiply.

$$= -x^2 - 4x - 4 + 4x + 8 + 1$$ Distributive Property

$$= -x^2 + 5$$ Simplify.

The Domain of a Function

The Domain of a Function

The domain of a function can be described explicitly or it can be *implied* by the expression used to define the function. The **implied domain** is the set of all real numbers for which the expression is defined. For instance, the function

$$f(x) = \frac{1}{x^2 - 4}$$

Domain excludes *x*-values that result in division by zero.

has an implied domain that consists of all real numbers *x* other than *x* = ±2. These two values are excluded from the domain because division by zero is undefined. Another common type of implied domain is that used to avoid even roots of negative numbers.

The Domain of a Function

For example, the function

$$f(x) = \sqrt{x}$$

Domain excludes *x*-values that results in even roots of negative numbers.

is defined only for $x \geq 0$. So, its implied domain is the interval $[0, \infty)$. In general, the domain of a function *excludes* values that would cause division by zero *or* result in the even root of a negative number.

Example 5 – *Finding the Domain of a Function*

Find the domain of each function.

a. f: {(–3, 0), (–1, 4), (0, 2), (2, 2), (4, –1)}

b. $g(x) = -3x^2 + 4x + 5$

c. $h(x) = \dfrac{1}{x + 5}$

Solution:

a. The domain of f consists of all first coordinates in the set of ordered pairs.

Domain = {–3, –1, 0, 2, 4}

Example 5 – *Solution*

cont'd

b. The domain of g is the set of all *real* numbers.

c. Excluding x-values that yield zero in the denominator, the domain of h is the set of all real numbers x except $x = -5$.

Difference Quotients

Difference Quotients

One of the basic definitions in calculus employs the ratio

$$\frac{f(x+h)-f(x)}{h}, \quad h \neq 0.$$

This ratio is called a **difference quotient** .

Example 10 – *Evaluating a Difference Quotient*

For $f(x) = x^2 - 4x + 7$, find $\dfrac{f(x+h) - f(x)}{h}$.

Solution:

$$\frac{f(x+h) - f(x)}{h} = \frac{[(x+h)^2 - 4(x+h) + 7] - (x^2 - 4x + 7)}{h}$$

$$= \frac{x^2 + 2xh + h^2 - 4x - 4h + 7 - x^2 + 4x - 7}{h}$$

$$= \frac{2xh + h^2 - 4h}{h}$$

$$= \frac{h(2x + h - 4)}{h}$$

$$= 2x + h - 4, \quad h \neq 0$$