Preface

Here are the solutions to the practice problems for my Calculus II notes. Some solutions will have more or less detail than other solutions. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you've reached the level of working the harder problems then you will probably already understand the basics fairly well and won't need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven't been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

Tangents with Parametric Equations

1. Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the following set of parametric equations.

$$x = 4t^3 - t^2 + 7t \qquad y = t^4 - 6$$

Step 1
The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 12t^2 - 2t + 7 \qquad \frac{dy}{dt} = 4t^3$$

Step 2
The first derivative is then,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \boxed{\frac{4t^3}{12t^2 - 2t + 7}}$$

Step 3
For the second derivative we'll now need,

$$\frac{d}{dt}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{4t^3}{12t^2-2t+7}\right)=\frac{\left(12t^2\right)\left(12t^2-2t+7\right)-4t^3\left(24t-2\right)}{\left(12t^2-2t+7\right)^2}=\boxed{\frac{48t^4-16t^3+84t^2}{\left(12t^2-2t+7\right)^2}}$$

Step 4
The second derivative is then,

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}=\frac{\frac{48t^4-16t^3+84t^2}{\left(12t^2-2t+7\right)^2}}{12t^2-2t+7}=\boxed{\frac{48t^4-16t^3+84t^2}{\left(12t^2-2t+7\right)^3}}$$

---

2. Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the following set of parametric equations.

$$x=\mathbf{e}^{-7t}+2 \qquad y=6\mathbf{e}^{2t}+\mathbf{e}^{-3t}-4t$$

Step 1
The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt}=-7\mathbf{e}^{-7t} \qquad \frac{dy}{dt}=12\mathbf{e}^{2t}-3\mathbf{e}^{-3t}-4$$

Step 2
The first derivative is then,

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{12\mathbf{e}^{2t}-3\mathbf{e}^{-3t}-4}{-7\mathbf{e}^{-7t}}=\boxed{-\tfrac{12}{7}\mathbf{e}^{9t}+\tfrac{3}{7}\mathbf{e}^{4t}+\tfrac{4}{7}\mathbf{e}^{7t}}$$

Step 3
For the second derivative we'll now need,

$$\frac{d}{dt}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(-\tfrac{12}{7}\mathbf{e}^{9t}+\tfrac{3}{7}\mathbf{e}^{4t}+\tfrac{4}{7}\mathbf{e}^{7t}\right)=\boxed{-\tfrac{108}{7}\mathbf{e}^{9t}+\tfrac{12}{7}\mathbf{e}^{4t}+4\mathbf{e}^{7t}}$$

Step 4
The second derivative is then,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-\frac{108}{7}\mathbf{e}^{9t} + \frac{12}{7}\mathbf{e}^{4t} + 4\mathbf{e}^{7t}}{-7\mathbf{e}^{-7t}} = \boxed{\frac{108}{49}\mathbf{e}^{16t} - \frac{12}{49}\mathbf{e}^{11t} - \frac{4}{7}\mathbf{e}^{14t}}$$

---

3. Find the equation of the tangent line(s) to the following set of parametric equations at the given point.

$$x = 2\cos(3t) - 4\sin(3t) \qquad y = 3\tan(6t) \quad \text{at} \quad t = \frac{\pi}{2}$$

Step 1
We'll need the first derivative for the set of parametric equations. We'll need the following derivatives,

$$\frac{dx}{dt} = -6\sin(3t) - 12\cos(3t) \qquad\qquad \frac{dy}{dt} = 18\sec^2(6t)$$

The first derivative is then,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{18\sec^2(6t)}{-6\sin(3t) - 12\cos(3t)} = \frac{3\sec^2(6t)}{-\sin(3t) - 2\cos(3t)}$$

Step 2
The slope of the tangent line at $t = \frac{\pi}{2}$ is then,

$$m = \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \frac{3(-1)^2}{-(-1) - 2(0)} = 3$$

At $t = \frac{\pi}{2}$ the parametric curve is at the point,

$$\left.x\right|_{t=\frac{\pi}{2}} = 2(0) - 4(-1) = 4 \qquad \left.y\right|_{t=\frac{\pi}{2}} = 3(0) = 0 \qquad\qquad \Rightarrow \qquad\qquad (4,0)$$

Step 3
The (only) tangent line for this problem is then,

$$y = 0 + 3(x - 4) \qquad \rightarrow \qquad \boxed{y = 3x - 12}$$

---

4. Find the equation of the tangent line(s) to the following set of parametric equations at the given point.

$$x = t^2 - 2t - 11 \qquad y = t\left(t-4\right)^3 - 3t^2\left(t-4\right)^2 + 7 \quad \text{at} \quad \left(-3,7\right)$$

Step 1
We'll need the first derivative for the set of parametric equations. We'll need the following derivatives,

$$\frac{dx}{dt} = 2t - 2$$
$$\frac{dy}{dt} = \left(t-4\right)^3 + 3t\left(t-4\right)^2 - 6t\left(t-4\right)^2 - 6t^2\left(t-4\right) = \left(t-4\right)^3 - 3t\left(t-4\right)^2 - 6t^2\left(t-4\right)$$

The first derivative is then,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\left(t-4\right)^3 - 3t\left(t-4\right)^2 - 6t^2\left(t-4\right)}{2t-2}$$

Hint : Don't forget that because the derivative we found above is in terms of t we need to determine the value(s) of t that put the parametric curve at the given point.
Step 2
Okay, the derivative we found above is in terms of t and we we'll need to next determine the value(s) of t that put the parametric curve at $\left(-3,7\right)$.

This is easy enough to do by setting the x and y coordinates equal to the known parametric equations and determining the value(s) of t that satisfy both equations.

Doing that gives,

$$\begin{aligned} -3 &= t^2 - 2t - 11 \\ 0 &= t^2 - 2t - 8 \\ 0 &= \left(t-4\right)\left(t+2\right) \qquad \rightarrow \qquad t = -2, t = 4 \end{aligned}$$

$$\begin{aligned} 7 &= t\left(t-4\right)^3 - 3t^2\left(t-4\right)^2 + 7 \\ 0 &= \left(t-4\right)^2\left[t\left(t-4\right) - 3t^2\right] \\ 0 &= \left(t-4\right)^2\left[-4t - 2t^2\right] \\ 0 &= -2t\left(t-4\right)^2\left[2+t\right] \qquad \rightarrow \qquad t = -2, t = 0, t = 4 \end{aligned}$$

We can see from this list that the parametric curve will be at $\left(-3,7\right)$ for $t = -2$ and $t = 4$.

Step 3
From the previous step we can see that we will in fact have two tangent lines at the point. Here are the slopes for each tangent line.

The slope of the tangent line at $t=-2$ is,

$$m=\left.\frac{dy}{dx}\right|_{t=-2}=-24$$

and the slope of the tangent line at $t=4$ is,

$$m=\left.\frac{dy}{dx}\right|_{t=4}=0$$

Step 4
The tangent line for $t=-2$ is then,

$$y=7-24(x+3) \qquad \rightarrow \qquad \boxed{y=-24x-65}$$

The tangent line for $t=4$ is then,

$$y=7-(0)(x+3) \qquad \rightarrow \qquad \boxed{y=7}$$

Do not get excited about the second tangent line! It is just saying that the second tangent line is a horizontal line.

---

5. Find the values of t that will have horizontal or vertical tangent lines for the following set of parametric equations.

$$x=t^5-7t^4-3t^3 \qquad y=2\cos(3t)+4t$$

Step 1
We'll need the following derivatives for this problem.

$$\frac{dx}{dt}=5t^4-28t^3-9t^2 \qquad\qquad \frac{dy}{dt}=-6\sin(3t)+4$$

Step 2
We know that horizontal tangent lines will occur where $\frac{dy}{dt}=0$, provided $\frac{dx}{dt}\neq 0$ at the same value of t.

So, to find the horizontal tangent lines we'll need to solve,

$$-6\sin(3t)+4=0 \qquad \rightarrow \qquad \sin(3t)=\tfrac{2}{3} \qquad \rightarrow \qquad 3t=\sin^{-1}\left(\tfrac{2}{3}\right)=0.7297$$

Also, a quick glance at a unit circle we can see that a second angle is,

$$3t = \pi - 0.7297 = 2.4119$$

All possible values of t that will give horizontal tangent lines are then,

$$\begin{array}{l} 3t = 0.7297 + 2\pi n \\ 3t = 2.4119 + 2\pi n \end{array} \quad \rightarrow \quad \begin{array}{l} t = 0.2432 + \frac{2}{3}\pi n \\ t = 0.8040 + \frac{2}{3}\pi n \end{array}, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Note that we don't officially know these do in fact give horizontal tangent lines until we also determine that $\frac{dx}{dt} \neq 0$ at these points. We'll be able to determine that after the next step.

Step 3

We know that vertical tangent lines will occur where $\frac{dx}{dt} = 0$, provided $\frac{dy}{dt} \neq 0$ at the same value of t.

So, to find the vertical tangent lines we'll need to solve,

$$5t^4 - 28t^3 - 9t^2 = 0$$

$$t^2\left(5t^2 - 28t - 9\right) = 0 \quad \rightarrow \quad t = 0,\ t = \frac{28 \pm \sqrt{964}}{10} \quad \rightarrow \quad t = 0,\ t = -0.3048,\ t = 5.9048$$

Step 4

From a quick inspection of the two lists of t values from Step 2 and Step 3 we can see there are no values in common between the two lists. Therefore, any values of t that gives $\frac{dy}{dt} = 0$ will not give $\frac{dx}{dt} = 0$ and visa-versa.

Therefore the values of t that gives horizontal tangent lines are,

$$\boxed{\begin{array}{l} t = 0.2432 + \frac{2}{3}\pi n \\ t = 0.8040 + \frac{2}{3}\pi n \end{array}, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots}$$

The values of t that gives vertical tangent lines are,

$$\boxed{t = 0,\ t = -0.3048,\ t = 5.9048}$$

Preface

Here are the solutions to the practice problems for my Calculus II notes. Some solutions will have more or less detail than other solutions. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you've reached the level of working the harder problems then you will probably already understand the basics fairly well and won't need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven't been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

Arc Length with Parametric Equations

1. Determine the length of the parametric curve given by the following set of parametric equations. You may assume that the curve traces out exactly once for the given range of t's.

$$x = 8t^{\frac{3}{2}} \qquad y = 3 + \left(8 - t\right)^{\frac{3}{2}} \qquad 0 \le t \le 4$$

Step 1
The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 12t^{\frac{1}{2}} \qquad \frac{dy}{dt} = -\tfrac{3}{2}\left(8 - t\right)^{\frac{1}{2}}$$

Step 2
We'll need the ds for this problem.

$$ds = \sqrt{\left[12t^{\frac{1}{2}}\right]^2 + \left[-\tfrac{3}{2}\left(8-t\right)^{\frac{1}{2}}\right]^2}\, dt = \sqrt{144t + \tfrac{9}{4}\left(8-t\right)}\, dt = \sqrt{\tfrac{567}{4}t + 18}\, dt$$

Step 3
The integral for the arc length is then,

$$L = \int ds = \int_0^4 \sqrt{\tfrac{567}{4}t + 18}\, dt$$

Step 4
This is a simple integral to compute with a quick substitution. Here is the integral work,

$$L=\int_0^4\sqrt{\tfrac{567}{4}t+18}\,dt=\tfrac{4}{567}\left(\tfrac{2}{3}\right)\left(\tfrac{567}{4}t+18\right)^{\frac{3}{2}}\Big|_0^4=\boxed{\tfrac{8}{1701}\left(585^{\frac{3}{2}}-18^{\frac{3}{2}}\right)=66.1865}$$

---

2. Determine the length of the parametric curve given by the following set of parametric equations. You may assume that the curve traces out exactly once for the given range of t's.

$$x=3t+1 \qquad y=4-t^2 \qquad -2\le t\le 0$$

Step 1
The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt}=3 \qquad\qquad \frac{dy}{dt}=-2t$$

Step 2
We'll need the ds for this problem.

$$ds=\sqrt{[3]^2+[-2t]^2}\,dt=\sqrt{9+4t^2}\,dt$$

Step 3
The integral for the arc length is then,

$$L=\int ds=\int_{-2}^{0}\sqrt{9+4t^2}\,dt$$

Step 4
This integral will require a trig substitution (as will quite a few arc length integrals!).

Here is the trig substitution we'll need for this integral.

$$t=\tfrac{3}{2}\tan\theta \qquad dt=\tfrac{3}{2}\sec^2\theta\,d\theta$$
$$\sqrt{9+4t^2}=\sqrt{9+9\tan^2\theta}=3\sqrt{1+\tan^2\theta}=3\sqrt{\sec^2\theta}=3\left|\sec\theta\right|$$

To get rid of the absolute value on the secant will need to convert the limits into θ limits.

$$t=-2: \qquad -2=\tfrac{3}{2}\tan\theta \quad\rightarrow\quad \tan\theta=-\tfrac{4}{3} \quad\rightarrow\quad \theta=\tan^{-1}\left(-\tfrac{4}{3}\right)=-0.9273$$
$$t=0: \qquad 0=\tfrac{3}{2}\tan\theta \quad\rightarrow\quad \tan\theta=0 \quad\rightarrow\quad \theta=0$$

Okay, the corresponding range of θ for this problem is $-0.9273\le\theta\le 0$ (fourth quadrant) and in this range we know that secant is positive. Therefore the root becomes,

$$\sqrt{9+4t^2}=3\sec\theta$$

The integral is then,

$$L = \int_{-2}^{0} \sqrt{9+4t^2}\,dt = \int_{-0.9273}^{0} \left(3\sec\theta\right)\left(\tfrac{3}{2}\sec^2\theta\right)d\theta$$
$$= \int_{-0.9273}^{0} \tfrac{9}{2}\sec^3\theta\,d\theta = \tfrac{9}{4}\left[\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\right]\Big|_{-0.9273}^{0} = \boxed{7.4719}$$

---

3. A particle travels along a path defined by the following set of parametric equations. Determine the total distance the particle travels and compare this to the length of the parametric curve itself.

$$x = 4\sin\left(\tfrac{1}{4}t\right) \qquad y = 1 - 2\cos^2\left(\tfrac{1}{4}t\right) \qquad\qquad -52\pi \le t \le 34\pi$$

Hint : Be very careful with this problem. Note the two quantities we are being asked to find, how they relate to each other and which of the two that we know how to compute from the material in this section.

Step 1
This is a problem that many students have issues with. First note that we are being asked to find both the total distance traveled by the particle AND the length of the curve. Also, recall that of these two quantities we only discussed how to determine the length of a curve in this section.

Therefore, let's concentrate on finding the length of the curve first, then we'll worry about the total distance traveled.

Step 2
To find the length we'll need the following two derivatives,

$$\frac{dx}{dt} = \cos\left(\tfrac{1}{4}t\right) \qquad\qquad \frac{dy}{dt} = \cos\left(\tfrac{1}{4}t\right)\sin\left(\tfrac{1}{4}t\right)$$

The *ds* for this problem is then,

$$ds = \sqrt{\left[\cos\left(\tfrac{1}{4}t\right)\right]^2 + \left[\cos\left(\tfrac{1}{4}t\right)\sin\left(\tfrac{1}{4}t\right)\right]^2}\,dt = \sqrt{\cos^2\left(\tfrac{1}{4}t\right) + \cos^2\left(\tfrac{1}{4}t\right)\sin^2\left(\tfrac{1}{4}t\right)}\,dt$$

Now, this is where many students run into issues with this problem. Many students use the following integral to determine the length of the curve.

$$\int_{-52\pi}^{34\pi} \sqrt{\cos^2\left(\tfrac{1}{4}t\right) + \cos^2\left(\tfrac{1}{4}t\right)\sin^2\left(\tfrac{1}{4}t\right)}\,dt$$

Can you see what is wrong with this integral?

Step 3
Remember from the discussion in this section that in order to use the arc length formula the curve can only trace out exactly once over the range of the limits in the integral.