12 Limits and an Introduction to Calculus

12.1 Introduction to Limits

Objectives

- Use the definition of limit to estimate limits.
- Determine whether limits of functions exist.
- Use properties of limits and direct substitution to evaluate limits.

The Limit Concept

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The notion of a limit is a *fundamental* concept of calculus.

In this chapter, you will learn how to evaluate limits and how to use them in the two basic problems of calculus: the tangent line problem and the area problem.

Example 1 – *Finding a Rectangle of Maximum Area*

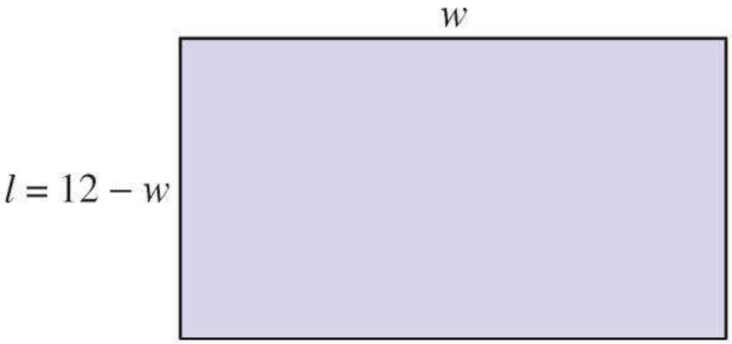
Find the dimensions of a rectangle that has a perimeter of 24 inches and a maximum area.

Solution:

Let w represent the width of the rectangle and let l represent the length of the rectangle. Because

$2w + 2l = 24$ Perimeter is 24.

it follows that $l = 12 – w$, as shown in the figure.



Example 1 – *Solution*

cont'd

So, the area of the rectangle is

$A = lw$ Formula for area

$= (12 - w)w$ Substitute $12 - w$ for l.

$= 12w - w^2$. Simplify.

Using this model for area, experiment with different values of w to see how to obtain the maximum area.

Example 1 – *Solution*

cont'd

After trying several values, it appears that the maximum area occurs when $w = 6$, as shown in the table.

Width, w	5.0	5.5	5.9	6.0	6.1	6.5	7.0
Area, A	35.00	35.75	35.99	36.00	35.99	35.75	35.00

In limit terminology, you can say that "the limit of A as w approaches 6 is 36." This is written as

$$\lim_{w \to 6} A = \lim_{w \to 6} (12w - w^2)$$
$$= 36.$$

Definition of Limit

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If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L. This is written as

$$\lim_{x \to c} f(x) = L.$$

Example 2 – *Estimating a Limit Numerically*

Use a table to estimate numerically the limit: $\lim_{x \to 2} (3x - 2)$

Solution:

Let $f(x) = 3x - 2$.

Then construct a table that shows values of $f(x)$ for two sets of x-values—one set that approaches 2 from the left and one that approaches 2 from the right.

x	1.9	1.99	1.999	2.0	2.001	2.01	2.1
$f(x)$	3.700	3.970	3.997	?	4.003	4.030	4.300

Example 2 – *Solution*

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From the table, it appears that the closer x gets to 2, the closer $f(x)$ gets to 4. So, you can estimate the limit to be 4. Figure 12.1 illustrates this conclusion.

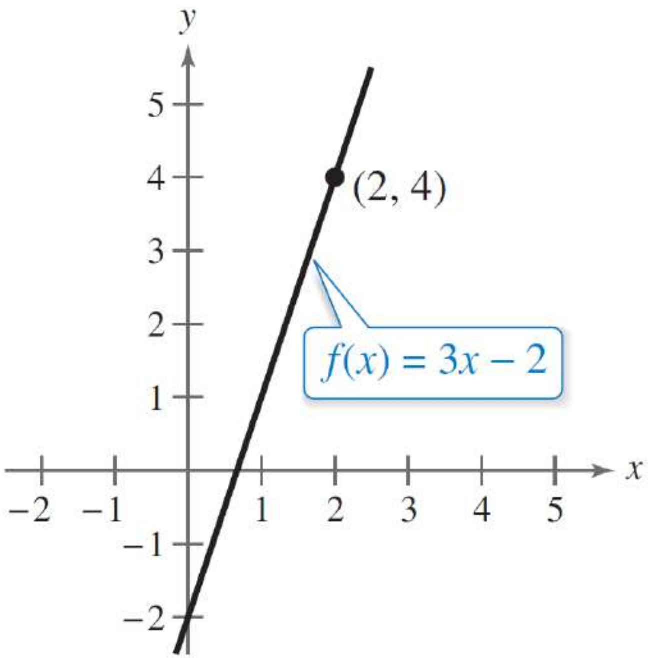

Figure 12.1

Limits That Fail to Exist

Limits That Fail to Exist

Next, you will examine some functions for which limits do not exist.

Example 6 – *Comparing Left and Right Behavior*

Show that the limit does not exist.

$$\lim_{x \to 0} \frac{|x|}{x}$$

Solution:

Consider the graph of $f(x) = |x|/x$.

From Figure 12.4, you can see that for positive x-values

$$\frac{|x|}{x} = 1, \quad x > 0$$

and for negative x-values

$$\frac{|x|}{x} = -1, \quad x < 0.$$

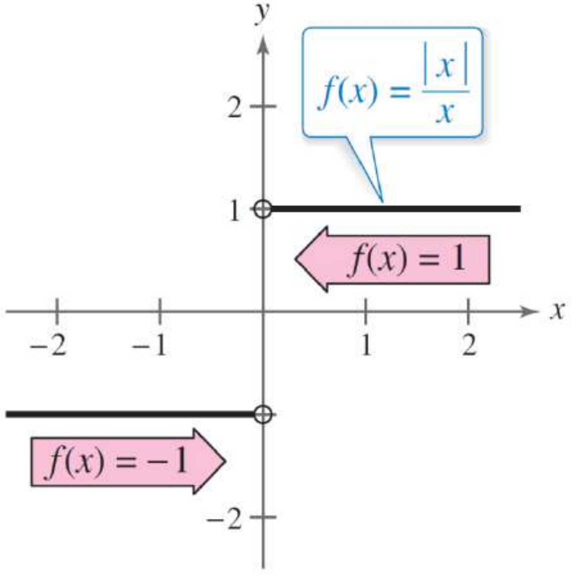

Figure 12.4

Example 6 – *Solution*

cont'd

This means that no matter how close x gets to 0, there will be both positive and negative x-values that yield $f(x) = 1$ and $f(x) = –1$.

This implies that the limit does not exist.

Limits That Fail to Exist

Following are the three most common types of behavior associated with the *nonexistence* of a limit.

Conditions Under Which Limits Do Not Exist

The limit of $f(x)$ as $x \to c$ does not exist when any of the following conditions are true.

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side of c.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

Properties of Limits and Direct Substitution

Properties of Limits and Direct Substitution

You have seen that sometimes the limit of $f(x)$ as $x \to c$ is simply $f(c)$, as shown in Example 2. In such cases, the limit can be evaluated by **direct substitution.**

That is,

$$\lim_{x \to c} f(x) = f(c).$$

Substitute *c* for *x*.

There are many "well-behaved" functions, such as polynomial functions and rational functions with nonzero denominators, that have this property.

Properties of Limits and Direct Substitution

The following list includes some basic limits.

Basic Limits

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x \to c} b = b$ — Limit of a constant function
2. $\lim_{x \to c} x = c$ — Limit of the identity function
3. $\lim_{x \to c} x^n = c^n$ — Limit of a power function
4. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$, for n even and $c > 0$ — Limit of a radical function

This list can also include trigonometric functions. For instance,

$\lim_{x \to \pi} \sin x = \sin \pi = 0$ anc $\lim_{x \to 0} \cos x = \cos 0 = 1.$

Properties of Limits and Direct Substitution

By combining the basic limits with the following operations, you can find limits for a wide variety of functions.

Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. Scalar multiple: $\lim_{x \to c} [bf(x)] = bL$
2. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x \to c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}, \quad K \neq 0$
5. Power: $\lim_{x \to c} [f(x)]^n = L^n$

Example 9 – *Direct Substitution and Properties of Limits*

Find each limit.

a. $\lim_{x \to 4} x^2$ $\lim_{x \to 4} 5x$ $\lim_{x \to \pi} \frac{\tan x}{x}$

d. $\lim_{x \to 9} \sqrt{x}$ $\lim_{x \to \pi} (x \cos x)$ $\lim_{x \to 3} (x + 4)^2$

Solution:

Use the properties of limits and direct substitution to evaluate each limit.

a. $\lim_{x \to 4} x^2 = (4)^2$

$= 16$

Example 9 – *Solution*

cont'd

b. $\lim_{x \to 4} 5x = 5 \lim_{x \to 4} x$ Property 1

$= 5(4)$

$= 20$

c. $\lim_{x \to \pi} \frac{\tan x}{x} = \frac{\lim_{x \to \pi} \tan x}{\lim_{x \to \pi} x}$ Property 4

$= \frac{0}{\pi}$

$= 0$

Example 9 – *Solution*

cont'd

d. $\lim_{x \to 9} \sqrt{x} = \sqrt{9}$

$= 3$

e. $\lim_{x \to \pi} (x \cos x) = \left(\lim_{x \to \pi} x\right)\left(\lim_{x \to \pi} \cos x\right)$ Property 3

$= \pi(\cos \pi)$

$= -\pi$

f. $\lim_{x \to 3} (x + 4)^2 = \left[\left(\lim_{x \to 3} x\right) + \left(\lim_{x \to 3} 4\right)\right]^2$ Properties 2 and 5

$= (3 + 4)^2$

$= 49$

Properties of Limits and Direct Substitution

Example 9 shows algebraic solutions. To verify the limit in Example 9(a) numerically, for instance, create a table that shows values of x^2 for two sets of x-values—one set that approaches 4 from the left and one that approaches 4 from the right, as shown below.

x	3.9	3.99	3.999	4.0	4.001	4.01	4.1
x^2	15.2100	15.9201	15.9920	?	16.0080	16.0801	16.8100

Properties of Limits and Direct Substitution

From the table, you can see that the limit as x approaches 4 is 16. To verify the limit graphically, sketch the graph of $y = x^2$. From the graph shown in Figure 12.7, you can determine that the limit as x approaches 4 is 16.

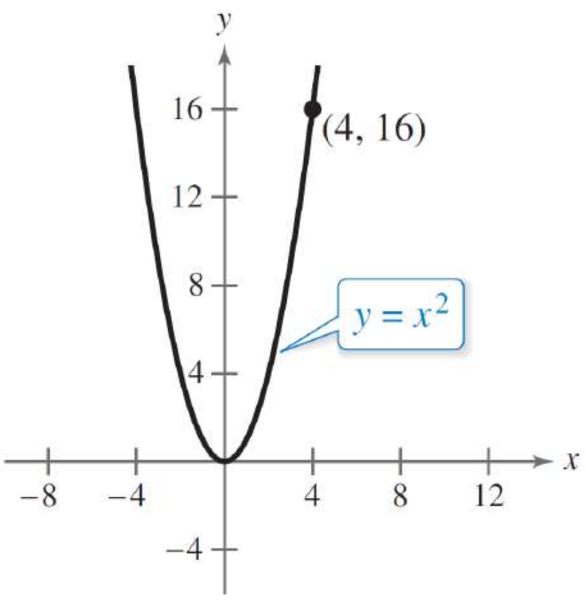

Figure 12.7

Properties of Limits and Direct Substitution

The following summarizes the results of using direct substitution to evaluate limits of polynomial and rational functions.

Limits of Polynomial and Rational Functions

1. If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c).$$

2. If r is a rational function $r(x) = p(x)/q(x)$, and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$