Justifications on the AP Calculus Exam

Students are expected to demonstrate their knowledge of calculus concepts in 4 ways.
1. Numerically (Tables/Data)
2. Graphically
3. Analytically (Algebraic equations)
4. Verbally

The verbal component occurs often on the free response portion of the exam and requires students to explain and/or justify their answers and work. It is important that students understand what responses are valid for their explanations and justifications.

General Tips and Strategies for Justifications
1. A quality explanation does not need to be too wordy or lengthy. A proper explanation is usually very precise and short. Once a statement is made, STOP WRITING!!! Too often, students give a correct explanation, but continue to further elaborate and end up contradicting themselves or making an incorrect assertion which forfeits any points they could have earned.
2. Students commonly mix ideas in their explanations which cause them to not earn points. For example: "a function $f(x)$ is increasing" is equivalent to writing " $f'(x) > 0$ ". However, students often write " $f'(x)$ is increasing" when they intended to write " $f'(x) > 0$ ".
3. Avoid using pronouns in descriptions. Be specific! Do not write statements that begin with "The function…", "It…", or "The graph…". These are too general and the reader will not assume which function or graph is referenced. Name the functions by starting your statement with the phrase " $f(x)\ldots$ " or " $f'(x)\ldots$ ", etc.
4. Know and understand proper mathematical reasons for the ideas covered in this session. Use the precise wording offered today and be assured that these are mathematically correct justifications that will earn points.
5. Make sure to show that the necessary conditions are met BEFORE using theorems like the Mean Value Theorem, Intermediate Value Theorem, Continuity, etc…

Here are several concepts that have required explanations and justifications on free response questions over the past several years.

1. Riemann Sums as an over/under approximation of area
2. Relative minimums/maximums of a function
3. Points of inflection on a function
4. Continuity of a function
5. Speed of a particle increasing/decreasing
6. Meaning of a definite integral in context of a problem
7. Absolute minimum/maximum of a function
8. Using Mean Value Theorem
9. Intervals when a function is increasing/decreasing (particle motion)
10. Tangent lines as an over/under approximation to a point on a function

Continuity

A function is continuous on an interval if it is continuous at every point of the interval. Intuitively, a function is continuous if its graph can be drawn without ever needing to pick up the pencil. This means that the graph of $y = f(x)$ has no "holes", no "jumps" and no vertical asymptotes at $x = a$. When answering free response questions on the AP exam, the formal definition of continuity is required. To earn all of the points on the free response question scoring rubric, all three of the following criteria need to be met, with work shown:

A function is *continuous* at a point $x = a$ if and only if:

1. $f(a)$ exists
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$ (i.e., the limit equals the function value)

Increasing/Decreasing Intervals of a Function

Remember: $f'(x)$ determines whether a function is increasing or decreasing, so always use the sign of $f'(x)$ when determining and justifying whether a function $f(x)$ is increasing or decreasing on (a, b).

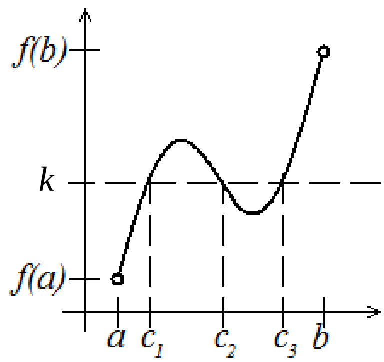
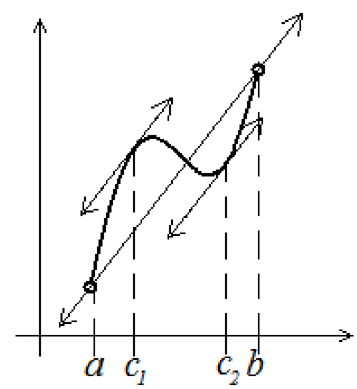
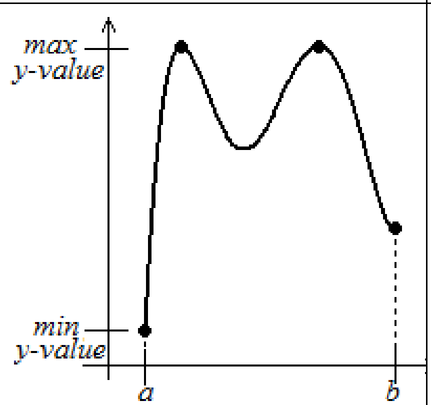
Situation	Explanation
$f(x)$ is increasing on the interval (a, b)	$f(x)$ is increasing on the interval (a, b) because $f'(x) > 0$
$f(x)$ is decreasing on the interval (a, b)	$f(x)$ is decreasing on the interval (a, b) because $f'(x) < 0$

Relative Minimums/Maximums and Points of Inflection

Sign charts are very commonly used in calculus classes and are a valuable tool for students to use when testing for relative extrema and points of inflection. However, a sign chart will never earn students any points on the AP exam. Students should use sign charts when appropriate to help make determinations, but they cannot be used as a justification or explanation on the exam.

Situation (at a point $x = a$ on the function $f(x)$)	Proper Explanation/Reasoning
Relative Minimum	$f(x)$ has a relative minimum at the point $x = a$ because $f'(x)$ changes signs from negative to positive when $x = a$.
Relative Maximum	$f(x)$ has a relative maximum at the point $x = a$ because $f'(x)$ changes signs from positive to negative when $x = a$.
Point of Inflection	$f(x)$ has a point of inflection at the point $x = a$ because $f''(x)$ changes sign when $x = a$

Intermediate Value, Mean Value, and Extreme Value Theorems

Name	Formal Statement	Restatement	Graph	Notes
IVT	If $f(x)$ is continuous on a closed interval $[a, b]$ and $f(a) \neq f(b)$, then for every value k between $f(a)$ and $f(b)$ there exists at least one value c in (a, b) such that $f(c) = k$.	On a continuous function, you will hit every *y*-value between two given *y*-values at least once.		When writing a justification using the IVT, you must state the function is continuous even if this information is provided in the question.
MVT	If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there must exist at least one value c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$	If conditions are met (very important!) there is at least one point where the slope of the tangent line equals the slope of the secant line.		When writing a justification using the MVT, you must state the function is differentiable (continuity is implied by differentiability) even if this information is provided in the question. (Questions may ask students to justify why the MVT cannot be applied often using piecewise functions that are not differentiable over an open interval.)
EVT	A continuous function $f(x)$ on a closed interval $[a, b]$ attains both an absolute maximum $f(c) \geq f(x)$ for all x in the interval and an absolute minimum $f(c) \leq f(x)$ for all x in the interval	Every continuous function on a closed interval has a highest *y*-value and a lowest *y*-value.		When writing a justification using the EVT, you must state the function is continuous on a closed interval even if this information is provided in the question.

Tangent Line Approximations

Unlike a Riemann Sum, determining whether a tangent line is an over/under approximation is not related to whether a function is increasing or decreasing. When determining (or justifying) whether a tangent line is an over or under approximation, the concavity of the function must be discussed. It is important to look at the concavity on the interval from the point of tangency to the x-value of the approximation, not just the concavity at the point of tangency.

Example Justification: The approximation of $f(1.1)$ using the tangent line of f(x) at the point $x=1$ is an over-approximation of the function because $f''(x)$ < 0 on the interval 1 < x < 1.1.

Speed Increasing/Decreasing (Particle Motion)

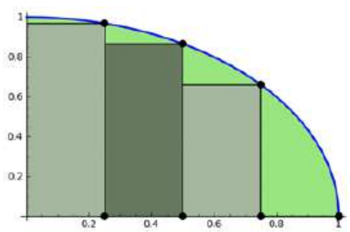
Many students struggle with the concept of speed in particle motion. The speed of a particle is the absolute value of velocity. If a particle's velocity and acceleration are in the same direction, then we know its speed will be increasing. In other words, if the velocity and acceleration have the same sign, then its speed is increasing. On the other hand, if the velocity and acceleration are in opposite directions (different signs), then the speed is decreasing.

When justifying an answer about whether the speed of a particle is increasing/decreasing at a given time, determine both the velocity and acceleration at that time and make reference to the signs of their values.

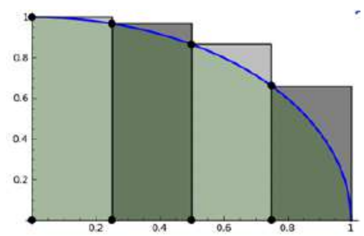
Answer	Possible Justification
Speed is increasing when $t=c$	Speed is increasing because $v(c)>0$ and $a(c)>0$
Speed is increasing when $t=c$	Speed is increasing because $v(c)<0$ and $a(c)<0$
Speed is decreasing when $t=c$	Speed is decreasing because $v(c)>0$ and $a(c)<0$
Speed is decreasing when $t=c$	Speed is decreasing because $v(c)<0$ and $a(c)>0$

Accumulation

Left and right Riemann sums



Right Riemann Sum



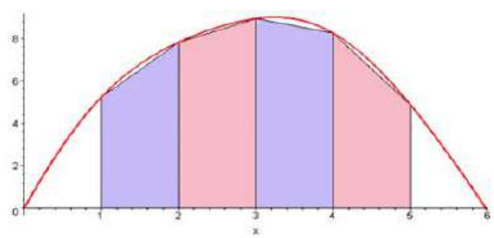
Left Riemann Sum

Correct justification for over and under approximations:

***f*(x)**	**Left Riemann Sum**	**Right Riemann Sum**
Increasing ($f'(x) > 0$)	Under approximates the area because *f*(x) is increasing	Over approximates the area because *f*(x) is increasing
Decreasing ($f'(x) < 0$)	Over approximates the area because *f*(x) is decreasing	Under approximates the area because *f*(x) is decreasing

Incorrect Reasoning: *The left Riemann Sum is an under approximation because the rectangles are all underneath or below the graph. Stating that the rectangles are below the function is not acceptable mathematical reasoning. It merely restates that it is an under approximation but does not explain WHY.*

Trapezoidal approximations



Over/Under Approximations with Trapezoidal Approximations

***f*(x)**	**Trapezoidal Sum**
Concave Up ($f''(x) > 0$)	Over approximates the area because $f''(x) > 0$
Concave Down ($f''(x) < 0$)	Under approximates the area because $f''(x) < 0$

Interpretation of a Definite Integral

When interpreting the meaning of a definite integral, remember the following:

1. Recognize that a definite integral gives an accumulation or total
2. Always give meaning to the integral in CONTEXT to the problem
3. Give the units of measurement
4. Reference the limits of integration with appropriate units in the context of the problem

Free Response

1. (calculator not allowed)

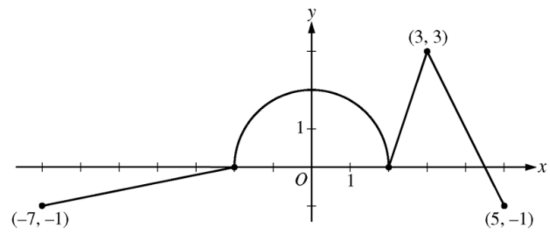
   Solutions to the differential equation $\frac{dy}{dx} = xy^3$ also satisfy $\frac{d^2y}{dx^2} = y^3(1+3x^2y^2)$. Let $y = f(x)$ be a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$.

   (a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.

   (b) Use the tangent line equation from part (a) to approximate $f(1.1)$. Given that $f(x) > 0$ for $1 < x < 1.1$, is the approximation for $f(1.1)$ greater than or less than $f(1.1)$? Explain your reasoning.

2. (calculator not allowed)

The function g is defined and differentiable on the closed interval $[-7, 5]$ and satisfies $g(0) = 5$. The graph of $y = g'(x)$, the derivative of g, consists of a semicircle and three line segments, as shown in the figure.



Graph of g'

(b) Find the x-coordinate of each point of inflection of the graph of $y = g(x)$ on the interval $-7 < x < 5$. Explain your reasoning.

(c) The function h is defined by $h(x) = g(x) - \frac{1}{2}x^2$. Find the x-coordinate of each critical point of h, where $-7 < x < 5$, and classify each critical point as the location of a relative minimum, relative maximum, or neither a minimum nor a maximum. Explain your reasoning.

3. (calculator allowed)

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

Concert tickets went on sale at noon $(t = 0)$ and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time t is modeled by a twice-differentiable function L for $0 \le t \le 9$. Values of $L(t)$ at various times t are shown in the table above.

(c) For $0 \le t \le 9$, what is the fewest number of times at which $L'(t)$ must equal 0? Give a reason for your answer.

4. (calculator allowed)
As a pot of tea cools, the temperature of the tea is modeled by a differentiable function H for $0 \le t \le 10$, where time t is measured in minutes and temperature $H(t)$ is measured in degrees Celsius. Values of $H(t)$ at selected values of time t are shown in the table.

t (minutes)	0	2	5	9	10
$H(t)$ (degrees Celsius)	66	60	52	44	43

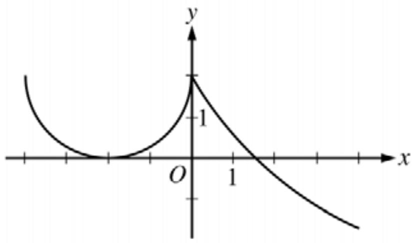
(b) Using correct units, explain the meaning of $\frac{1}{10}\int_0^{10} H(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the four subintervals indicated by the table to estimate $\frac{1}{10}\int_0^{10} H(t)\,dt$.

(c) Evaluate $\int_0^{10} H'(t)\,dt$. Using correct units, explain the meaning of the expression in the context of this problem

5. (calculator not allowed)

The derivative of a function f is defined by

$$f'(x)=\begin{cases} g(x) & \text{for } -4\le x\le 0 \\ 5e^{\frac{-x}{3}}-3 & \text{for } 0<x\le 4 \end{cases}.$$



Graph of f'

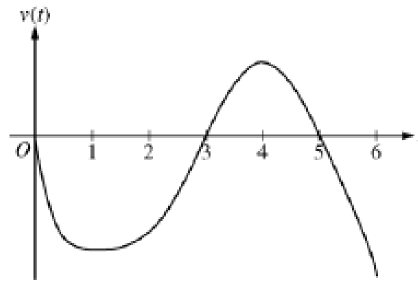
The graph of the continuous function f', shown in the figure above, has x-intercepts at $x=-2$ and $x=3\ln\left(\frac{5}{3}\right)$. The graph g on $-4\le x\le 0$ is a semicircle, and $f(0)=5$.

(a) For $-4\le x\le 4$, find all the values of x at which the graph of f has a point of inflection. Justify your answer.

(b) Find $f(-4)$ and $f(4)$.

(c) For $-4\le x\le 4$, find the value of x at which f has an absolute maximum. Justify your answer.

6. (calculator not allowed)
A particle moves along the *x*-axis so that the velocity at time, *t*, for $0 \le t \le 6$, is given by a differentiable function *v* whose graph is shown. The velocity is 0 at $t = 0$, $t = 3$ and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the *t*-axis and the graph of *v* on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.



Graph of *v*

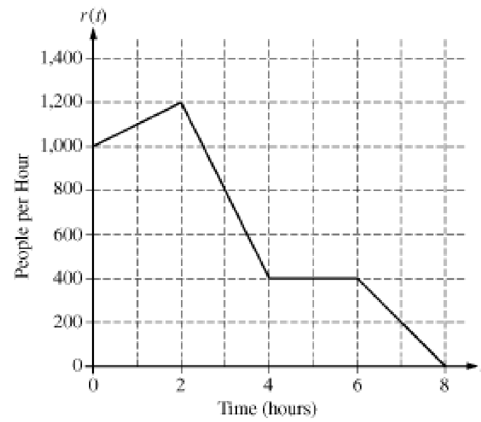
(a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

(b) For how many values of *t*, where $0 \le t \le 6$, is the particle at $x = -8$? Explain your reasoning.

(c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

(d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

7. (calculator allowed)
There are 700 people in line for a popular amusement-park ride when the ride begins operation in the morning. Once it begins operation, the ride accepts passengers until the park closes 8 hours later. While there is a line, people move onto the ride at a rate of 800 people per hour. The graph above shows the rate, $r(t)$, at which people arrive at the ride throughout the day. Time t is measured in hours from the time the ride begins operation.



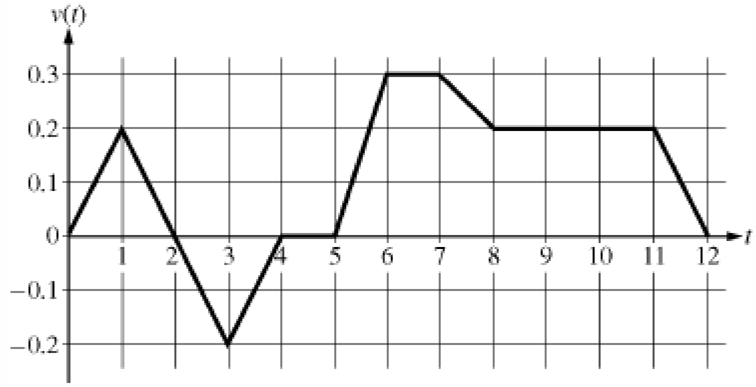
(a) How many people arrive at the ride between $t=0$ and $t=3$? Show the computations that lead to your answer.

(b) Is the number of people waiting in line to get on the ride increasing or decreasing between $t=2$ and $t=3$? Justify your answer.

(c) At what time t is the line for the ride the longest? How many people are in line at that time? Justify your answers.

8. **(**calculator allowed)

Caren rides her bicycle along a straight road from home to school, starting at home at time $t = 0$ minutes and arriving at school at time $t = 12$ minutes. During the time interval $0 \le t \le 12$ minutes, her velocity $v(t)$, in miles per minute, is modeled by the piecewise-linear function whose graph is shown.



(b) Using the correct units, explain the meaning of $\int_0^{12} |v(t)|\,dt$ in terms of Caren's trip. Find the value of $\int_0^{12} |v(t)|\,dt$.

(c) Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. At what time does she turn around to go back home? Give a reason for your answer.

(d) Larry also rides his bicycle along a straight road from home to school in 12 minutes. His velocity is modeled by the function w given by $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)$, where $w(t)$ is in miles per minute for $0 \le t \le 12$ minutes. Who lives closer to school: Caren or Larry? Show the work that leads to your answer.