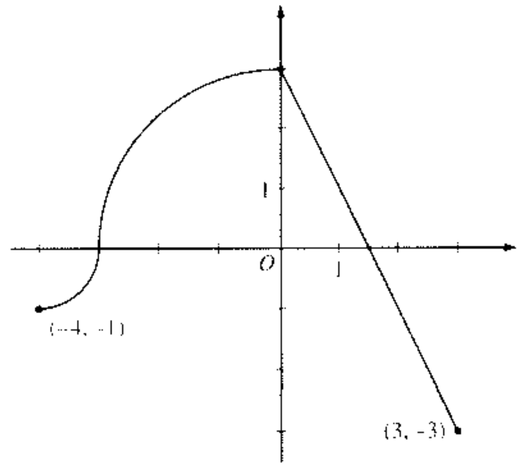
Analysis CW

AP® CALCULUS AB
2011 SCORING GUIDELINES

Question 4

The continuous function f is defined on the interval $-4 \le x \le 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above.



Graph of f

Let $g(x) = 2x + \int_0^x f(t)\,dt$.

(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

(b) Determine the x-coordinate of the point at which g has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

(c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.

(d) Find the average rate of change of f on the interval $-4 \le x \le 3$. There is no point c, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

(a) $g(-3) = 2(-3) + \int_0^{-3} f(t)\,dt = -6 - \frac{9\pi}{4}$ $g'(x) = 2 + f(x)$ $g'(-3) = 2 + f(-3) = 2$	3 : { 1 : $g(-3)$ 1 : $g'(x)$ 1 : $g'(-3)$ }
(b) $g'(x) = 0$ when $f(x) = -2$. This occurs at $x = \frac{5}{2}$. $g'(x) > 0$ for $-4 < x < \frac{5}{2}$ and $g'(x) < 0$ for $\frac{5}{2} < x < 3$. Therefore g has an absolute maximum at $x = \frac{5}{2}$.	3 : { 1 : considers $g'(x) = 0$ 1 : identifies interior candidate 1 : answer with justification }
(c) $g''(x) = f'(x)$ changes sign only at $x = 0$. Thus the graph of g has a point of inflection at $x = 0$.	1 : answer with reason
(d) The average rate of change of f on the interval $-4 \le x \le 3$ is $\frac{f(3) - f(-4)}{3 - (-4)} = -\frac{2}{7}$. To apply the Mean Value Theorem, f must be differentiable at each point in the interval $-4 < x < 3$. However, f is not differentiable at $x = -3$ and $x = 0$.	2 : { 1 : average rate of change 1 : explanation }

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2011 SCORING GUIDELINES (Form B)

Question 4

Consider a differentiable function f having domain all positive real numbers, and for which it is known that $f'(x) = (4 - x)x^{-3}$ for $x > 0$.

(a) Find the x-coordinate of the critical point of f. Determine whether the point is a relative maximum, a relative minimum, or neither for the function f. Justify your answer.

(b) Find all intervals on which the graph of f is concave down. Justify your answer.

(c) Given that $f(1) = 2$, determine the function f.

(a) $f'(x) = 0$ at $x = 4$

$f'(x) > 0$ for $0 < x < 4$

$f'(x) < 0$ for $x > 4$

Therefore f has a relative maximum at $x = 4$.

$$3 : \begin{cases} 1 : x = 4 \\ 1 : \text{relative maximum} \\ 1 : \text{justification} \end{cases}$$

(b) $$\begin{aligned} f''(x) &= -x^{-3} + (4 - x)\left(-3x^{-4}\right) \\ &= -x^{-3} - 12x^{-4} + 3x^{-3} \\ &= 2x^{-4}(x - 6) \\ &= \frac{2(x-6)}{x^4} \end{aligned}$$

$f''(x) < 0$ for $0 < x < 6$

The graph of f is concave down on the interval $0 < x < 6$.

$$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with justification} \end{cases}$$

(c) $$\begin{aligned} f(x) &= 2 + \int_1^x \left(4t^{-3} - t^{-2}\right) dt \\ &= 2 + \left[-2t^{-2} + t^{-1}\right]_{t=1}^{t=x} \\ &= 3 - 2x^{-2} + x^{-1} \end{aligned}$$

$$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$$

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Question 2

The function g is defined for $x > 0$ with $g(1) = 2$, $g'(x) = \sin\left(x + \frac{1}{x}\right)$, and $g''(x) = \left(1 - \frac{1}{x^2}\right)\cos\left(x + \frac{1}{x}\right)$.

(a) Find all values of x in the interval $0.12 \le x \le 1$ at which the graph of g has a horizontal tangent line.

(b) On what subintervals of $(0.12, 1)$, if any, is the graph of g concave down? Justify your answer.

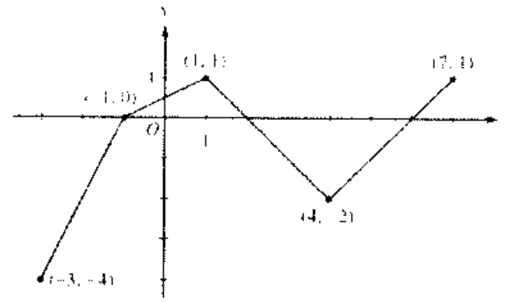
(c) Write an equation for the line tangent to the graph of g at $x = 0.3$.

(d) Does the line tangent to the graph of g at $x = 0.3$ lie above or below the graph of g for $0.3 < x < 1$? Why?

(a) The graph of g has a horizontal tangent line when $g'(x) = 0$. This occurs at $x = 0.163$ and $x = 0.359$.	$2 : \begin{cases} 1 : \text{sets } g'(x) = 0 \\ 1 : \text{answer} \end{cases}$
(b) $g''(x) = 0$ at $x = 0.129458$ and $x = 0.222734$ The graph of g is concave down on $(0.1295, 0.2227)$ because $g''(x) < 0$ on this interval.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$
(c) $g'(0.3) = -0.472161$ $g(0.3) = 2 + \int_1^{0.3} g'(x)\,dx = 1.546007$ An equation for the line tangent to the graph of g is $y = 1.546 - 0.472(x - 0.3)$.	$4 : \begin{cases} 1 : g'(0.3) \\ 1 : \text{integral expression} \\ 1 : g(0.3) \\ 1 : \text{equation} \end{cases}$
(d) $g''(x) > 0$ for $0.3 < x < 1$ Therefore the line tangent to the graph of g at $x = 0.3$ lies below the graph of g for $0.3 < x < 1$.	1 : answer with reason

Question 5

Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$.



Graph of g'

(a) Find the x-coordinate of all points of inflection of the graph of $y = g(x)$ for $-3 < x < 7$. Justify your answer.

(b) Find the absolute maximum value of g on the interval $-3 \le x \le 7$. Justify your answer.

(c) Find the average rate of change of $g(x)$ on the interval $-3 \le x \le 7$.

(d) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

(a) g' changes from increasing to decreasing at $x = 1$;
g' changes from decreasing to increasing at $x = 4$.

Points of inflection for the graph of $y = g(x)$ occur at $x = 1$ and $x = 4$.

$2: \begin{cases} 1: x\text{-values} \\ 1: \text{justification} \end{cases}$

(b) The only sign change of g' from positive to negative in the interval is at $x = 2$.

$$g(-3) = 5 + \int_2^{-3} g'(x)\,dx = 5 + \left(-\frac{3}{2}\right) + 4 = \frac{15}{2}$$
$$g(2) = 5$$
$$g(7) = 5 + \int_2^{7} g'(x)\,dx = 5 + (-4) + \frac{1}{2} = \frac{3}{2}$$

The maximum value of g for $-3 \le x \le 7$ is $\frac{15}{2}$.

$3: \begin{cases} 1: \text{identifies } x = 2 \text{ as a candidate} \\ 1: \text{considers endpoints} \\ 1: \text{maximum value and justification} \end{cases}$

(c) $\dfrac{g(7) - g(-3)}{7 - (-3)} = \dfrac{\frac{3}{2} - \frac{15}{2}}{10} = -\dfrac{3}{5}$

$2: \begin{cases} 1: \text{difference quotient} \\ 1: \text{answer} \end{cases}$

(d) $\dfrac{g'(7) - g'(-3)}{7 - (-3)} = \dfrac{1 - (-4)}{10} = \dfrac{1}{2}$

No, the MVT does *not* guarantee the existence of a value c with the stated properties because g' is not differentiable for at least one point in $-3 < x < 7$.

$2: \begin{cases} 1: \text{average value of } g'(x) \\ 1: \text{answer "No" with reason} \end{cases}$