Limits and an Introduction to Calculus

12

Bruce Low/Eye Ubiquitous/CORBIS

This Ferris wheel is part of the Oktoberfest celebration held each fall in Munich, Germany. This festival began as a celebration of the royal wedding in 1810.

The Big Picture

In this chapter you will learn how to

- estimate limits and use properties and operations of limits.
- find limits by direct substitution and by using the dividing out and rationalizing techniques.
- approximate slopes of tangent lines, use the limit definition of slope, and use derivatives to find slopes of graphs.
- evaluate limits at infinity and find limits of sequences.
- find limits of summations and use them to find areas of regions bounded by graphs of functions.

Important Vocabulary

As you encounter each new vocabulary term in this chapter, add the term and its definition to your notebook glossary.

- limit (p. 807)
- indeterminate form (p. 818)
- one-sided limit (p. 821)
- tangent line (p. 826)
- secant line (p. 828)
- difference quotient (p. 828)
- derivative (p. 831)
- limits at infinity (p. 835)
- area of a plane region (p. 847)

Additional Resources Text-specific additional resources are available to help you do well in this course. See page xvi for details.

12.1 Introduction to Limits

The Limit Concept

The notion of a limit is a *fundamental* concept of calculus. In this chapter, you will learn how to evaluate limits and how they are used in the two basic problems of calculus: the tangent line problem and the area problem.

What You Should Learn:

- How to use the definition of a limit to estimate limits
- How to decide whether limits of functions exist
- How to use properties and operations of limits to find limits

Why You Should Learn It:

The concept of a limit is useful in applications involving maximization. For instance, in Exercise 1 on page 813, the concept of a limit is used to verify the maximum volume of an open box.

Dick Luria/FPG International

EXAMPLE 1 Finding a Rectangle of Maximum Area

You are given 24 inches of wire and are asked to form a rectangle whose area is as large as possible. What dimensions should the rectangle have?

Solution

Let w represent the width of the rectangle and let l represent the length of the rectangle. Because

$2w + 2l = 24$ Perimeter is 24.

it follows that $l = 12 - w$, as shown in Figure 12.1. So, the area of the rectangle is

$$A = lw \qquad \text{Formula for area}$$
$$= (12 - w)w \qquad \text{Substitute } 12 - w \text{ for } l.$$
$$= 12w - w^2. \qquad \text{Simplify.}$$

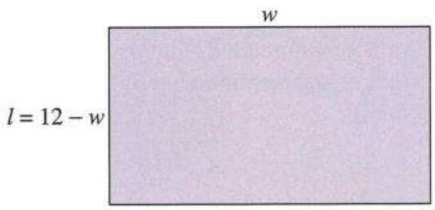


Figure 12.1

Using this model for area, you can experiment with different values of w to see how to obtain the maximum area. After trying several values, it appears that the maximum area occurs when $w = 6$.

Width, w	5.0	5.5	5.9	6.0	6.1	6.5	7.0
Area, A	35.0	35.75	35.99	36.0	35.99	35.75	35.0

In limit terminology, you can say that "the limit of A as w approaches 6 is 36." This is written as

$$\lim_{w \to 6} A = 36.$$

Definition of Limit

The *Interactive* CD-ROM and *Internet* versions of this text show every example with its solution; clicking on the *Try It!* button brings up similar problems. Guided Examples and Integrated Examples show step-by-step solutions to additional examples. Integrated Examples are related to several concepts in the section.

Definition of Limit

If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, the **limit** of $f(x)$ as x approaches c is L. This is written as

$$\lim_{x \to c} f(x) = L.$$

EXAMPLE 2 Estimating a Limit Numerically

Use a table to estimate numerically the limit $\lim_{x \to 2} (3x - 2)$.

Solution

Let $f(x) = 3x - 2$. Then construct a table that shows values of $f(x)$ when x is close to 2.

x	1.9	1.99	1.999	2.0	2.001	2.01	2.1
$f(x)$	3.7	3.97	3.997	?	4.003	4.03	4.3

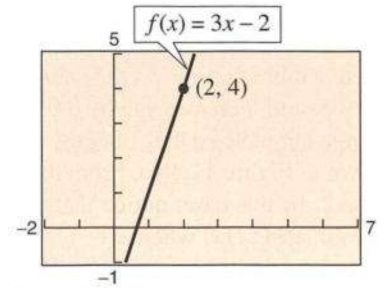


Figure 12.2

From the table, it appears that the closer x gets to 2, the closer $f(x)$ gets to 4. So, you can estimate the limit to be 4. For this particular function, you can obtain the limit simply by substituting 2 for x to obtain

$$\lim_{x \to 2} (3x - 2) = 3(2) - 2 = 4.$$

Figure 12.2 adds further support to this conclusion.

In Figure 12.2, note that the graph of $f(x) = 3x - 2$ is continuous. For graphs that are not continuous, finding a limit can be more difficult.

EXAMPLE 3 Estimating a Limit Numerically

Use a table to estimate numerically the limit $\lim_{x \to 0} \dfrac{x}{\sqrt{x+1} - 1}$.

Solution

Let $f(x) = x/\left(\sqrt{x+1} - 1\right)$. Then construct a table that shows values of $f(x)$ when x is close to 0.

x	−0.01	−0.001	−0.0001	0	0.0001	0.001	0.01
$f(x)$	1.995	1.9995	1.99995	?	2.00005	2.0005	2.005

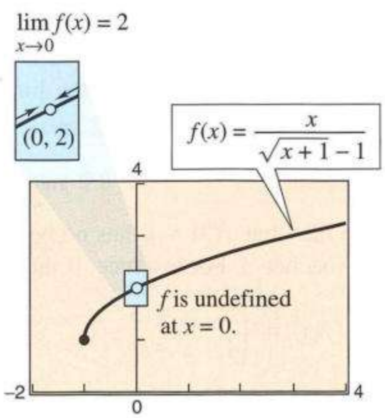


Figure 12.3

From the table, it appears that the limit is 2. Try using the *zoom* and *trace* features of a graphing utility to verify graphically that the limit is 2 (see Figure 12.3).

In Example 3, note that $f(x)$ has a limit when $x \to 0$, even though the function is not defined when $x = 0$. This often happens, and it is important to realize that *the existence or nonexistence of $f(x)$ when $x = c$ has no bearing on the existence of the limit of $f(x)$ as x approaches c.*

EXAMPLE 4 Using a Graphing Utility to Find a Limit

Estimate the limit $\lim\limits_{x \to 1} \dfrac{x^3 - x^2 + x - 1}{x - 1}$.

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Numerical Solution

Let $f(x) = (x^3 - x^2 + x - 1)/(x - 1)$. Because you are finding the limit when $x = 1$, use the *table* feature of a graphing utility to create a table that shows the value of the function for x beginning at $x = 0.9$ and incrementing by 0.01, as shown in Figure 12.4(a). Then change the table so that x begins at 0.99 and increments by 0.001, as shown in Figure 12.4(b). From the tables, you can estimate the limit to be 2. In this case, notice that you cannot obtain the limit simply by evaluating $f(x)$ when $x = 1$.

X	Y1
.97	1.9409
.98	1.9604
.99	1.9801
1	ERROR
1.01	2.0201
1.02	2.0404
1.03	2.0609
X=1	

(a)

X	Y1
.997	1.994
.998	1.996
.999	1.998
1	ERROR
1.001	2.002
1.002	2.004
1.003	2.006
X=1	

(b)

Figure 12.4

Graphical Solution

Use a graphing utility to graph $y = (x^3 - x^2 + x - 1)/(x - 1)$. Then use the *zoom* and *trace* features to determine that as x gets closer and closer to 1, y gets closer and closer to 2 from the left and from the right, as shown in Figure 12.5. Using the *trace* feature, notice that there is no value given for y when $x = 1$.

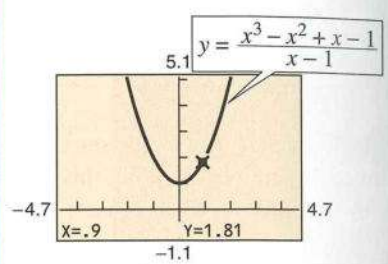


Figure 12.5

EXAMPLE 5 Using a Graph to Find a Limit

Find the limit of $f(x)$ as x approaches 3, where

$$f(x) = \begin{cases} 2, & x \neq 3 \\ 0, & x = 3 \end{cases}.$$

Solution

Because $f(x) = 2$ for all x other than $x = 3$ and because the value of $f(3)$ is immaterial, it follows that the limit is 2, as shown in Figure 12.6. So, you can write

$$\lim_{x \to 3} f(x) = 2.$$

The fact that $f(3) = 0$ has no bearing on the existence or value of the limit as x approaches 3. For instance, if the function were defined as

$$f(x) = \begin{cases} 2, & x \neq 3 \\ 4, & x = 3 \end{cases}$$

the limit would be the same.

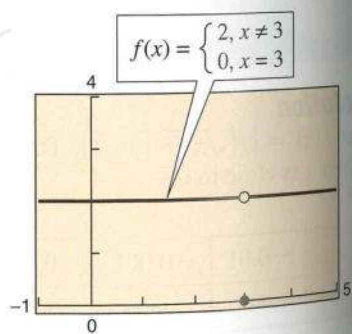


Figure 12.6

Limits That Fail to Exist

In the next three examples, you will examine some functions for which limits do not exist.

EXAMPLE 6 Comparing Left and Right Behavior

Show that the following limit does not exist.

$$\lim_{x \to 0} \frac{|x|}{x}$$

Solution

Consider the graph of the function $f(x) = |x|/x$. From Figure 12.7, you can see that for positive x-values

$$\frac{|x|}{x} = 1, \qquad x > 0$$

and for negative x-values

$$\frac{|x|}{x} = -1, \qquad x < 0.$$

This means that no matter how close x gets to 0, there will be both positive and negative x-values that yield $f(x) = 1$ and $f(x) = -1$. This implies that the limit does not exist.

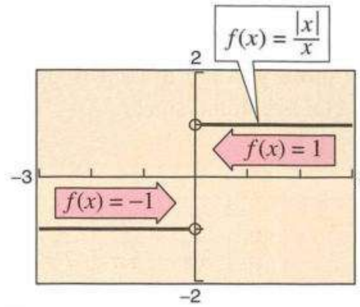


Figure 12.7

EXAMPLE 7 Unbounded Behavior

Discuss the existence of the limit

$$\lim_{x \to 0} \frac{1}{x^2}.$$

Solution

Let $f(x) = 1/x^2$. In Figure 12.8, note that as x approaches 0 from either the right or the left, $f(x)$ increases without bound. This means that by choosing x close enough to 0, you can force $f(x)$ to be as large as you want. For instance, $f(x)$ will be larger than 100 if you choose x that is within $\frac{1}{10}$ of 0. That is,

$$0 < |x| < \frac{1}{10} \quad \Longrightarrow \quad f(x) = \frac{1}{x^2} > 100.$$

Similarly, you can force $f(x)$ to be larger than 1,000,000, as follows.

$$0 < |x| < \frac{1}{1000} \quad \Longrightarrow \quad f(x) = \frac{1}{x^2} > 1{,}000{,}000$$

Because $f(x)$ is not approaching a real number L as x approaches 0, you can conclude that the limit does not exist.

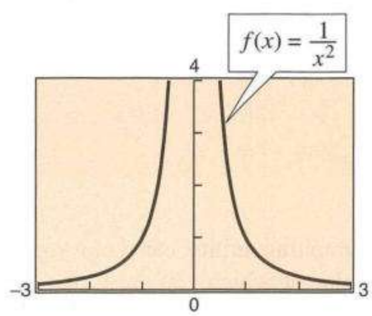


Figure 12.8

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 8 Oscillating Behavior

Discuss the existence of the limit

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right).$$

Solution

Let $f(x) = \sin(1/x)$. In Figure 12.9, you can see that as x approaches 0, $f(x)$ oscillates between -1 and 1. Therefore, the limit does not exist because no matter how close you are to 0, it is possible to choose values of x_1 and x_2 such that $\sin(1/x_1) = 1$ and $\sin(1/x_2) = -1$, as indicated in the table.

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$x \to 0$
$\sin\left(\frac{1}{x}\right)$	1	-1	1	-1	1	-1	Limit does not exist.

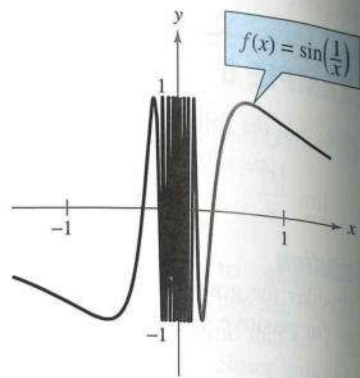


Figure 12.9

Examples 6, 7, and 8 show three of the most common types of behavior associated with the nonexistence of a limit.

Conditions Under Which Limits Do Not Exist

The limit of $f(x)$ as $x \to c$ does not exist if any of the following conditions is true.

1. $f(x)$ approaches a different number from the right side of c than from the left side of c. Example 6
2. $f(x)$ increases or decreases without bound as x approaches c. Example 7
3. $f(x)$ oscillates between two fixed values as x approaches c. Example 8

A graphing utility can help you discover the behavior of a function near the x-value at which you are trying to evaluate a limit. When you do this, however, you should realize that you can't always trust the graphs that graphing utilities show. For instance, if you use a graphing utility to sketch the graph of the function in Example 8 over an interval containing 0, you will most likely obtain an incorrect graph—such as that shown in Figure 12.10. The reason that a graphing utility can't show the correct graph is that the graph has infinitely many oscillations over any interval that contains 0.

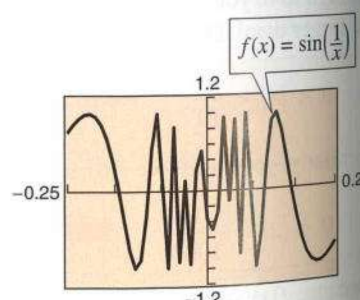


Figure 12.10

Properties of Limits

You have seen that sometimes the limit of $f(x)$ as $x \to c$ is simply $f(c)$. In such cases, it is said that the limit can be evaluated by *direct substitution*. That is,

$$\lim_{x \to c} f(x) = f(c). \qquad \text{Substitute } c \text{ for } x.$$

There are many "well-behaved" functions that have this property. Some of the basic ones are included in the following list.

Properties of Limits

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x \to c} b = b$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} x^n = c^n$
4. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$, for n even and $c > 0$

Trigonometric functions could also be included in this list. For instance,

$$\lim_{x \to \pi} \sin x = \sin \pi$$
$$= 0$$

and

$$\lim_{x \to 0} \cos x = \cos 0$$
$$= 1.$$

By combining the properties of limits with the following operations, you can find limits for a wide variety of functions.

Operations with Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. Scalar multiple: $\lim_{x \to c} [bf(x)] = bL$
2. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x \to c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$
5. Power: $\lim_{x \to c} [f(x)]^n = L^n$

Exploration

Use a graphing utility to graph the tangent function. What are $\lim_{x \to 0} \tan x$ and $\lim_{x \to \pi/4} \tan x$? What can you say about the existence of the limit $\lim_{x \to \pi/2} \tan x$?

EXAMPLE 9 Finding Limits by Direct Substitution

Find each of the following limits.

a. $\lim\limits_{x\to4} x^2$ **b.** $\lim\limits_{x\to4} 5$

c. $\lim\limits_{x\to4} (x^2 + 2x - 5)$ **d.** $\lim\limits_{x\to4} \dfrac{x^2 + 2x - 5}{x - 1}$

Solution

You can use the Properties of Limits and direct substitution to evaluate each limit.

a. $$\lim_{x\to4} x^2 = (4)^2 = 16$$

b. $$\lim_{x\to4} 5 = 5$$

c. $$\lim_{x\to4} (x^2 + 2x - 5) = (4)^2 + 2(4) - 5 = 16 + 8 - 5 = 19$$

d. $$\lim_{x\to4} \frac{x^2 + 2x - 5}{x - 1} = \frac{(4)^2 + 2(4) - 5}{4 - 1} = \frac{16 + 8 - 5}{3} = \frac{19}{3}$$

STUDY T!P

When evaluating limits, remember that there are several ways to solve most problems. Often, a problem can be solved *numerically, graphically*, or *algebraically*. For instance, the limits in Examples 1 to 3 were found numerically (by constructing tables). The limit in Example 4 was found numerically and graphically. The limit is Example 5 was found graphically, and the limits in Example 9 were found algebraically.

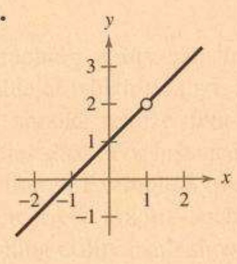
Writing About Math *Graphs with Gaps*

Match each graph with one of the following functions.

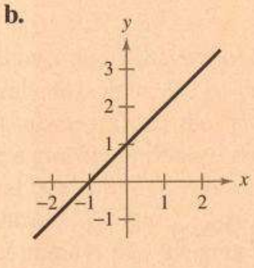
$$f(x) = x + 1 \qquad g(x) = \frac{x^2 - 1}{x - 1} \qquad h(x) = \frac{x^3 - 2x^2 - x + 2}{x^2 - 3x + 2}$$

Find the limits of each function as x approaches 1 and as x approaches 2. What conclusion can you make?

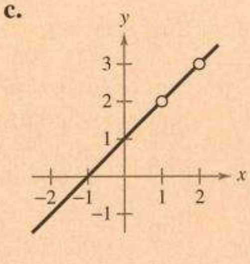
a.



b.



c.



Graph these functions on a graphing utility to see whether it distinguishes among the graphs. Write a short explanation of your findings.

12.1 Exercises

1. ***Maximum Volume*** Consider an open box that is to be made from a square piece of material, 24 centimeters on a side, by cutting equal squares from the corners and turning up the sides.

   (a) Draw a diagram that represents the box.

   (b) Verify that the volume of the box is $V = 4x(12 - x)^2$.

   (c) The box has a maximum volume when $x = 4$. Use a graphing utility to complete the table and observe the behavior of the function as x approaches 4. Use the table to find $\lim_{x \to 4} V$.

x	3	3.5	3.9	4	4.1	4.5	5
V							

   (d) Use a graphing utility to graph the volume function. Verify that the volume is maximum when $x = 4$.

2. ***Maximum Area*** You are given wire and are asked to form a right triangle with a hypotenuse of $\sqrt{18}$ inches whose area is as large as possible.

   (a) Draw and label a diagram that shows the base x and height y of the triangle.

   (b) Verify that the area of the triangle is

   $A = \frac{1}{2}x\sqrt{18 - x^2}$.

   (c) The triangle has a maximum area when $x = 3$ inches. Use a graphing utility to complete the table and observe the behavior of the function as x approaches 3. Use the table to find $\lim_{x \to 3} A$.

x	2	2.5	2.9	3	3.1	3.5	4
A							

   (d) Use a graphing utility to graph the area function. Verify that the area is maximum when $x = 3$ inches.

In Exercises 3–6, complete the table and use the result to numerically estimate the limit. Determine whether or not the limit can be reached.

3. $\lim_{x \to 3} (4 - 3x)$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$				?			

4. $\lim_{x \to 4} \left(\frac{1}{2}x^2 - 2x + 3\right)$

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$				?			

5. $\lim_{x \to 3} \dfrac{x - 3}{x^2 - 9}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$				?			

6. $\lim_{x \to -1} \dfrac{x + 1}{x^2 - x - 2}$

x	-1.1	-1.01	-1.001	-1	-0.999
$f(x)$				?	

x	-0.99	-0.9
$f(x)$		

In Exercises 7–14, use a graphing utility to construct a table and use the result to estimate the limit numerically. Use the graphing utility to graph the corresponding function to confirm your result graphically.

7. $\lim_{x \to 1} \dfrac{x - 1}{x^2 + 2x - 3}$

8. $\lim_{x \to -2} \dfrac{x + 2}{x^2 + 5x + 6}$

9. $\lim_{x \to 0} \dfrac{\sqrt{x + 5} - \sqrt{5}}{x}$

10. $\lim_{x \to -3} \dfrac{\sqrt{1 - x} - 2}{x + 3}$

The *Interactive* CD-ROM and *Internet* versions of this text contain step-by-step solutions to all odd-numbered Section and Review Exercises. They also provide Tutorial Exercises, which link to Guided Examples for additional help.

11. $\displaystyle\lim_{x\to -4} \frac{\frac{x}{x+2} - 2}{x+4}$

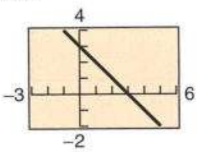
12. $\displaystyle\lim_{x\to 2} \frac{\frac{1}{x+2} - \frac{1}{4}}{x-2}$

13. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$

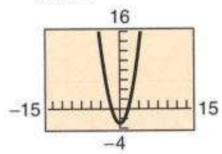
14. $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x}$

In Exercises 15–22, use the graph to find the limit (if it exists).

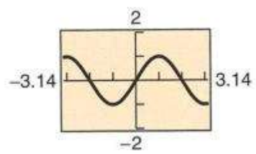
15. $\displaystyle\lim_{x\to 2} (3 - x)$



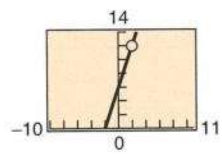
16. $\displaystyle\lim_{x\to -4} (x^2 - 3)$



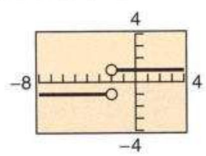
17. $\displaystyle\lim_{x\to -1} \sin\frac{\pi x}{2}$



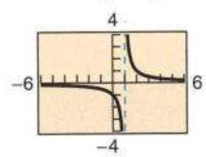
18. $\displaystyle\lim_{x\to 2} \frac{3x^2 - 12}{x - 2}$



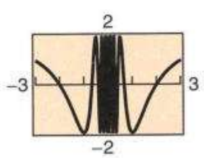
19. $\displaystyle\lim_{x\to -2} \frac{|x+2|}{x+2}$



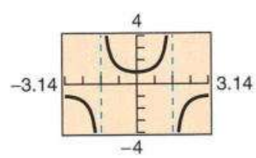
20. $\displaystyle\lim_{x\to 1} \frac{1}{x-1}$



21. $\displaystyle\lim_{x\to 0} 2\cos\frac{\pi}{x}$



22. $\displaystyle\lim_{x\to \pi/2} \sec x$



In Exercises 23–32, use a graphing utility to graph the function and use the graph to determine whether the specified limit exists.

23. $f(x) = \dfrac{5}{2 + e^{1/x}}, \quad \displaystyle\lim_{x\to 0} f(x)$

24. $f(x) = \dfrac{e^x - 1}{x}, \quad \displaystyle\lim_{x\to 0} f(x)$

25. $f(x) = \cos\dfrac{1}{x}, \quad \displaystyle\lim_{x\to 0} f(x)$

26. $f(x) = \sin \pi x, \quad \displaystyle\lim_{x\to 1} f(x)$

27. $f(x) = \dfrac{\sqrt{x+3} - 1}{x - 4}, \quad \displaystyle\lim_{x\to 4} f(x)$

28. $f(x) = \dfrac{\sqrt{x+5} - 4}{x - 2}, \quad \displaystyle\lim_{x\to 2} f(x)$

29. $f(x) = \dfrac{x - 1}{x^2 - 4x + 3}, \quad \displaystyle\lim_{x\to 1} f(x)$

30. $f(x) = \dfrac{7}{x - 3}, \quad \displaystyle\lim_{x\to 3} f(x)$

31. $f(x) = \ln(x + 2), \quad \displaystyle\lim_{x\to 4} f(x)$

32. $f(x) = \ln(x + 6), \quad \displaystyle\lim_{x\to -1} f(x)$

In Exercises 33–46, find the limit by direct substitution.

33. $\displaystyle\lim_{x\to 1}(x^2 + 3x - 4)$

34. $\displaystyle\lim_{x\to -2} (x^3 - 6x + 5)$

35. $\displaystyle\lim_{x\to 2} \sqrt[3]{10x + 7}$

36. $\displaystyle\lim_{x\to -3} \sqrt[3]{10 + 3x}$

37. $\displaystyle\lim_{x\to 3} \frac{15}{x}$

38. $\displaystyle\lim_{x\to -5} \frac{6}{x + 2}$

39. $\displaystyle\lim_{x\to -1} \frac{\sqrt{x^2 - 1}}{x}$

40. $\displaystyle\lim_{x\to 8} \frac{\sqrt{x+1}}{x - 4}$

41. $\displaystyle\lim_{x\to 3} e^x$

42. $\displaystyle\lim_{x\to e} \ln x$

43. $\displaystyle\lim_{x\to \pi} \sin 2x$

44. $\displaystyle\lim_{x\to 1/4} \frac{\tan \pi x}{2}$

45. $\displaystyle\lim_{x\to 1/2} \arcsin x$

46. $\displaystyle\lim_{x\to 1} \arccos\frac{x}{2}$

In Exercises 47 and 48, find the limit (if it exists) as x approaches 2.

47. $f(x) = \begin{cases} 2x + 1, & x < 2 \\ 2x + 2, & x \geq 2 \end{cases}$

48. $f(x) = \begin{cases} x^2 - 4, & x < 2 \\ x - 2, & x \geq 2 \end{cases}$

49. If $\lim_{x \to c} f(x) = 3$ and $\lim_{x \to c} g(x) = 6$, find
   (a) $\lim_{x \to c} [-2g(x)]$.
   (b) $\lim_{x \to c} [f(x) + g(x)]$.
   (c) $\lim_{x \to c} \dfrac{f(x)}{g(x)}$.
   (d) $\lim_{x \to c} \sqrt{f(x)}$.

50. If $\lim_{x \to c} f(x) = 3$ and $\lim_{x \to c} g(x) = -2$, find
   (a) $\lim_{x \to c} [f(x) + g(x)]^2$.
   (b) $\lim_{x \to c} [6f(x)g(x)]$.
   (c) $\lim_{x \to c} \dfrac{5g(x)}{4f(x)}$.
   (d) $\lim_{x \to c} \dfrac{1}{\sqrt{f(x)}}$.

In Exercises 51 and 52, find (a) $\lim_{x \to 2} f(x)$, (b) $\lim_{x \to 2} g(x)$, (c) $\lim_{x \to 2} [f(x)g(x)]$, and (d) $\lim_{x \to 2} [g(x) - f(x)]$.

51. $f(x) = x^3, \qquad g(x) = \dfrac{\sqrt{x^2 + 5}}{2x^2}$

52. $f(x) = \dfrac{x}{3 - x}, \qquad g(x) = \sin \pi x$

Synthesis

True or False? **In Exercises 53 and 54, determine whether the statement is true or false. Justify your answer.**

53. The limit of a function as x approaches c does not exist if the function approaches -3 from the left of c and 3 from the right of c.

54. The limit of the product of two functions is equal to the product of limits of the two functions.

55. ***Think About It*** From Exercises 3 to 6, select a limit that can be reached and one that cannot be reached.
   (a) Use a graphing utility to graph the corresponding functions in the standard viewing window. Do the graphs reveal whether or not the limit can be reached? Explain.
   (b) Use a graphing utility to graph the corresponding functions using a decimal window. Do the graphs reveal whether or not the limit can be reached? Explain.

56. ***Think About It*** Use the results from Exercise 55 to draw a conclusion as to whether or not you can use the graph generated by a graphing utility to determine reliably whether or not a limit can be reached.

57. ***Think About It*** If $f(2) = 4$, can you conclude anything about $\lim_{x \to 2} f(x)$? Give reasons for your answer.

58. ***Think About It*** If $\lim_{x \to 2} f(x) = 4$, can you conclude anything about $f(2)$? Give reasons for your answer.

59. ***Writing*** Write a brief description of the meaning of the notation $\lim_{x \to 5} f(x) = 12$.

Writing **In Exercises 60 and 61, use a graphing utility to graph the function and estimate the limit (if it exists). What is the domain of the function? Can you detect a possible danger in determining the domain of the function solely by analyzing the graph generated by a graphing utility? Write a short paragraph about the importance of examining a function algebraically as well as graphically.**

60. $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}$
   $\lim_{x \to 9} f(x)$

61. $f(x) = \dfrac{x - 3}{x^2 - 9}$
   $\lim_{x \to 3} f(x)$

Review

In Exercises 62–67, simplify the rational expression.

62. $\dfrac{x^2 - 81}{9 - x}$

63. $\dfrac{5 - x}{3x - 15}$

64. $\dfrac{x^2 - 12x + 36}{x^2 - 7x + 6}$

65. $\dfrac{15x^2 + 7x - 4}{15x^2 + x - 2}$

66. $\dfrac{x^3 - 8}{x^2 - 4}$

67. $\dfrac{x^3 + 27}{x^2 + x - 6}$

In Exercises 68–73, find the distance between the indicated points.

68. $(1, 0, 3)$ and $(5, 2, 6)$

69. $(3, 2, 7)$ and $(3, 2, 8)$

70. $(-2, 5, -3)$ and $(4, -2, 6)$

71. $(2, 2, -5)$ and $(-3, -4, -8)$

72. $(0, -4, 0)$ and $(2, 0, -9)$

73. $(3, -3, 0)$ and $(0, 5, -5)$

12.2 Techniques for Evaluating Limits

Limits of Polynomial and Rational Functions

In Section 12.1, you saw how direct substitution and operations with limits can be used to evaluate limits of certain well-behaved functions, such as polynomial functions and rational functions with nonzero denominators. This result is summarized as follows.

Limits of Polynomial and Rational Functions

1. If p is a polynomial function and c is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$

2. If r is a rational function given by $r(x) = p(x)/q(x)$, and c is a real number such that $q(c) \neq 0$, then

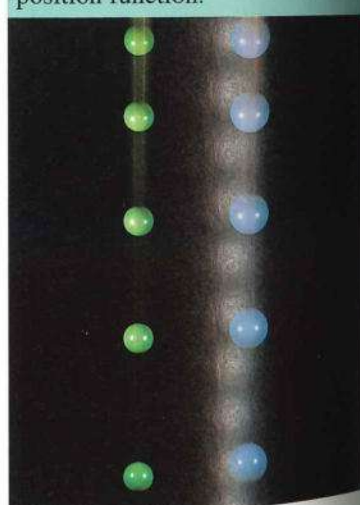
$$\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}, \qquad q(c) \neq 0.$$

What You Should Learn:

- How to find limits of polynomial and rational functions by direct substitution
- How to use the dividing out technique to find limits of functions
- How to use the rationalizing technique to find limits of functions
- How to approximate limits of functions graphically and numerically
- How to evaluate one-sided limits of functions
- How to evaluate the limits of difference quotients from calculus

Why You Should Learn It:

Many definitions in calculus involve the limit of a function. For instance, in Exercises 61 and 62 on page 825, the definition of the velocity of a free-falling object at any instant in time involves finding the limit of a position function.



Superstock

EXAMPLE 1 Evaluating Limits by Direct Substitution

Find each of the following limits.

a. $\lim_{x\to -1} (x^2 + x - 6)$

b. $\lim_{x\to -1} \dfrac{x^2 + x - 6}{x + 3}$

Solution

The first function is a polynomial function and the second is a rational function (with a nonzero denominator at $x = -1$). So, you can evaluate the limits by direct substitution.

a. $\lim_{x\to -1} (x^2 + x - 6) = (-1)^2 + (-1) - 6 = -6$

b. $\lim_{x\to -1} \dfrac{x^2 + x - 6}{x + 3} = \dfrac{(-1)^2 + (-1) - 6}{-1 + 3} = -\dfrac{6}{2} = -3$

Use a graphing utility to graph the function

$$f(x) = \frac{x^2 + x - 6}{x + 3}$$

in Example 1(b). Use the *trace* feature to approximate $\lim_{x\to -1} f(x)$. What do you think $\lim_{x\to -3} f(x)$ equals? Is f defined at $x = -3$? Does this affect the existence of the limit as x approaches -3?

Dividing Out Technique

In Example 1(b), suppose you were asked to find the limit as $x \to -3$.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$$

Direct substitution would fail because -3 is a zero of the denominator. By using a table, however, it appears that the limit of the function as $x \to -3$ is -5.

x	-3.01	-3.001	-3.0001	-3	-2.9999	-2.999	-2.99
$\dfrac{x^2 + x - 6}{x + 3}$	-5.01	-5.001	-5.0001	?	-4.9999	-4.999	-4.99

Another way to find the limit of this function is to factor the numerator and divide out common factors, as shown in Example 2.

EXAMPLE 2 Dividing Out Technique

Find the following limit.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$$

Solution

Begin by factoring the numerator and dividing out any common factors.

$$\begin{aligned}
\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} &= \lim_{x \to -3} \frac{(x - 2)(x + 3)}{x + 3} && \text{Factor numerator.} \\
&= \lim_{x \to -3} \frac{(x - 2)\cancel{(x + 3)}}{\cancel{x + 3}} && \text{Divide out common factor.} \\
&= \lim_{x \to -3} (x - 2) && \text{Simplify.} \\
&= -3 - 2 && \text{Direct substitution} \\
&= -5 && \text{Simplify.}
\end{aligned}$$

This procedure for evaluating a limit is called the *dividing out technique*. The validity of the procedure stems from the fact that if two functions agree at all but a single number c, they must have identical limit behavior at $x = c$. In Example 2, the functions

$$f(x) = \frac{x^2 + x - 6}{x + 3} \quad \text{and} \quad g(x) = x - 2$$

agree at all values of x other than $x = -3$. So, you can use $g(x)$ to find the limit of $f(x)$.

The *Interactive* CD-ROM and *Internet* versions of this text offer a built-in graphing calculator, which can be used with the Examples, Explorations, and Exercises.

Exploration

Use a graphing utility to compare the graphs of

$$y = \frac{x^2 + x - 6}{x + 3}$$

and

$$y = x - 2.$$

Use the *value* feature to evaluate the first graph at $x = -3$. Then evaluate the second graph at $x = -3$. What can you conclude?

The dividing out technique should be applied only when direct substitution produces 0 in both the numerator *and* the denominator. The resulting fraction, $\frac{0}{0}$, has no meaning as a real number. It is called an **indeterminate form** because you cannot, from the form alone, determine the limit. When you encounter this form by direct substitution into a rational function, you can conclude that the numerator and denominator must have a common factor. After factoring and dividing out, you should try direct substitution again.

EXAMPLE 3 Dividing Out Technique

Find the following limit.

$$\lim_{x \to 1} \frac{x - 1}{x^3 - x^2 + x - 1}$$

Solution

Begin by substituting $x = 1$ into the numerator and denominator.

$$1 - 1 = 0 \qquad \text{Numerator is 0 when } x = 1.$$

$$(1)^3 - (1)^2 + 1 - 1 = 0 \qquad \text{Denominator is 0 when } x = 1.$$

Because both the numerator and denominator are zero when $x = 1$, direct substitution will not yield the limit. To find the limit, you should factor the numerator and denominator, divide out any common factors, and then try direct substitution again.

$$\lim_{x \to 1} \frac{x - 1}{x^3 - x^2 + x - 1} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(x^2 + 1)} \qquad \text{Factor denominator.}$$

$$= \lim_{x \to 1} \frac{\cancel{x - 1}}{\cancel{(x - 1)}(x^2 + 1)} \qquad \text{Divide out common factor.}$$

$$= \lim_{x \to 1} \frac{1}{x^2 + 1} \qquad \text{Simplify.}$$

$$= \frac{1}{(1)^2 + 1} \qquad \text{Substitute.}$$

$$= \frac{1}{2} \qquad \text{Simplify.}$$

This result is shown graphically in Figure 12.11.

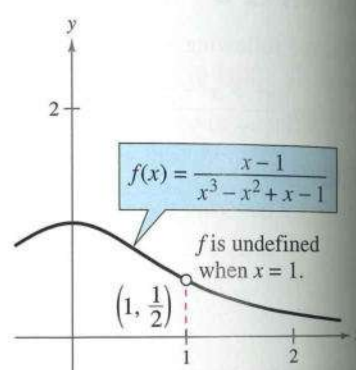


Figure 12.11

In Example 3, the factorization of the denominator can be obtained by dividing by $(x - 1)$ or by grouping as follows.

$$x^3 - x^2 + x - 1 = x^2(x - 1) + (x - 1)$$

$$= (x - 1)(x^2 + 1)$$

Rationalizing Technique

Another way to find the limit of a function is to first rationalize the numerator of the function. This is called the *rationalizing technique*.

EXAMPLE 4 Rationalizing Technique

Find the following limit.

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$$

Solution

By direct substitution, you obtain the indeterminate form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \frac{0}{0} \qquad \text{Indeterminate form}$$

In this case, you can rewrite the fraction by rationalizing the numerator.

$$\begin{aligned}\frac{\sqrt{x+1}-1}{x} &= \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right) \\ &= \frac{(x+1)-1}{x\left(\sqrt{x+1}+1\right)} && \text{Multiply.} \\ &= \frac{x}{x\left(\sqrt{x+1}+1\right)} && \text{Simplify.} \\ &= \frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1}+1\right)} && \text{Divide out common factor.} \\ &= \frac{1}{\sqrt{x+1}+1}, \quad x \neq 0 && \text{Simplify.}\end{aligned}$$

Now you can evaluate the limit by direct substitution.

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{1+1} = \frac{1}{2}$$

You can reinforce your conclusion that the limit is $\frac{1}{2}$ by constructing a table, or by sketching a graph, as shown in Figure 12.12.

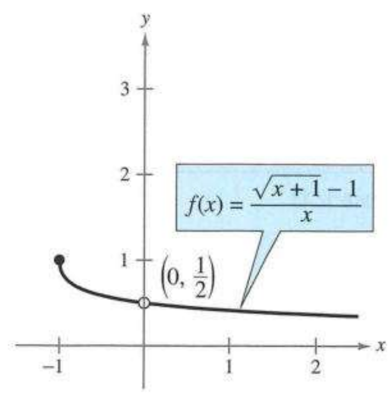


Figure 12.12

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881

The rationalizing technique for evaluating limits is based on multiplication by a convenient form of 1. In Example 4, the convenient form is

$$1 = \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}.$$

Using Technology

The dividing out and rationalizing techniques work well for finding the limit of a rational function or for finding the limit of a function involving a radical. To find limits of nonalgebraic functions, you often need to use more sophisticated analytic techniques, as shown in Examples 5 and 6.

EXAMPLE 5 Approximating a Limit

Approximate the limit $\lim_{x\to 0} (1 + x)^{1/x}$.

Numerical Solution

Let $f(x) = (1 + x)^{1/x}$. Because you are finding the limit when $x = 0$, use the *table* feature of a graphing utility to create a table that shows the value of f for x beginning at $x = -0.01$ and incrementing by 0.001, as shown in Figure 12.13. Because 0 is halfway between -0.001 and 0.001, use the average of the values of f at these two x-coordinates as an estimate of the limit as follows.

$$\lim_{x\to 0} (1 + x)^{1/x} \approx \frac{2.7196 + 2.7169}{2} = 2.71825$$

The actual limit can be found algebraically to be $e \approx 2.71828$.

X	Y1	
-.003	2.7224	
-.002	2.721	
-.001	7.7196	
0	ERROR	
.001	2.7169	
.002	2.7156	
.003	2.7142	
X=0		

Figure 12.13

Graphical Solution

To approximate the limit graphically, enter the function $f(x) = (1 + x)^{1/x}$ in a graphing utility, as shown in Figure 12.14. Using the *zoom* and *trace* features of the graphing utility, choose two points on the graph of f, such as

$$(-0.001645, 2.7205) \quad \text{and} \quad (0.001645, 2.7160).$$

Because the x-coordinates of these two points are equidistant from 0, you can approximate the limit to be the average of the y-coordinates. That is,

$$\lim_{x\to 0} (1 + x)^{1/x} \approx \frac{2.7205 + 2.7160}{2} = 2.71825.$$

The actual limit can be found algebraically to be $e \approx 2.71828$.

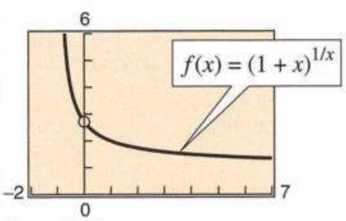


Figure 12.14

EXAMPLE 6 Approximating a Limit Graphically

Approximate the limit $\lim_{x\to 0} \dfrac{\sin x}{x}$.

Solution

Notice that direct substitution does not work because it produces the indeterminate form $\frac{0}{0}$. To approximate the limit, use the procedure described in Example 5. Begin by sketching the graph of $f(x) = (\sin x)/x$, as shown in Figure 12.15. Then use the *zoom* and *trace* features of the graphing utility to choose a point on each side of 0, such as

$$(-0.001234, 0.9999997) \quad \text{and} \quad (0.001234, 0.9999997).$$

Finally, use interpolation to approximate the limit as the average of the y-coordinates of these two points, $\lim_{x\to 0} (\sin x)/x \approx 0.9999997$. It can be shown algebraically that this limit is exactly 1.

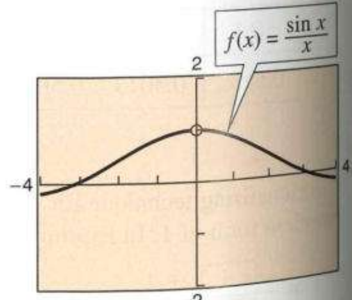


Figure 12.15

One-Sided Limits

In Section 12.1, you saw that one way in which a limit can fail to exist is when a function approaches a different value from the left side of c than it approaches from the right side of c. This type of behavior can be described more concisely with the concept of a **one-sided limit.**

$$\lim_{x \to c^-} f(x) = L \qquad \text{Limit from the left}$$

$$\lim_{x \to c^+} f(x) = L \qquad \text{Limit from the right}$$

EXAMPLE 7 Evaluating One-Sided Limits

Find the limit as $x \to 0$ from the left and the limit as $x \to 0$ from the right for

$$f(x) = \frac{|2x|}{x}.$$

Solution

From the graph of f, shown in Figure 12.16, you can see that $f(x) = -2$ for all $x < 0$. Therefore, the limit from the left is

$$\lim_{x \to 0^-} \frac{|2x|}{x} = -2. \qquad \text{Limit from the left}$$

Because $f(x) = 2$ for all $x > 0$, the limit from the right is

$$\lim_{x \to 0^+} \frac{|2x|}{x} = 2. \qquad \text{Limit from the right}$$

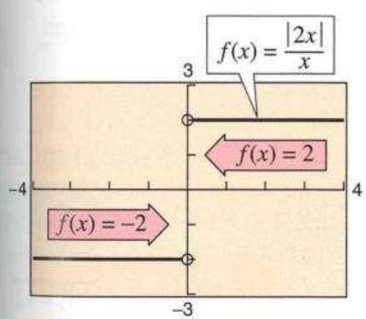


Figure 12.16

In Example 7, note that the function approaches different limits from the left and from the right. In such cases, the limit of $f(x)$ as $x \to c$ does not exist. For the limit of a function to exist as $x \to c$, it must be true that both one-sided limits exist and are equal.

Existence of a Limit

If f is a function and c and L are real numbers, then

$$\lim_{x \to c} f(x) = L$$

if and only if both the left and right limits exist and are equal to L.

EXAMPLE 8 Evaluating One-Sided Limits

Find the limit of $f(x)$ as x approaches 1.

$$f(x) = \begin{cases} 4 - x, & x < 1 \\ 4x - x^2, & x > 1 \end{cases}$$

Solution

Remember that you are concerned about the value of f near $x = 1$ rather than at $x = 1$. So, for $x < 1$, $f(x)$ is given by $4 - x$, and you can use direct substitution to obtain

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} (4 - x) \\ &= 4 - 1 \\ &= 3. \end{aligned}$$

For $x > 1$, $f(x)$ is given by $4x - x^2$, and you can use direct substitution to obtain

$$\begin{aligned} \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} (4x - x^2) \\ &= 4 - 1 \\ &= 3. \end{aligned}$$

Because the one-sided limits both exist and are equal to 3, it follows that

$$\lim_{x\to 1} f(x) = 3.$$

The graph in Figure 12.17 confirms this conclusion.

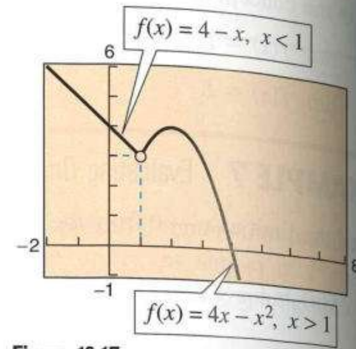


Figure 12.17

EXAMPLE 9 Comparing Limits from the Left and Right

An overnight delivery service charges \$8 for the first pound and \$2 for each additional pound. Let x represent the weight of a parcel and let $f(x)$ represent the shipping cost. Show that the limit of $f(x)$ as $x \to 2$ does not exist.

$$f(x) = \begin{cases} 8, & 0 < x \le 1 \\ 10, & 1 < x \le 2 \\ 12, & 2 < x \le 3 \end{cases}$$

Solution

The graph of f is shown in Figure 12.18. The limit of $f(x)$ as x approaches 2 from the left is

$$\lim_{x\to 2^-} f(x) = 10$$

whereas the limit of $f(x)$ as x approaches 2 from the right is

$$\lim_{x\to 2^+} f(x) = 12.$$

Because these one-sided limits are not equal, the limit of $f(x)$ as $x \to 2$ does not exist.

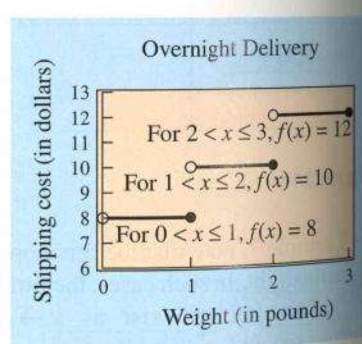


Figure 12.18

A Limit from Calculus

In the next section, you will study an important type of limit from calculus—the limit of a *difference quotient*.

EXAMPLE 10 Evaluating a Limit from Calculus

For the function $f(x) = x^2 - 1$, find

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}.$$

Solution

Direct substitution produces an indeterminant form.

$$\begin{aligned}
\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} &= \lim_{h \to 0} \frac{[(3+h)^2 - 1] - [(3)^2 - 1]}{h} \\
&= \lim_{h \to 0} \frac{9 + 6h + h^2 - 1 - 9 + 1}{h} \\
&= \lim_{h \to 0} \frac{6h + h^2}{h} \\
&= \frac{0}{0}
\end{aligned}$$

By factoring and dividing out, you can obtain the following.

$$\begin{aligned}
\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} &= \lim_{h \to 0} \frac{h(6+h)}{h} \\
&= \lim_{h \to 0} (6 + h) = 6
\end{aligned}$$

So, the limit is 6.

Note that the limit of a difference quotient is an expression of the form

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Direct substitution into the difference quotient always produces the indeterminate form $\frac{0}{0}$.

Writing About Math *Limits of Rational Functions*

Consider the limit of the rational function $p(x)/q(x)$. What conclusion can you make if direct substitution produces the given expression? Write a short paragraph explaining your reasoning.

a. $\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{0}{1}$ **b.** $\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{1}{1}$ **c.** $\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{1}{0}$ **d.** $\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{0}{0}$

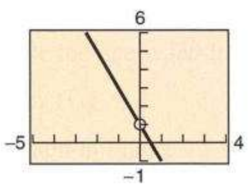
12.2 Exercises

In Exercises 1–8, evaluate the limit by direct substitution.

1. $\lim_{x\to5} (10 - x^2)$
2. $\lim_{x\to-2} \left(\frac{1}{2}x^3 - 5x\right)$
3. $\lim_{x\to-3} \dfrac{3x}{x^2+1}$
4. $\lim_{x\to4} \dfrac{x-1}{x^2+2x+3}$
5. $\lim_{x\to-2} \dfrac{5x+3}{2x-9}$
6. $\lim_{x\to3} \dfrac{x^2+1}{x}$
7. $\lim_{x\to-1} \sqrt{x+2}$
8. $\lim_{x\to3} \sqrt[3]{x^2-1}$

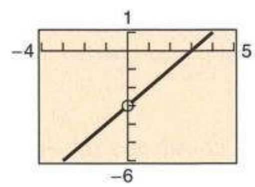
In Exercises 9–12, use the graph to determine the limit visually (if it exists). Then identify another function that agrees with the given function at all but one point.

9. $g(x) = \dfrac{-2x^2+x}{x}$



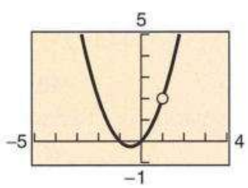
(a) $\lim_{x\to0} g(x)$
(b) $\lim_{x\to-1} g(x)$
(c) $\lim_{x\to-2} g(x)$

10. $h(x) = \dfrac{x^2-3x}{x}$



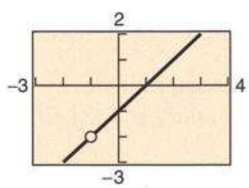
(a) $\lim_{x\to-2} h(x)$
(b) $\lim_{x\to0} h(x)$
(c) $\lim_{x\to3} h(x)$

11. $g(x) = \dfrac{x^3-x}{x-1}$



(a) $\lim_{x\to1} g(x)$
(b) $\lim_{x\to-1} g(x)$
(c) $\lim_{x\to0} g(x)$

12. $f(x) = \dfrac{x^2-1}{x+1}$



(a) $\lim_{x\to1} f(x)$
(b) $\lim_{x\to2} f(x)$
(c) $\lim_{x\to-1} f(x)$

In Exercises 13–24, find the limit (if it exists). Use a graphing utility to verify your result graphically.

13. $\lim_{x\to6} \dfrac{x-6}{x^2-36}$
14. $\lim_{x\to5} \dfrac{5-x}{x^2-25}$
15. $\lim_{x\to-1} \dfrac{1-2x-3x^2}{1+x}$
16. $\lim_{t\to-3} \dfrac{t^3+27}{t+3}$
17. $\lim_{y\to0} \dfrac{\sqrt{5+y}-\sqrt{5}}{y}$
18. $\lim_{z\to0} \dfrac{\sqrt{7-z}-\sqrt{7}}{z}$
19. $\lim_{x\to-3} \dfrac{\sqrt{x+7}-2}{x+3}$
20. $\lim_{x\to2} \dfrac{4-\sqrt{18-x}}{x-2}$
21. $\lim_{x\to0} \dfrac{\dfrac{1}{x+1}-1}{x}$
22. $\lim_{x\to0} \dfrac{\dfrac{1}{4-x}-\dfrac{1}{4}}{x}$
23. $\lim_{x\to0} \dfrac{\sec x}{\tan x}$
24. $\lim_{x\to\pi/2} \dfrac{1-\sin x}{\cos x}$

In Exercises 25–28, use a graphing utility to graph the function and approximate the limit.

25. $\lim_{x\to0} \dfrac{\sqrt{x+3}-\sqrt{3}}{x}$
26. $\lim_{x\to9} \dfrac{3-\sqrt{x}}{x-9}$
27. $\lim_{x\to2} \dfrac{x^5-32}{x-2}$
28. $\lim_{x\to0} \dfrac{\dfrac{1}{2+x}-\dfrac{1}{2}}{x}$

Graphical, Numerical, and Algebraic Analysis **In Exercises 29–32, graphically approximate the limit (if it exists) by using a graphing utility to graph the function. Numerically approximate the limit (if it exists) by constructing a table. Algebraically evaluate the limit (if it exists) by the appropriate techniques.**

29. $\lim_{x\to1^-} \dfrac{(x-1)}{x^2-1}$
30. $\lim_{x\to5^+} \dfrac{5-x}{25-x^2}$
31. $\lim_{x\to16^+} \dfrac{4-\sqrt{x}}{x-16}$
32. $\lim_{x\to0^-} \dfrac{\sqrt{x+2}-\sqrt{2}}{x}$

In Exercises 33–38, use a graphing utility to graph the function. Determine the limit (if it exists) by evaluating the corresponding one-sided limits.

33. $\lim_{x\to6} \dfrac{|x-6|}{x-6}$
34. $\lim_{x\to2} |x-2|$
35. $\lim_{x\to1} \dfrac{1}{x^2+1}$
36. $\lim_{x\to1} \dfrac{1}{x^2-1}$
37. $\lim_{x\to2} f(x)$ where $f(x) = \begin{cases} x-1, & x \le 2 \\ 2x-3, & x > 2 \end{cases}$
38. $\lim_{x\to1} f(x)$ where $f(x) = \begin{cases} 4-x^2, & x \le 1 \\ 3-x, & x > 1 \end{cases}$

In Exercises 39–46, use a graphing utility to graph the function and approximate the limit. Write an approximation that is accurate to three decimal places.

39. $\lim_{x\to 0^+} (x \ln x)$ **40.** $\lim_{x\to 0^+} (x^2 \ln x)$

41. $\lim_{x\to 0} \dfrac{\sin 2x}{x}$ **42.** $\lim_{x\to 0} \dfrac{\sin 3x}{x}$

43. $\lim_{x\to 0} \dfrac{\tan x}{x}$ **44.** $\lim_{x\to 0} \dfrac{1 - \cos 2x}{x}$

45. $\lim_{x\to 1} \dfrac{1 - \sqrt[3]{x}}{1 - x}$ **46.** $\lim_{x\to 0} (1 + 2x)^{1/x}$

In Exercises 47–52, use a graphing utility to graph the function and the equations $y = x$ and $y = -x$ in the same viewing window. Use the graph to find $\lim_{x\to 0} f(x)$.

47. $f(x) = x \cos x$ **48.** $f(x) = |x \sin x|$

49. $f(x) = |x| \sin x$ **50.** $f(x) = |x| \cos x$

51. $f(x) = x \sin \dfrac{1}{x}$ **52.** $f(x) = x \cos \dfrac{1}{x}$

In Exercises 53 and 54, state which limit can be evaluated using direct substitution. Then evaluate or approximate each limit.

53. (a) $\lim_{x\to 0} x^2 \sin x^2$ (b) $\lim_{x\to 0} \dfrac{\sin x^2}{x^2}$

54. (a) $\lim_{x\to 0} \dfrac{x}{\cos x}$ (b) $\lim_{x\to 0} \dfrac{1 - \cos x}{x}$

In Exercises 55–60, find $\lim_{h\to 0} \dfrac{f(x + h) - f(x)}{h}$.

55. $f(x) = 3x - 1$ **56.** $f(x) = 5 - 6x$

57. $f(x) = \sqrt{x}$ **58.** $f(x) = \sqrt{2x - 2}$

59. $f(x) = x^2 - 3x$ **60.** $f(x) = 4 - 2x - x^2$

In Exercises 61 and 62, use the position function $s(t) = -16t^2 + 128$, which gives the height (in feet) of a free-falling object. The velocity at time $t = a$ seconds is given by $\lim_{t\to a} [s(a) - s(t)]/(a - t)$.

61. Find the velocity when $t = 1$ second.

62. Find the velocity when $t = 2$ seconds.

63. ***Salary Contract*** A union contract guarantees a 10% salary increase yearly for 3 years. Let t represent the time in years and $f(t)$ represent the salary (in thousands of dollars). For a current salary of \$28,000, the salary for the next 3 years is

$$f(t) = \begin{cases} 28.00, & 0 < t \le 1 \\ 30.80, & 1 < t \le 2. \\ 33.88, & 2 < t \le 3 \end{cases}$$

Show that the limit of f as $t \to 2$ does not exist.

64. ***Cost of Overnight Delivery*** The cost of sending an overnight package is \$10.75 for the first pound and \$3.95 for each additional pound. A plastic mailing bag can hold up to 3 pounds. Let x represent the weight of a package and let $f(x)$ represent the cost of sending a package in a plastic mailing bag. Show that the limit of f as $x \to 1$ does not exist.

$$f(x) = \begin{cases} 10.75, & 0 < x \le 1 \\ 14.70, & 1 < x \le 2 \\ 18.65, & 2 < x \le 3 \end{cases}$$

Synthesis

True or False? **In Exercises 65 and 66, determine whether the statement is true or false. Justify your answer.**

65. When evaluating the limit of a rational function yields an indeterminate form, the rational function's numerator and denominator have a common factor.

66. If $f(c) = L$, then $\lim_{x\to c} f(x) = L$.

67. ***Think About It*** Sketch the graph of a function for which $f(2)$ is defined but for which the limit of $f(x)$ as x approaches 2 does not exist.

68. ***Think About It*** Sketch the graph of a function for which the limit of $f(x)$ as x approaches 1 is 4 but for which $f(1) \neq 4$.

69. ***Writing*** Write a short paragraph explaining several reasons why the limit of a function may not exist.

Review

70. Write an equation of the line that passes through $(6, -10)$ and is perpendicular to the line that passes through $(4, -6)$ and $(3, -4)$.

71. Write an equation of the line that passes through $(1, -1)$ and is parallel to the line that passes through $(3, -3)$ and $(5, -2)$.

In Exercises 72–75, determine whether the vectors are orthogonal, parallel, or neither.

72. $\langle 7, -2, 3\rangle, \langle -1, 4, 5\rangle$ **73.** $\langle 5, 5, 0\rangle, \langle 0, 5, 1\rangle$

74. $\langle -4, 3, -6\rangle, \langle 12, -9, 18\rangle$

75. $\langle 2, -3, 1\rangle, \langle -2, 2, 2\rangle$

12.3 The Tangent Line Problem

Tangent Line to a Graph

Calculus is a branch of mathematics that studies rates of change of functions. If you go on to take a course in calculus, you will learn that rates of change have many applications in real life.

Earlier in the text, you learned how the slope of a line indicates the rate at which a line rises or falls. For a line, this rate (or slope) is the same at every point on the line. For graphs other than lines, the rate at which the graph rises or falls changes from point to point. For instance, in Figure 12.19, the parabola is rising more quickly at the point (x_1, y_1) than it is at the point (x_2, y_2). At the vertex (x_3, y_3), the graph levels off, and at the point (x_4, y_4), the graph is falling.

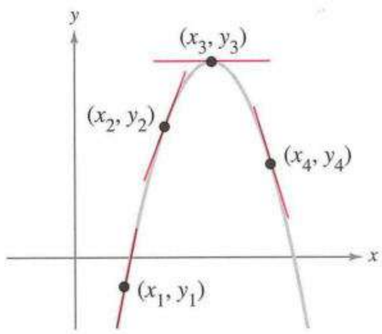


Figure 12.19

To determine the rate at which a graph rises or falls at a *single point*, you can find the slope of the tangent line at that point. In simple terms, the **tangent line** to a graph of a function f at a point $P(x_1, y_1)$ is the line that best approximates the slope of the graph at the point. Figure 12.20 shows other examples of tangent lines.

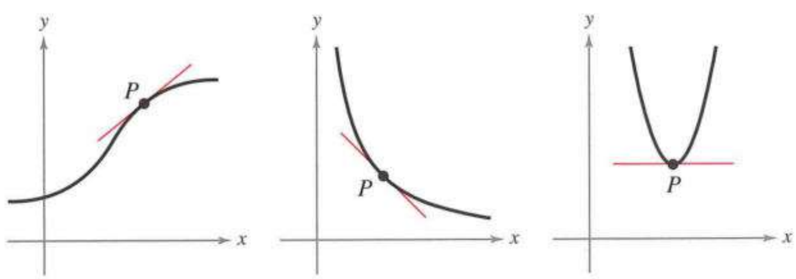


Figure 12.20

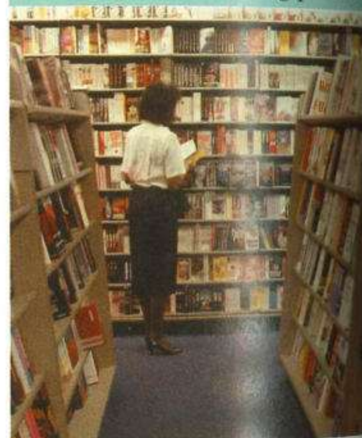
From geometry, you know that a line is tangent to a circle if the line intersects the circle at only one point. Tangent lines to noncircular graphs can intersect the graph at more than one point. For instance, in the first graph in Figure 12.20, if the tangent line were extended, it would intersect the graph at a point other than the point of tangency.

What You Should Learn:

- How to use a tangent line to approximate the slope of a graph at a point
- How to use the limit definition of slope to find exact slopes of graphs
- How to find derivatives of functions and use derivatives to find slopes of graphs

Why You Should Learn It:

The slope of the graph of a function can be used to analyze rates of change at particular points on the graph. For instance, in Exercise 40 on page 834, the slope of the graph is used to analyze the rate of change in book sales for particular selling prices.



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Slope of a Graph

Because a tangent line approximates the slope of the graph at a point, the problem of finding the slope of a graph at a point becomes one of finding the slope of the tangent line at the point.

A computer simulation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 1 Visually Approximating the Slope of a Graph

Use the graph in Figure 12.21 to approximate the slope of the graph of $f(x) = x^2$ at the point $(1, 1)$.

Solution

From the graph of $f(x) = x^2$, you can see that the tangent line at $(1, 1)$ rises approximately two units for each unit change in x. So, the slope of the tangent line at $(1, 1)$ is

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} \approx \frac{2}{1} = 2.$$

Because the tangent line at the point $(1, 1)$ has a slope of about 2, you can conclude that the graph has a slope of about 2 at the point $(1, 1)$.

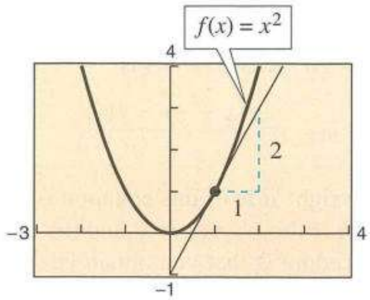


Figure 12.21

When you are visually approximating the slope of a graph, remember that the scales on the horizontal and vertical axes may differ. When this happens (as it frequently does in applications), the slope of the tangent line is distorted, and you must be careful to account for the difference in scales.

EXAMPLE 2 Approximating the Slope of a Graph

Figure 12.22 graphically depicts the average daily temperature (in degrees Fahrenheit) for each month in Dallas, Texas. Approximate the slope of this graph at the indicated point and give a physical interpretation of the result.

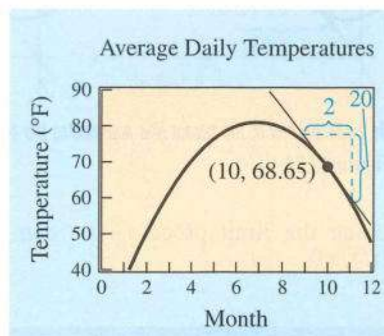


Figure 12.22

Solution

From the graph, you can see that the tangent line at the given point falls approximately 20 units for each two-unit change in x. So, you can estimate the slope at the given point to be

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} \approx \frac{-20}{2} = -10 \text{ degrees per month.}$$

This means that you can expect the average daily temperature in November to be about 10 degrees lower than the corresponding temperature in October.

Slope and the Limit Process

In Examples 1 and 2, you approximated the slope of a graph at a point by making a careful graph and then "eyeballing" the tangent line at the point of tangency. A more precise method of approximating tangent lines makes use of a **secant line** through the point of tangency and a second point on the graph, as shown in Figure 12.23. If $(x, f(x))$ is the point of tangency and $(x + h, f(x + h))$ is a second point on the graph of f, the slope of the secant line through the two points is

$$m_{\text{sec}} = \frac{f(x + h) - f(x)}{h}. \qquad \text{Slope of secant line}$$

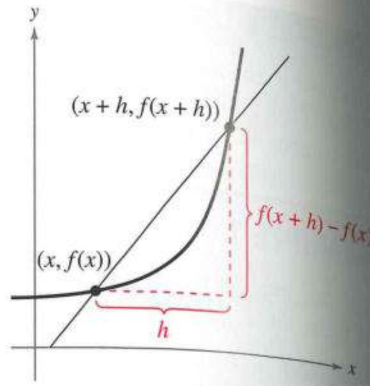
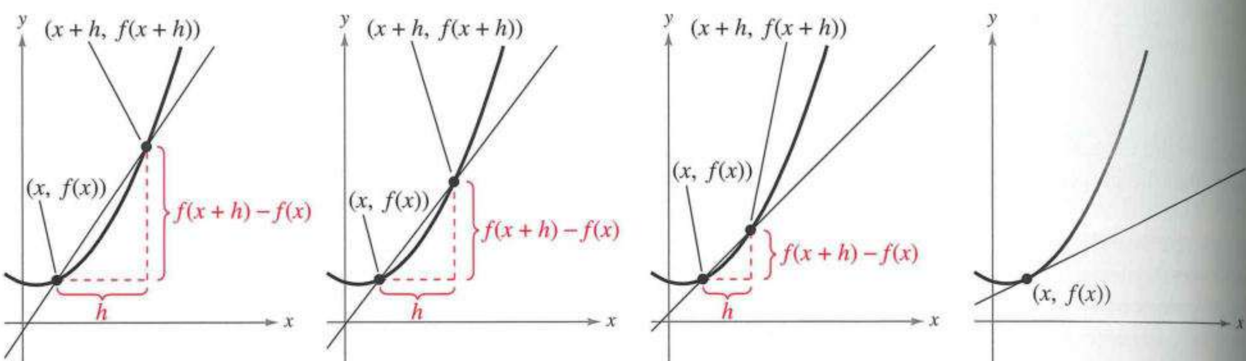


Figure 12.23

The right side of this equation is called the **difference quotient.** The denominator h is the *change in x*, and the numerator is the *change in y*. The beauty of this procedure is that you obtain better and better approximations of the slope of the tangent line by choosing the second point closer and closer to the point of tangency, as shown in Figure 12.24.



As h approaches 0, the secant line approaches the tangent line.
Figure 12.24

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Using the limit process, you can find the *exact* slope of the tangent line at $(x, f(x))$.

Definition of the Slope of a Graph

The **slope** m of the graph of f at the point $(x, f(x))$ is equal to the slope of its tangent line at $(x, f(x))$, and is given by

$$m = \lim_{h \to 0} m_{\text{sec}}$$

$$= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

provided this limit exists.

EXAMPLE 3 Finding the Slope of a Graph

Find the slope of the graph of $f(x) = x^2$ at the point $(-2, 4)$.

Solution

Begin by finding an expression that represents the slope of a secant line at the point $(-2, 4)$.

$$m_{\text{sec}} = \frac{f(-2 + h) - f(-2)}{h} \qquad \text{Set up difference quotient.}$$

$$= \frac{(-2 + h)^2 - (-2)^2}{h} \qquad \text{Use } f(x) = x^2.$$

$$= \frac{4 - 4h + h^2 - 4}{h} \qquad \text{Expand terms.}$$

$$= \frac{-4h + h^2}{h} \qquad \text{Simplify.}$$

$$= \frac{\cancel{h}(-4 + h)}{\cancel{h}} \qquad \text{Factor and divide out.}$$

$$= -4 + h, \quad h \neq 0 \qquad \text{Simplify.}$$

Next, take the limit of m_{sec} as h approaches 0.

$$m = \lim_{h \to 0} m_{\text{sec}}$$

$$= \lim_{h \to 0} (-4 + h)$$

$$= -4$$

The graph has a slope of -4 at the point $(-2, 4)$, as shown in Figure 12.25.

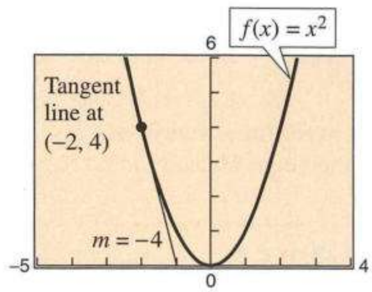


Figure 12.25

EXAMPLE 4 Finding the Slope of a Graph

Find the slope of $f(x) = -2x + 4$.

Solution

You know from your study of linear functions that the line given by $f(x) = -2x + 4$ has a slope of -2, as shown in Figure 12.26. This conclusion is consistent with that obtained by the limit definition of slope.

$$m = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[-2(x + h) + 4] - (-2x + 4)}{h}$$

$$= \lim_{h \to 0} \frac{-2x - 2h + 4 + 2x - 4}{h}$$

$$= \lim_{h \to 0} \frac{-2\cancel{h}}{\cancel{h}}$$

$$= -2$$

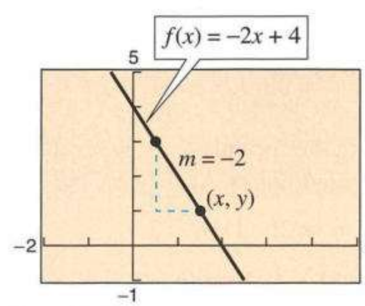


Figure 12.26

It is important that you see the difference between the ways the difference quotients were set up in Examples 3 and 4. In Example 3, you were finding the slope of a graph at a specific point $(c, f(c))$. To find the slope, you can use the following form of the difference quotient.

$$m = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \qquad \text{Slope at specific point}$$

In Example 4, however, you were finding a formula for the slope at *any* point on the graph. In such cases, you should use x, rather than c, in the difference quotient.

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{Formula for slope}$$

Except for linear functions, this form will always produce a function of x, which can then be evaluated to find the slope at any desired point.

EXAMPLE 5 Finding a Formula for the Slope of a Graph

Find a formula for the slope of the graph of $f(x) = x^2 + 1$. What is the slope at the points $(-1, 2)$ and $(2, 5)$?

Solution

$$m_{\text{sec}} = \frac{f(x+h) - f(x)}{h} \qquad \text{Set up difference quotient.}$$

$$= \frac{[(x+h)^2 + 1] - (x^2+1)}{h} \qquad \text{Use } f(x) = x^2 + 1.$$

$$= \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} \qquad \text{Expand terms.}$$

$$= \frac{2xh + h^2}{h} \qquad \text{Simplify.}$$

$$= \frac{\cancel{h}(2x + h)}{\cancel{h}} \qquad \text{Factor and divide out.}$$

$$= 2x + h, \quad h \neq 0 \qquad \text{Simplify.}$$

Next, take the limit of m_{sec} as h approaches 0.

$$m = \lim_{h \to 0} m_{\text{sec}}$$
$$= \lim_{h \to 0} (2x + h) = 2x$$

Using the formula $m = 2x$ for the slope at $(x, f(x))$, you can find the slope at the specified points. At $(-1, 2)$, the slope is

$$m = 2(-1) = -2$$

and at $(2, 5)$, the slope is

$$m = 2(2) = 4.$$

The graph of f is shown in Figure 12.27.

STUDY TIP

Try verifying the result in Example 5 by graphing the function and the tangent lines at $(-1, 2)$ and $(2, 5)$ as

$$y_1 = x^2 + 1$$
$$y_2 = -2x$$
$$y_3 = 4x - 3$$

in the same viewing window. Some graphing utilities even have the capability of automatically graphing the tangent line to a curve at a given point. If you have such a graphing utility, try verifying Example 5.

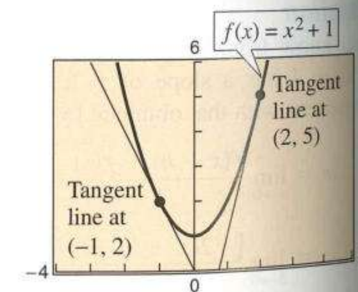
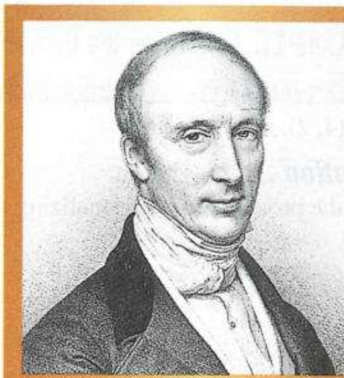


Figure 12.27

The Derivative of a Function

In Example 5, you started with the function $f(x) = x^2 + 1$ and used the limit process to derive another function, $m = 2x$, that represents the slope of the graph of f at the point $(x, f(x))$. This derived function is called the **derivative** of f at x. It is denoted by $f'(x)$, which is read as "f prime of x."



Augustin Louis Cauchy (1789–1857), a leading French mathematician, developed many of the basic concepts of limits and continuity. He defined the derivative.

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Definition of the Derivative

The **derivative** of f at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists.

Remember that the derivative $f'(x)$ is a formula for the slope of the tangent line to the graph of f at the point $(x, f(x))$.

EXAMPLE 6 Finding a Derivative

Find the derivative of $f(x) = 3x^2 - 2x$.

Solution

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{[3(x+h)^2 - 2(x+h)] - (3x^2 - 2x)}{h} \\
&= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 3x^2 + 2x}{h} \\
&= \lim_{h \to 0} \frac{6xh + 3h^2 - 2h}{h} \\
&= \lim_{h \to 0} \frac{\cancel{h}(6x + 3h - 2)}{\cancel{h}} \\
&= \lim_{h \to 0} (6x + 3h - 2) \\
&= 6x - 2
\end{aligned}$$

So, the derivative of $f(x) = 3x^2 - 2x$ is

$$f'(x) = 6x - 2.$$

Exploration

Use a graphing utility to graph the function $f(x) = 3x^2 - 2x$. Use the *trace* feature to approximate the coordinates of the vertex of this parabola. Then use the derivative of $f(x) = 3x^2 - 2x$ to find the slope of the tangent line at the vertex. Make a conjecture about the slope of the tangent line at the vertex of an arbitrary parabola.

Note that in addition to $f'(x)$, other notations can be used to denote the derivative of $y = f(x)$. The most common are

$$\frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad \text{and} \quad D_x[y].$$

EXAMPLE 7 Using the Derivative

Find $f'(x)$ for $f(x) = \sqrt{x}$. Then find the slope of the graph of f at the points $(1, 1)$ and $(4, 2)$.

Solution

Use the procedure for rationalizing numerators, as discussed in Section 12.2.

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right) \\
&= \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\
&= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}
$$

At the point $(1, 1)$, the slope is

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

At the point $(4, 2)$, the slope is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The graph of f is shown in Figure 12.28.

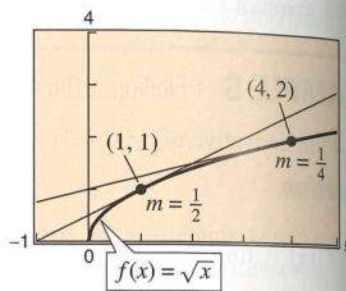


Figure 12.28

Writing About Math *Using a Derivative to Find Slope*

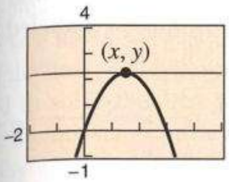
In many applications, it is convenient to use a variable other than x as the independent variable. Complete the following limit process to find the derivative of $f(t) = 3/t$. Then use the result to find the slope of the graph of $f(t) = 3/t$ at the point $(3, 1)$.

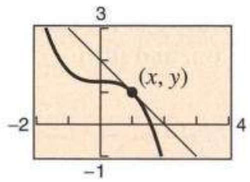
$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h \to 0} \frac{\dfrac{3}{t+h} - \dfrac{3}{t}}{h} = \cdots$$

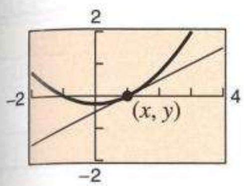
Write a short paragraph summarizing your findings.

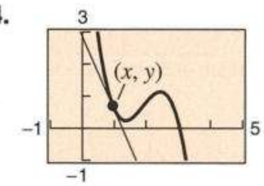
12.3 Exercises

In Exercises 1–4, approximate the slope of the curve at the point (x, y).

1. 

2. 

3. 

4. 

In Exercises 5–8, use a graphing utility to graph the function and the tangent line at the point $(1, f(1))$. Use the graph to approximate the slope of the tangent line.

5. $f(x) = x^2 - 2$
6. $f(x) = x^2 - 2x + 1$
7. $f(x) = \sqrt{2 - x}$
8. $f(x) = \dfrac{3}{2 - x}$

In Exercises 9–16, use the limit process to find the slope of the graph of the function at the specified point. Use a graphing utility to confirm your result.

9. $g(x) = 5 - 2x,\ (1, 3)$
10. $h(x) = 2x + 5,\ (-1, -3)$
11. $g(x) = x^2 - 4x,\ (3, -3)$
12. $f(x) = 10x - 2x^2,\ (3, 12)$
13. $g(x) = \dfrac{4}{x},\ (2, 2)$
14. $g(x) = \dfrac{1}{x - 2},\ \left(4, \dfrac{1}{2}\right)$
15. $h(x) = \sqrt{x},\ (9, 3)$
16. $h(x) = \sqrt{x + 10},\ (-1, 3)$

In Exercises 17–20, find a formula for the slope of the graph. Then use it to find the slope at the two points.

17. $g(x) = 4 - x^2$
 (a) $(0, 4)$
 (b) $(-1, 3)$
18. $g(x) = x^3$
 (a) $(1, 1)$
 (b) $(-2, -8)$
19. $g(x) = \dfrac{1}{x + 4}$
 (a) $\left(0, \frac{1}{4}\right)$
 (b) $\left(-2, \frac{1}{2}\right)$
20. $g(x) = \sqrt{x - 1}$
 (a) $(5, 2)$
 (b) $(10, 3)$

In Exercises 21–26, find the derivative of the function.

21. $f(x) = 5$
22. $f(x) = -5x + 2$
23. $g(x) = 9 - \frac{1}{3}x$
24. $f(x) = x^2 - 3x + 4$
25. $f(x) = \dfrac{1}{x^2}$
26. $h(s) = \dfrac{1}{\sqrt{s + 1}}$

In Exercises 27–30, find the slope of the graph of f at the given point. Use the result to find an equation of the tangent line to the graph at the point. Use a graphing utility to graph the function and the tangent line.

27. $f(x) = x^2 - 1,\ (2, 3)$
28. $f(x) = x^3 - x,\ (2, 6)$
29. $f(x) = \sqrt{x + 1},\ (3, 2)$
30. $f(x) = 2x + \dfrac{4}{x},\ (2, 6)$

In Exercises 31–34, use a graphing utility to graph f over the interval $[-2, 2]$ and complete the table. Compare the value of the first derivative with a visual approximation of the slope of the graph.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$									
$f'(x)$									

31. $f(x) = \frac{1}{2}x^2$
32. $f(x) = \frac{1}{4}x^3$
33. $f(x) = \sqrt{x + 3}$
34. $f(x) = \dfrac{x^2 - 4}{x + 4}$

In Exercises 35–38, find the derivative of f. Use the derivative to determine any points on the graph of f where the tangent line is horizontal. Use a graphing utility to verify your results.

35. $f(x) = x^2 - 4x + 3$
36. $f(x) = x^3 + 3x$
37. $f(x) = 3x^3 - 9x$
38. $f(x) = 3x^4 + 4x^3$

39. ***Federal Debt*** The table shows the per capita federal debt (in dollars) for the United States for selected years. (Source: U.S. Treasury Department)

Year	1950	1960	1970
Per Capita Debt	\$1688	\$1572	\$1807

Year	1980	1990	2000
Per Capita Debt	\$3981	\$12,848	\$20,788

(a) Use the regression capabilities of a graphing utility to fit a third-degree polynomial to the data. Let t be the time in years, with $t = 0$ corresponding to 1950.

(b) Use a graphing utility to graph the model found in part (a). Estimate the slope of the graph when $t = 30$ and give an interpretation of the result.

(c) Use a graphing utility to graph the tangent line to the model when $t = 30$. Compare the slope given by the graphing utility with the estimate in part (b).

40. ***Market Research*** The data in the table shows the number N (in thousands) of books sold when the price is p (in dollars).

p	\$10	\$15	\$20	\$25	\$30	\$35
N	900	630	396	227	102	36

(a) Use the regression capabilities of a graphing utility to fit a second-degree polynomial to the data.

(b) Use a graphing utility to graph the model found in part (a). Estimate the slope of the graph when $p = \$15$ and $p = \$30$.

(c) Use a graphing utility to graph the tangent line to the model when $p = \$15$ and $p = \$30$. Compare the slopes given by the graphing utility with your estimates in part (b).

(d) The slopes of the tangent lines at $p = \$15$ and $p = \$30$ are not the same. Explain what this means to the company selling the books.

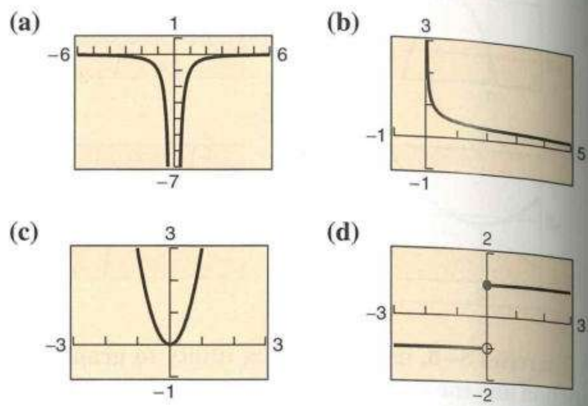
Synthesis

True or False? **In Exercises 41 and 42, determine whether the statement is true or false. Justify your answer.**

41. The slope of the graph of $y = x^2$ is different at every point of the graph of f.

42. A tangent line to a graph can intersect the graph only at the point of tangency.

In Exercises 43–46, match the function with the graph of its *derivative*. It is not necessary to find the derivative of the function. [The graphs are labeled (a), (b), (c), and (d).]



43. $f(x) = \sqrt{x}$

44. $f(x) = \dfrac{1}{x}$

45. $f(x) = |x|$

46. $f(x) = x^3$

47. ***Think About It*** Sketch the graph of a function whose derivative is always positive.

48. ***Think About It*** Sketch the graph of a function whose derivative is always negative.

49. ***Think About It*** Sketch the graph of a function for which $f'(x) < 0$ for $x < 1$, $f'(x) \geq 0$ for $x > 1$, and $f'(1) = 0$.

Review

In Exercises 50–53, sketch the graph of the rational function. As sketching aids, check for intercepts, symmetry, vertical asymptotes, and horizontal asymptotes. Use a graphing utility to verify your graph.

50. $f(x) = \dfrac{1}{x^2 - x - 2}$

51. $f(x) = \dfrac{x - 2}{x^2 - 4x + 3}$

52. $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

53. $f(x) = \dfrac{x^2 - 16}{x + 4}$

In Exercises 54–57, find the cross product of the vectors.

54. $\langle 1, 1, 1\rangle, \langle 2, 1, -1\rangle$

55. $\langle -10, 0, 6\rangle, \langle 7, 0, 0\rangle$

56. $\langle -4, 10, 0\rangle, \langle 4, -1, 0\rangle$

57. $\langle 8, -7, 14\rangle, \langle -1, 8, 4\rangle$

12.4 Limits at Infinity and Limits of Sequences

Limits at Infinity and Horizontal Asymptotes

As pointed out at the beginning of this chapter, there are two basic problems in calculus: finding **tangent lines** and finding the **area** of a region. In Section 12.3, you saw how limits can be used to solve the tangent line problem. In this section and the next, you will see how a different type of limit, a *limit at infinity*, can be used to solve the area problem. To get an idea of what is meant by a limit at infinity, consider the function $f(x) = (x + 1)/2x$. The graph of f is shown in Figure 12.29. From earlier work, you know that $y = \frac{1}{2}$ is a horizontal asymptote of the graph of this function. Using limit notation, this can be written as follows.

$$\lim_{x \to -\infty} f(x) = \frac{1}{2} \qquad \text{Horizontal asymptote to the left}$$

$$\lim_{x \to \infty} f(x) = \frac{1}{2} \qquad \text{Horizontal asymptote to the right}$$

These limits mean that the value of $f(x)$ gets arbitrarily close to $\frac{1}{2}$ as x decreases or increases without bound.

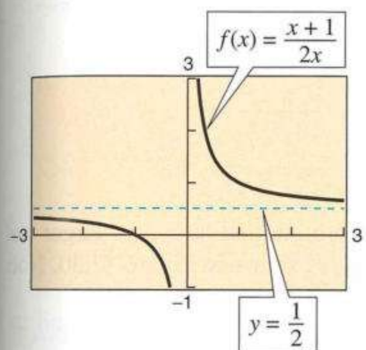


Figure 12.29

What You Should Learn:

- How to evaluate limits of functions at infinity
- How to find limits of sequences

Why You Should Learn It:

Finding limits at infinity is useful in business applications. For instance, in Exercise 40 on page 842, you are asked to find a limit at infinity to determine the average cost of a pen as the number of pens sold approaches infinity.

Jim Pickerell/Stock Boston

Definition of Limits at Infinity

If f is a function and L_1 and L_2 are real numbers, the statements

$$\lim_{x \to -\infty} f(x) = L_1 \qquad \text{Limit as } x \text{ approaches } -\infty$$

and

$$\lim_{x \to \infty} f(x) = L_2 \qquad \text{Limit as } x \text{ approaches } \infty$$

denote the **limits at infinity.** The first is read "*the limit of $f(x)$ as x approaches $-\infty$ is L_1*," and the second is read "*the limit of $f(x)$ as x approaches ∞ is L_2*."

To help evaluate limits at infinity, you can use the following.

Limits at Infinity

If r is a positive real number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0. \qquad \text{Limit toward the right}$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0. \qquad \text{Limit toward the left}$$

Limits at infinity share many of the properties of limits listed in Section 12.1. Some of these properties are demonstrated in the next example.

Exploration

Use a graphing utility to graph the two functions

$$y_1 = \frac{1}{\sqrt{x}}$$

$$y_2 = \frac{1}{\sqrt[3]{x}}$$

in the standard viewing window. Why doesn't y_1 appear to the left of the y-axis? How does this relate to the statement at the left about the infinite limit

$$\lim_{x \to -\infty} \frac{1}{x^r}?$$

EXAMPLE 1 Evaluating a Limit at Infinity

Find the limit.

$$\lim_{x \to \infty} \left(4 - \frac{3}{x^2}\right)$$

Algebraic Solution

Use the properties of limits listed in Section 12.1.

$$\begin{aligned}
\lim_{x \to \infty} \left(4 - \frac{3}{x^2}\right) &= \lim_{x \to \infty} 4 - \lim_{x \to \infty} \frac{3}{x^2} \\
&= \lim_{x \to \infty} 4 - 3\left(\lim_{x \to \infty} \frac{1}{x^2}\right) \\
&= 4 - 3(0) \\
&= 4
\end{aligned}$$

So, the limit of $f(x) = 4 - (3/x^2)$ as x approaches ∞ is 4.

Graphical Solution

Use a graphing utility to graph $y = 4 - 3/x^2$.

Then use the *trace* feature to determine that as x gets larger and larger, y gets closer and closer to 4, as shown in Figure 12.30. Note that the line $y = 4$ is a horizontal asymptote to the right.

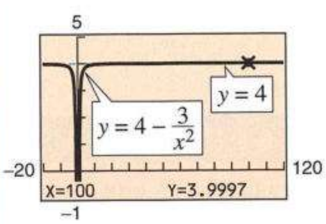


Figure 12.30

In Figure 12.30, it appears that the line $y = 4$ is also a horizontal asymptote *to the left*. You can verify this by showing that

$$\lim_{x \to -\infty} \left(4 - \frac{3}{x^2}\right) = 4.$$

The graph of a rational function need not have a horizontal asymptote. If it does, however, its left and right asymptotes must be the same.

EXAMPLE 2 Comparing Limits at Infinity

Find the limit as x approaches ∞ for each of the following functions.

a. $f(x) = \dfrac{-2x + 3}{3x^2 + 1}$ **b.** $f(x) = \dfrac{-2x^2 + 3}{3x^2 + 1}$ **c.** $f(x) = \dfrac{-2x^3 + 3}{3x^2 + 1}$

Solution

In each case, begin by dividing both the numerator and denominator by x^2, the highest-powered term in the denominator.

a.
$$\lim_{x\to\infty} \frac{-2x + 3}{3x^2 + 1} = \lim_{x\to\infty} \frac{-\dfrac{2}{x} + \dfrac{3}{x^2}}{3 + \dfrac{1}{x^2}}$$
$$= \frac{-0 + 0}{3 + 0}$$
$$= 0$$

b.
$$\lim_{x\to\infty} \frac{-2x^2 + 3}{3x^2 + 1} = \lim_{x\to\infty} \frac{-2 + \dfrac{3}{x^2}}{3 + \dfrac{1}{x^2}}$$
$$= \frac{-2 + 0}{3 + 0}$$
$$= -\frac{2}{3}$$

c.
$$\lim_{x\to\infty} \frac{-2x^3 + 3}{3x^2 + 1} = \lim_{x\to\infty} \frac{-2x + \dfrac{3}{x^2}}{3 + \dfrac{1}{x^2}}$$

In this case, you can conclude that the limit does not exist because the numerator decreases without bound as the denominator approaches 3.

Exploration

Use a graphing utility to graph the rational functions

$$y_1 = \frac{-2x + 3}{3x^2 + 1}$$
$$y_2 = \frac{-2x^2 + 3}{3x^2 + 1}$$
$$y_3 = \frac{-2x^3 + 3}{3x^2 + 1}.$$

In each case, make a conjecture about the limit as x approaches ∞. Compare your answers with the algebraic results shown in Example 2.

In Example 2, observe that when the degree of the numerator is less than the degree of the denominator, the limit is 0. When the degrees of the numerator and denominator are equal, the limit is the ratio of the coefficients of the highest-powered terms. When the degree of the numerator is greater than the degree of the denominator, the limit does not exist.

This result seems reasonable when you realize that for large values of x, the highest-powered term of a polynomial is the most "influential" term. That is, a polynomial tends to behave as its highest-powered term behaves as x approaches positive or negative infinity.

Limits at Infinity for Rational Functions

For the rational function $f(x) = N(x)/D(x)$, where

$$N(x) = a_n x^n + \cdot \cdot \cdot + a_0$$

and

$$D(x) = b_m x^m + \cdot \cdot \cdot + b_0$$

the limit as x approaches positive or negative infinity is as follows.

$$\lim_{x \to \pm\infty} f(x) = \begin{cases} 0, & n < m \\ \dfrac{a_n}{b_m}, & n = m \end{cases}$$

If $n > m$, the limit does not exist.

EXAMPLE 3 Finding the Average Cost

You are manufacturing a product that costs \$0.50 per unit to produce. Your initial investment is \$5000, which implies that the total cost of producing x units is $C = 0.5x + 5000$. The average cost per unit is

$$\overline{C} = \frac{C}{x} = \frac{0.5x + 5000}{x}.$$

Find the average cost per unit when (a) $x = 1000$, (b) $x = 10{,}000$, and (c) $x = 100{,}000$. (d) What is the limit of $\overline{C}$ when x approaches infinity?

Solution

a. When $x = 1000$, the average cost per unit is

$$\overline{C} = \frac{0.5(1000) + 5000}{1000} \qquad x = 1000$$

$$= \$5.50.$$

b. When $x = 10{,}000$, the average cost per unit is

$$\overline{C} = \frac{0.5(10{,}000) + 5000}{10{,}000} \qquad x = 10{,}000$$

$$= \$1.00.$$

c. When $x = 100{,}000$, the average cost per unit is

$$\overline{C} = \frac{0.5(100{,}000) + 5000}{100{,}000} \qquad x = 100{,}000$$

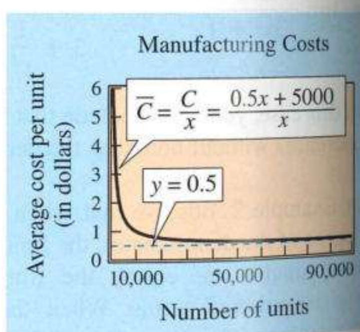
$$= \$0.55.$$

d. As x approaches infinity, the limit of $\overline{C}$ is

$$\lim_{x \to \infty} \frac{0.5x + 5000}{x} \qquad x \to \infty$$

$$= \$0.50.$$

The graph of $\overline{C}$ is shown in Figure 12.31.



As $x \to \infty$ the average cost per unit approaches \$0.50.

Figure 12.31

Limits of Sequences

Limits of sequences have many of the same properties as limits of functions. For instance, consider the sequence whose nth term is $a_n = 1/2^n$.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

As n increases without bound, the terms of this sequence gets closer and closer to 0, and the sequence is said to **converge** to 0. Using limit notation, you can write

$$\lim_{n\to\infty} \frac{1}{2^n} = 0.$$

The following relationship shows how limits of functions of x can be used to evaluate the limit of a sequence.

Limit of a Sequence

Let f be a function of a real variable, such that

$$\lim_{x\to\infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n\to\infty} a_n = L.$$

A sequence that does not converge is said to **diverge.** For instance, the sequence $1, -1, 1, -1, 1, \ldots$ diverges.

STUDY TIP

There are a number of ways to use a graphing utility to generate the terms of a sequence. For instance, you can display the first 10 terms of the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

using the *sequence* command or using the *table* feature. Consult your user's manual for instructions on how to generate the terms of a sequence.

EXAMPLE 4 Finding the Limit of a Sequence

Find the limit of each of the following sequences.

a. $a_n = \dfrac{2n+1}{n+4}$

b. $b_n = \dfrac{2n+1}{n^2+4}$

c. $c_n = \dfrac{2n^2+1}{4n^2}$

Solution

a. $\displaystyle\lim_{n\to\infty} \frac{2n+1}{n+4} = 2$ $\quad \frac{3}{5}, \frac{5}{6}, \frac{7}{7}, \frac{9}{8}, \frac{11}{9}, \frac{13}{10}, \ldots \to 2$

b. $\displaystyle\lim_{n\to\infty} \frac{2n+1}{n^2+4} = 0$ $\quad \frac{3}{5}, \frac{5}{8}, \frac{7}{13}, \frac{9}{20}, \frac{11}{29}, \frac{13}{40}, \ldots \to 0$

c. $\displaystyle\lim_{n\to\infty} \frac{2n^2+1}{4n^2} = \frac{1}{2}$ $\quad \frac{3}{4}, \frac{9}{16}, \frac{19}{36}, \frac{33}{64}, \frac{51}{100}, \frac{73}{144}, \ldots \to \frac{1}{2}$

In the next section, you will encounter limits of sequences such as that shown in Example 5. A strategy for evaluating such limits is to begin by writing the nth term in standard rational function form. Then you can determine the limit by comparing the degrees of the numerator and denominator.

EXAMPLE 5 Finding the Limit of a Sequence

Find the limit of the sequence whose nth term is

$$a_n = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right].$$

Algebraic Solution

Begin by writing the nth term in standard rational function form—as the ratio of two polynomials.

$$a_n = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \qquad \text{Write original } n\text{th term.}$$

$$= \frac{8(n)(n+1)(2n+1)}{6n^3} \qquad \text{Multiply fractions.}$$

$$= \frac{8n^3 + 12n^2 + 4n}{3n^3} \qquad \text{Write in standard rational form.}$$

From this form, you can see that the degree of the numerator is equal to the degree of the denominator. So, the limit of the sequence is the ratio of the coefficients of the highest-powered terms.

$$\lim_{n\to\infty} \frac{8n^3 + 12n^2 + 4n}{3n^3} = \frac{8}{3}$$

Numerical Solution

Construct a table such as the one shown below, which shows the value of a_n as n becomes larger and larger.

n	a_n
1	8
10	3.08
100	2.707
1000	2.671
10,000	2.667

From the table, you can estimate that as n approaches ∞, a_n gets closer and closer to $\frac{8}{3} \approx 2.667$.

Writing About Math *Comparing Rates of Convergence*

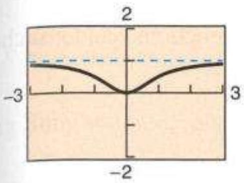
In the table in Example 5 above, the values of a_n are approaching their limit of $\frac{8}{3}$ rather slowly. (The first term to be accurate to three decimal places is $a_{4801} \approx 2.667$.) Each of the following sequences converges to 0. Which converges most rapidly? Which converges most slowly? Why? Write a short paragraph discussing your conclusions.

a. $a_n = \dfrac{1}{n}$ **b.** $b_n = \dfrac{1}{n^2}$ **c.** $c_n = \dfrac{1}{2^n}$ **d.** $d_n = \dfrac{1}{n!}$ **e.** $h_n = \dfrac{2^n}{n!}$

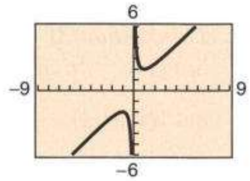
12.4 Exercises

In Exercises 1–4, match the function with its graph using horizontal asymptotes as aids. [The graphs are labeled (a), (b), (c), and (d).]

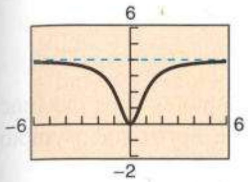
(a)



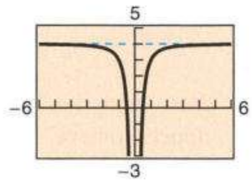
(b)



(c)



(d)



1. $f(x) = \dfrac{4x^2}{x^2+1}$ **2.** $f(x) = \dfrac{x^2}{x^2+1}$

3. $f(x) = 4 - \dfrac{1}{x^2}$ **4.** $f(x) = x + \dfrac{1}{x}$

In Exercises 5–18, find the limit (if it exists). Use a graphing utility to graphically verify your result.

5. $\displaystyle\lim_{x\to\infty} \frac{3}{x^2}$ **6.** $\displaystyle\lim_{x\to\infty} \frac{5}{2x}$

7. $\displaystyle\lim_{x\to\infty} \frac{3+x}{3-x}$ **8.** $\displaystyle\lim_{x\to\infty} \frac{1-6x}{1+5x}$

9. $\displaystyle\lim_{x\to-\infty} \frac{4x-3}{2x+1}$ **10.** $\displaystyle\lim_{x\to-\infty} \frac{3x^2+1}{4x^2-5}$

11. $\displaystyle\lim_{t\to\infty} \frac{t^2}{t+3}$ **12.** $\displaystyle\lim_{y\to\infty} \frac{4y^4}{y^2+3}$

13. $\displaystyle\lim_{x\to-\infty} \frac{-(x^2+3)}{(2-x)^2}$ **14.** $\displaystyle\lim_{x\to\infty} \frac{2x^2-6}{(x-1)^2}$

15. $\displaystyle\lim_{x\to-\infty} \left[\frac{x}{(x+1)^2} - 4\right]$ **16.** $\displaystyle\lim_{x\to\infty} \left[7 + \frac{2x^2}{(x+3)^2}\right]$

17. $\displaystyle\lim_{t\to\infty} \left(\frac{1}{3t^2} - \frac{5t}{t+2}\right)$

18. $\displaystyle\lim_{x\to\infty} \left[\frac{x}{2x+1} + \frac{3x^2}{(x-3)^2}\right]$

In Exercises 19–22, use a graphing utility to graph the function and verify that the horizontal asymptote corresponds to the limit at infinity.

19. $y = \dfrac{3x}{1-x}$ **20.** $y = \dfrac{x^2}{x^2+4}$

21. $y = \dfrac{2x}{1-x^2}$ **22.** $y = 1 - \dfrac{3}{x^2}$

Numerical and Graphical Analysis **In Exercises 23–26, complete the table and estimate the limit numerically as x approaches infinity. Then use the graphing utility to graph the function and graphically estimate the limit.**

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

23. $f(x) = x - \sqrt{x^2+2}$ **24.** $f(x) = 3x - \sqrt{9x^2+1}$

25. $f(x) = 3\left(2x - \sqrt{4x^2+x}\right)$

26. $f(x) = 4\left(4x - \sqrt{16x^2-x}\right)$

In Exercises 27–34, write the first five terms of the sequence and find the limit of the sequence (if it exists). Assume n begins with 1.

27. $a_n = \dfrac{n+1}{n^2+1}$ **28.** $a_n = \dfrac{4n-1}{n+3}$

29. $a_n = \dfrac{n^2}{3n+2}$ **30.** $a_n = \dfrac{4n-5}{3}$

31. $a_n = \dfrac{(n+1)!}{n!}$ **32.** $a_n = \dfrac{(3n-1)!}{(3n+1)!}$

33. $a_n = \dfrac{(-1)^n}{n}$ **34.** $a_n = \dfrac{(-1)^{n+1}}{n^2}$

In Exercises 35–38, find the limit of the sequence. Then verify the limit numerically by using a graphing utility to complete the table.

n	10^0	10^1	10^2	10^3	10^4	10^5	10^6
a_n							

35. $a_n = \dfrac{1}{n}\left(n + \dfrac{1}{n}\left[\dfrac{n(n+1)}{2}\right]\right)$

36. $a_n = \dfrac{4}{n}\left(n + \dfrac{4}{n}\left[\dfrac{n(n+1)}{2}\right]\right)$

37. $a_n = \dfrac{16}{n^3}\left[\dfrac{n(n+1)(2n+1)}{6}\right]$

38. $a_n = \dfrac{n(n+1)}{n^2} - \dfrac{1}{n^4}\left[\dfrac{n(n+1)}{2}\right]^2$

39. ***Demographics*** The table shows the number N (in thousands) of high school graduates at the end of each decade for the years 1900 through 2000. (Source: U.S. Department of Education)

Year	1900	1910	1920	1930	1940
N	62	111	231	592	1140

Year	1950	1960	1970	1980	1990	2000
N	1063	1627	2589	2748	2503	2856

A model for this data is

$$N = \frac{292.78 + 8218.50t}{1000 - 20.18t + 0.13t^2}, \quad 0 \le t \le 100$$

where t is the time (in years), with $t = 0$ corresponding to 1900.

(a) Use a graphing utility to create a scatter plot of the data and graph the model in the same viewing window. How do they compare?

(b) Use the model to estimate the number of high school graduates in 1980.

(c) Discuss why this model should *not* be used to predict the numbers of graduates in future years.

40. ***Average Cost*** The cost function for a certain model of pen is

$$C = 1.35x + 4570$$

where C is measured in dollars and x is the number of pens produced.

(a) Find the average cost per unit when $x = 100$ and $x = 1000$.

(b) Determine the limit of the average cost function as x approaches infinity. Explain the meaning of the limit in the context of the problem.

Synthesis

True or False? **In Exercises 41 and 42, determine whether the statement is true or false. Justify your answer.**

41. Every rational function has a horizontal asymptote.

42. If $\lim f(x) = \infty$, then the limit of $f(x)$ exists.

43. ***Think About It*** Find the functions f and g such that $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$ but $\lim_{x\to c}[f(x) - g(x)] \neq \infty$.

44. ***Think About It*** Graph the function

$$f(x) = \frac{x}{\sqrt{x^2+1}}.$$

How many horizontal asymptotes does the function appear to have? What are the horizontal asymptotes?

Exploration **In Exercises 45–48, use a graphing utility to create a scatter plot of the terms of the sequence. Decide whether the sequence converges or diverges. If it converges, estimate its limit.**

45. $a_n = 4\left(\frac{2}{3}\right)^n$

46. $a_n = 3\left(\frac{3}{2}\right)^n$

47. $a_n = \dfrac{3[1 - (1.5)^n]}{1 - 1.5}$

48. $a_n = \dfrac{3[1 - (0.5)^n]}{1 - 0.5}$

Review

In Exercises 49–52, find all the real zeros of the polynomial function. Use a graphing utility to graph the function and verify that the real zeros are the x-intercepts of the graph of the function.

49. $f(x) = x^4 - x^3 - 20x^2$

50. $f(x) = x^5 + x^3 - 6x$

51. $f(x) = x^3 - 3x^2 + 2x - 6$

52. $f(x) = x^3 - 4x^2 - 25x + 100$

In Exercises 53–56, find the sum.

53. $\sum_{i=1}^{6}(2i + 3)$

54. $\sum_{i=0}^{4} 5i^2$

55. $\sum_{k=1}^{10} 15$

56. $\sum_{k=0}^{8} \dfrac{3}{k^2+1}$

12.5 The Area Problem

Limits of Summations

Earlier in the text, you used the concept of a limit to obtain a formula for the sum S of an infinite geometric sequence

$$S = a_1 + a_1 r + a_1 r^2 + \cdots = \sum_{i=1}^{\infty} a_1 r^{i-1} = \frac{a_1}{1-r}, \quad |r| < 1.$$

Using limit notation, this sum can be written as

$$\begin{aligned} S &= \lim_{n\to\infty} \sum_{i=1}^{n} a_1 r^{i-1} \\ &= \lim_{n\to\infty} \frac{a_1(1-r^n)}{1-r} \\ &= \frac{a_1}{1-r}. \end{aligned}$$

The following summation formulas and properties are used to evaluate finite and infinite summations.

Summation Formulas and Properties

1. $\displaystyle\sum_{i=1}^{n} c = cn$
2. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$
5. $\displaystyle\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$
6. $\displaystyle\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$

EXAMPLE 1 Evaluating a Summation

Using the second summation formula with $n = 200$, you can write

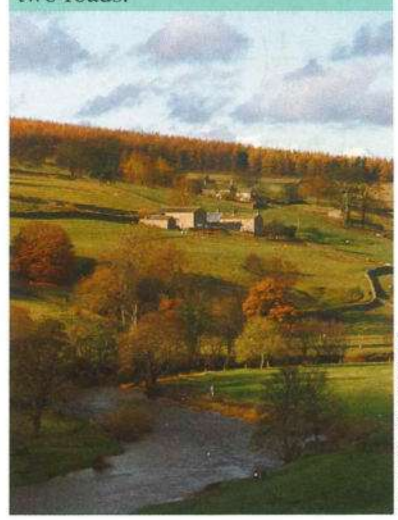
$$\begin{aligned} \sum_{i=1}^{200} i &= 1 + 2 + 3 + 4 + \cdots + 200 \\ &= \frac{n(n+1)}{2} \\ &= \frac{200(201)}{2} \\ &= 20{,}100. \end{aligned}$$

What You Should Learn:

- How to find limits of summations
- How to use rectangles to approximate areas of plane regions
- How to use limits of summations to find areas of plane regions

Why You Should Learn It:

The limit of a summation is useful in determining areas of plane regions. For instance, in Exercise 36 on page 850, you are asked to find the limit of a summation to determine the area of a parcel of land bounded by a stream and two roads.



Adam Woolfitt/CORBIS

EXAMPLE 2 Evaluating a Summation

Evaluate the summation

$$S = \sum_{i=1}^{n} \frac{i+2}{n^2} = \frac{3}{n^2} + \frac{4}{n^2} + \frac{5}{n^2} + \cdot \cdot \cdot + \frac{n+2}{n^2}$$

for $n = 10, 100, 1000$, and $10{,}000$.

Solution

Begin by applying summation formulas and properties to simplify S. In the second line of the solution, note that $1/n^2$ can be factored out of the sum because n is considered to be constant. You could not factor i out of the summation because i is the (variable) index of summation.

$$S = \sum_{i=1}^{n} \frac{i+2}{n^2} \qquad \text{Write original form of summation.}$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} (i+2) \qquad \text{Factor constant } 1/n^2 \text{ out of sum.}$$

$$= \frac{1}{n^2} \left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} 2 \right) \qquad \text{Write as two sums.}$$

$$= \frac{1}{n^2} \left[\frac{n(n+1)}{2} + 2n \right] \qquad \text{Apply Formulas 1 and 2.}$$

$$= \frac{1}{n^2} \left(\frac{n^2 + 5n}{2} \right) \qquad \text{Add fractions.}$$

$$= \frac{n+5}{2n} \qquad \text{Simplify.}$$

Now you can evaluate the sum by substituting the appropriate values of n, as shown in the following table.

n	10	100	1000	10,000
$\sum_{i=1}^{n} \frac{i+2}{n^2} = \frac{n+5}{2n}$	0.75	0.525	0.5025	0.50025

In Example 2, note that the sum appears to approach a limit as n increases. To find the limit of $(n + 5)/2n$ as n approaches infinity, you can use the techniques from Section 12.4 to write

$$\lim_{n \to \infty} \frac{n+5}{2n} = \frac{1}{2}.$$

STUDY TIP

Some graphing utilities have the capability of computing summations. Consult your user's manual for instructions.

Be sure you notice the strategy used in Example 2. Rather than separately evaluating the sums

$$\sum_{i=1}^{10} \frac{i+2}{n^2}, \qquad \sum_{i=1}^{100} \frac{i+2}{n^2}, \qquad \sum_{i=1}^{1000} \frac{i+2}{n^2}, \qquad \sum_{i=1}^{10,000} \frac{i+2}{n^2}$$

it was more efficient first to convert to rational form.

$$S = \underbrace{\sum_{i=1}^{n} \frac{i+2}{n^2}}_{\text{Summation form}} = \underbrace{\frac{n+5}{2n}}_{\text{Rational form}}$$

With this rational form, each sum can be evaluated by simply substituting appropriate values of n.

EXAMPLE 3 Finding the Limit of a Summation

Find the limit of $S(n)$ as $n \to \infty$.

$$S(n) = \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^2 \left(\frac{1}{n}\right)$$

Solution

Begin by rewriting the summation in rational form.

$$S(n) = \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^2 \left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n} \left(\frac{n^2 + 2ni + i^2}{n^2}\right)\left(\frac{1}{n}\right) \qquad \text{Square } (1 + i/n) \text{ and write as a single fraction.}$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} (n^2 + 2ni + i^2) \qquad \text{Factor constant } 1/n^3 \text{ out of sum.}$$

$$= \frac{1}{n^3}\left(\sum_{i=1}^{n} n^2 + \sum_{i=1}^{n} 2ni + \sum_{i=1}^{n} i^2 \right) \qquad \text{Write as three sums.}$$

$$= \frac{1}{n^3}\left\{ n^3 + 2n\left[\frac{n(n+1)}{2}\right] + \frac{n(n+1)(2n+1)}{6} \right\} \qquad \text{Use summation formulas.}$$

$$= \frac{14n^3 + 9n^2 + n}{6n^3} \qquad \text{Simplify.}$$

In this rational form, you can now find the limit as $n \to \infty$.

$$\lim_{n\to\infty} S(n) = \lim_{n\to\infty} \frac{14n^3 + 9n^2 + n}{6n^3}$$

$$= \frac{14}{6}$$

$$= \frac{7}{3}$$

The Area Problem

You now have the tools needed to solve the second basic problem of calculus: the area problem. The problem is to find the *area* of the region R bounded by the graph of a nonnegative, continuous function f, the x-axis, and the vertical lines $x = a$ and $x = b$, as shown in Figure 12.32.

If the region R is a square, triangle, trapezoid, or semicircle, you can find its area by using a geometric formula. For more general regions, however, you must use a different approach—one that involves the limit of a summation. The basic strategy is to use a collection of rectangles that approximates the region R, as illustrated in Example 4.

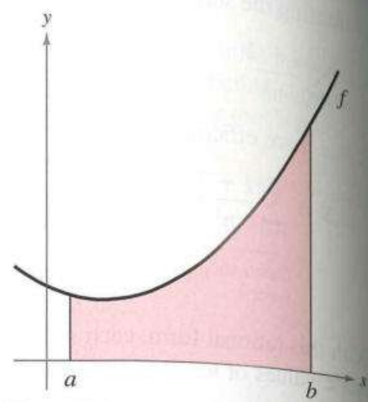


Figure 12.32

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 4 Approximating the Area of a Region

Use the five rectangles in Figure 12.33 to approximate the area of the region bounded by the graph of $f(x) = 6 - x^2$, the x-axis, and the lines $x = 0$ and $x = 2$.

Solution

Because the length of the interval along the x-axis is 2 and there are five rectangles, the width of each rectangle is $\frac{2}{5}$. The height of each rectangle can be obtained by evaluating f at the right endpoint of each interval. The five intervals are

$$\left[0, \frac{2}{5}\right], \quad \left[\frac{2}{5}, \frac{4}{5}\right], \quad \left[\frac{4}{5}, \frac{6}{5}\right], \quad \left[\frac{6}{5}, \frac{8}{5}\right], \quad \left[\frac{8}{5}, \frac{10}{5}\right].$$

Notice that the right endpoint of each interval is $\frac{2}{5}i$, for $i = 1, 2, 3, 4, 5$.

The sum of the areas of the five rectangles is

$$\sum_{i=1}^{5} \underbrace{f\left(\frac{2i}{5}\right)}_{\text{Height}} \underbrace{\left(\frac{2}{5}\right)}_{\text{Width}} = \sum_{i=1}^{5} \left[6 - \left(\frac{2i}{5}\right)^2\right]\left(\frac{2}{5}\right)$$

$$= \frac{212}{25}$$

$$= 8.48.$$

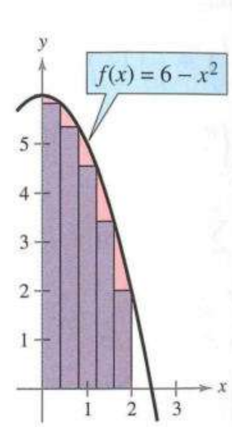


Figure 12.33

So, you can approximate the area of R as 8.48.

By increasing the number of rectangles used in Example 4, you can obtain closer and closer approximations of the area of the region. For instance, using 25 rectangles of width $\frac{2}{25}$ each, you can approximate the area to be $A \approx 9.17$. The following table shows even better approximations.

n	5	25	100	1000	5000
Approximate Area	8.48	9.17	9.29	9.33	9.33

The *exact* **area of a plane region** R is given by the limit of the sum of n rectangles as n approaches ∞.

Area of a Plane Region

Let f be continuous and nonnegative on the interval $[a, b]$. The **area** A of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \underbrace{f\left(a + \frac{(b-a)i}{n}\right)}_{\text{Height}} \underbrace{\left(\frac{b-a}{n}\right)}_{\text{Width}}.$$

EXAMPLE 5 Finding the Area of a Region

Find the area of the region bounded by the graph of $f(x) = x^2$ and the x-axis between $x = 0$ and $x = 1$.

Solution

Begin by finding the dimensions of the rectangles.

$$\textit{Width: } \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$\textit{Height: } f\left(a + \frac{(b-a)i}{n}\right) = f\left(0 + \frac{(1-0)i}{n}\right) = f\left(\frac{i}{n}\right) = \frac{i^2}{n^2}$$

Next, approximate the area as the sum of the areas of n rectangles.

$$\begin{aligned}
A &\approx \sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right)\left(\frac{b-a}{n}\right) \\
&= \sum_{i=1}^{n} \left(\frac{i^2}{n^2}\right)\left(\frac{1}{n}\right) \\
&= \sum_{i=1}^{n} \frac{i^2}{n^3} \\
&= \frac{1}{n^3} \sum_{i=1}^{n} i^2 \\
&= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \\
&= \frac{2n^3 + 3n^2 + n}{6n^3}
\end{aligned}$$

Finally, find the exact area by taking the limit as n approaches ∞.

$$A = \lim_{n \to \infty} \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{1}{3}$$

Figure 12.34 shows the region R.

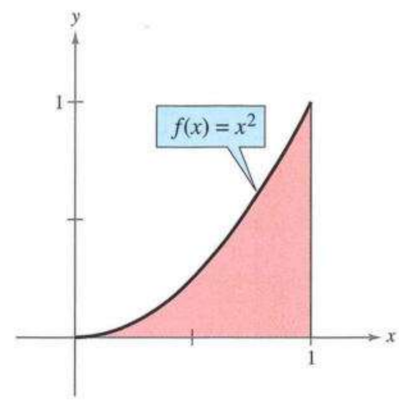


Figure 12.34

EXAMPLE 6 Finding the Area of a Region

Find the area of the region bounded by the graph of $f(x) = 3x - x^2$ and the x-axis between $x = 1$ and $x = 2$.

Solution

Begin by finding the dimensions of the rectangles.

$$\textit{Width: } \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$\begin{aligned}\textit{Height: } f\left(a + \frac{(b-a)i}{n}\right) &= f\left(1 + \frac{i}{n}\right)\\ &= 3\left(1 + \frac{i}{n}\right) - \left(1 + \frac{i}{n}\right)^2\\ &= 3 + \frac{3i}{n} - \left(1 + \frac{2i}{n} + \frac{i^2}{n^2}\right)\\ &= 2 + \frac{i}{n} - \frac{i^2}{n^2}\end{aligned}$$

Next, approximate the area as the sum of the areas of n rectangles.

$$\begin{aligned}A &\approx \sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right)\left(\frac{b-a}{n}\right)\\ &= \sum_{i=1}^{n}\left(2 + \frac{i}{n} - \frac{i^2}{n^2}\right)\left(\frac{1}{n}\right)\\ &= \frac{1}{n}\sum_{i=1}^{n} 2 + \frac{1}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^3}\sum_{i=1}^{n} i^2\\ &= \frac{1}{n}(2n) + \frac{1}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\\ &= 2 + \frac{n^2+n}{2n^2} - \frac{2n^3+3n^2+n}{6n^3}\\ &= 2 + \frac{1}{2} + \frac{1}{2n} - \frac{1}{3} - \frac{1}{2n} - \frac{1}{6n^2}\\ &= \frac{13}{6} - \frac{1}{6n^2}\end{aligned}$$

Finally, find the exact area by taking the limit as n approaches ∞.

$$\begin{aligned}A &= \lim_{n\to\infty}\left(\frac{13}{6} - \frac{1}{6n^2}\right)\\ &= \frac{13}{6}\end{aligned}$$

Figure 12.35 shows the region R.

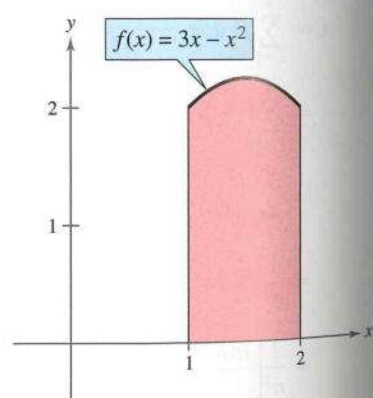


Figure 12.35

12.5 Exercises

In Exercises 1–6, evaluate the sum using the summation formulas and properties.

1. $\sum_{i=1}^{60} i$
2. $\sum_{i=1}^{30} i^2$
3. $\sum_{k=1}^{20} k^3$
4. $\sum_{k=1}^{50} (2k + 1)$
5. $\sum_{j=1}^{25} (j^2 + j)$
6. $\sum_{j=1}^{10} (j^3 - 3j^2)$

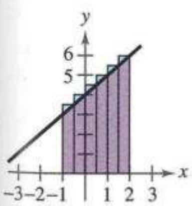
In Exercises 7–14, use the summation formulas and properties to rewrite the sum as a rational function $S(n)$. Use $S(n)$ to complete the table. Then find $\lim_{n\to\infty} S(n)$.

n	10^0	10^1	10^2	10^3	10^4
$S(n)$					

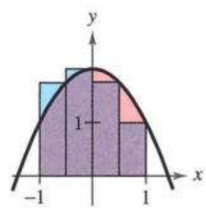
7. $\sum_{i=1}^{n} \frac{i^3}{n^4}$
8. $\sum_{i=1}^{n} \frac{i}{n^2}$
9. $\sum_{i=1}^{n} \frac{3}{n^3}(1 + i^2)$
10. $\sum_{i=1}^{n} \frac{2i + 3}{n^2}$
11. $\sum_{i=1}^{n} \left(\frac{i^2}{n^3} + \frac{2}{n}\right)\left(\frac{1}{n}\right)$
12. $\sum_{i=1}^{n} \left[3 - 2\left(\frac{i}{n}\right)\right]\left(\frac{1}{n}\right)$
13. $\sum_{i=1}^{n} \left[1 - \left(\frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right)$
14. $\sum_{i=1}^{n} \left[\frac{4}{n} + \left(\frac{2i}{n^2}\right)\right]\left(\frac{2i}{n}\right)$

In Exercises 15–18, approximate the area of the region using the indicated number of rectangles of equal width.

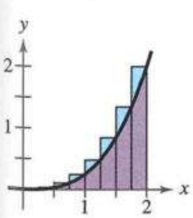
15. $f(x) = x + 4$



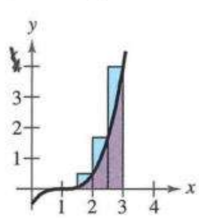
16. $f(x) = 2 - x^2$



17. $f(x) = \frac{1}{4}x^3$



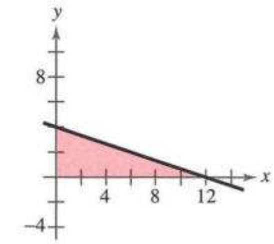
18. $f(x) = \frac{1}{2}(x - 1)^3$



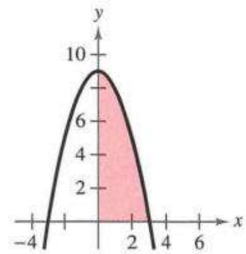
In Exercises 19–22, complete the table showing the approximate area of the region in the graph using n rectangles of equal width.

n	4	8	20	50
Approximate Area				

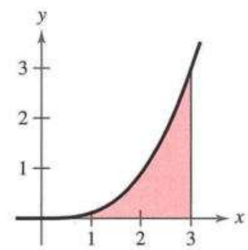
19. $f(x) = -\frac{1}{3}x + 4$



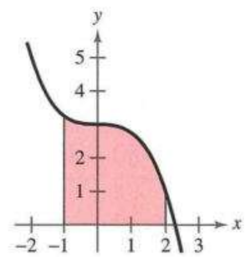
20. $f(x) = 9 - x^2$



21. $f(x) = \frac{1}{9}x^3$



22. $f(x) = 3 - \frac{1}{4}x^3$



In Exercises 23–30, use the limit process to find the area of the region between the graph of the function and the x-axis over the specified interval.

	Function	*Interval*
23.	$f(x) = 4x$	$[0, 1]$
24.	$f(x) = x + 2$	$[-2, 2]$
25.	$f(x) = 2 - x^2$	$[-1, 1]$
26.	$f(x) = \frac{1}{4}x^2 + 2$	$[1, 4]$
27.	$g(x) = 8 - x^3$	$[1, 2]$
28.	$g(x) = 4x - x^3$	$[0, 2]$
29.	$f(x) = \frac{1}{4}(x^2 + 4x)$	$[1, 4]$
30.	$f(x) = x^2 - x^3$	$[-1, 1]$

In Exercises 31–34, use the *integrate* feature of a graphing utility to find the area of the region between the graph of the function and the x-axis over the specified interval.

Function	*Interval*
31. $y = x - x^2$	$[0, 1]$
32. $y = (3 - x)\sqrt{x}$	$[0, 3]$
33. $y = \cos x$	$\left[0, \frac{\pi}{2}\right]$
34. $y = 6e^{-x}$	$[0, 2]$

35. ***Real Estate*** The boundaries of a parcel of land are two edges modeled by the coordinate axes and a stream modeled by the equation

$$y = (-3.0 \times 10^{-6})x^3 + 0.002x^2 - 1.05x + 400.$$

Use a graphing utility to graph the equation and find the area of the property. Assume all distances are measured in feet.

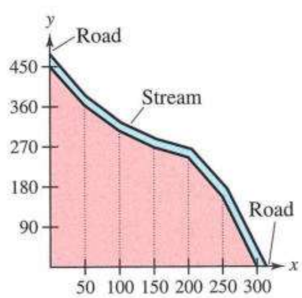
36. ***Real Estate*** The table lists the measurements (in feet) of a lot bounded by a stream and two straight roads that meet at right angles (see figure).

x	0	50	100	150	200	250	300
y	450	362	305	268	245	156	0

(a) Use the regression capabilities of a graphing utility to find a model of the form $y = ax^3 + bx^2 + cx + d$.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the model in part (a) to estimate the area of the lot.



Synthesis

True or False? **In Exercises 37 and 38, determine whether the statement is true or false. Justify your answer.**

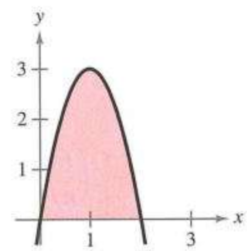
37. The sum of the first n positive integers is $n(n + 1)/2$.

38. The exact area of a region is given by the limit of the sum of n rectangles as n approaches 0.

39. ***Writing*** Describe the process of finding the area of a region bounded by the graph of a nonnegative, continuous function f, the x-axis, and the vertical lines $x = a$ and $x = b$.

40. ***Think About It*** Determine which value best approximates the area of the region shown in the graph. (Make your selection on the basis of the sketch of the region and *not* by performing any calculations.)

(a) -2 (b) 1 (c) 4 (d) 6 (e) 9



Review

In Exercises 41 and 42, sketch the graphs of $y = x^n$ and the specified transformations.

41. $y = x^4$

(a) $f(x) = (x + 3)^4$ (b) $f(x) = x^4 - 1$

(c) $f(x) = -2 + x^4$ (d) $f(x) = \frac{1}{2}(x - 4)^4$

42. $y = x^3$

(a) $f(x) = (x + 2)^3$ (b) $f(x) = 3 + x^3$

(c) $f(x) = 2 - \frac{1}{4}x^3$ (d) $f(x) = 3(x + 1)^3$

In Exercises 43–46, use the vectors $\mathbf{u} = \langle 4, -5 \rangle$ and $\mathbf{v} = \langle -1, -2 \rangle$ to find the indicated quantity.

43. $(\mathbf{u} \cdot \mathbf{v})\mathbf{u}$ **44.** $\|\mathbf{v}\| - 2$

45. $3\mathbf{u} \cdot \mathbf{v}$ **46.** $\|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$

12 Chapter Summary

What did you learn?

Section 12.1	Review Exercises
☐ How to use the definition of a limit to estimate limits	1, 2
☐ How to decide whether limits of functions exist	3–6
☐ How to use properties and operations of limits to find limits	7–14

Section 12.2	
☐ How to find limits of polynomial and rational functions by direct substitution	15–20
☐ How to use the dividing out technique to find limits of functions	21–28
☐ How to use the rationalizing technique to find limits of functions	29–32
☐ How to approximate limits of functions graphically and numerically	33–40
☐ How to evaluate one-sided limits of functions	41–48
☐ How to evaluate limits of difference quotients from calculus	49, 50

Section 12.3	
☐ How to use a tangent line to approximate the slope of a graph at a point	51–56
☐ How to use the limit definition of slope to find exact slopes of graphs	57–60
☐ How to find derivatives of functions and use derivatives to find slopes of graphs	61–70

Section 12.4	
☐ How to evaluate limits of functions at infinity	71–77
☐ How to find limits of sequences	78–81

Section 12.5	
☐ How to find limits of summations	82, 83
☐ How to use rectangles to approximate areas of plane regions	84–87
☐ How to use limits of summations to find areas of plane regions	88–94

12 Review Exercises

12.1 In Exercises 1 and 2, use a graphing utility to complete the table and use the result to estimate the limit.

1. $\displaystyle\lim_{x\to 2} \frac{x-2}{3x^2-4x-4}$

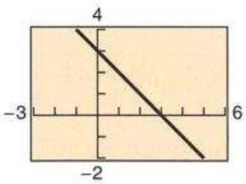
x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$				?			

2. $\displaystyle\lim_{x\to 3} \frac{\frac{x}{x+1}-\frac{3}{4}}{x-3}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$				?			

In Exercises 3–6, use the graph to determine whether the limit exists as x approaches 2.

3.



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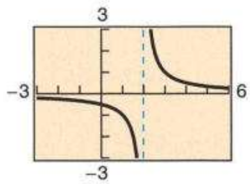

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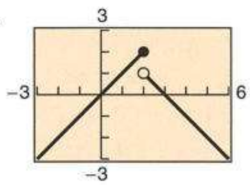

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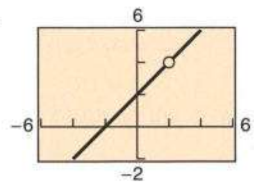


In Exercises 7–12, find the limit by direct substitution.

7. $\displaystyle\lim_{x\to 4}\left(\tfrac{1}{2}x+3\right)$ **8.** $\displaystyle\lim_{x\to -1}\sqrt{5-x}$ **9.** $\displaystyle\lim_{x\to 2}\frac{x^2-1}{x^3+2}$

10. $\displaystyle\lim_{x\to e} 7$ **11.** $\displaystyle\lim_{x\to \pi} \sin 3x$ **12.** $\displaystyle\lim_{x\to 0} \tan x$

In Exercises 13 and 14, use the given information to evaluate the limits.

13. $\displaystyle\lim_{x\to c} f(x) = 4$ $\displaystyle\lim_{x\to c} g(x) = 5$

(a) $\displaystyle\lim_{x\to c}[f(x)]^3$ (b) $\displaystyle\lim_{x\to c}[3f(x)-g(x)]$

(c) $\displaystyle\lim_{x\to c}[f(x)g(x)]$ (d) $\displaystyle\lim_{x\to c}\frac{f(x)}{g(x)}$

14. $\displaystyle\lim_{x\to c} f(x) = 27$ $\displaystyle\lim_{x\to c} g(x) = 12$

(a) $\displaystyle\lim_{x\to c}\sqrt[3]{f(x)}$ (b) $\displaystyle\lim_{x\to c}\frac{f(x)}{18}$

(c) $\displaystyle\lim_{x\to c}[f(x)g(x)]$ (d) $\displaystyle\lim_{x\to c}[f(x)-2g(x)]$

12.2 In Exercises 15–32, find the limit (if it exists). Use a graphing utility to verify your answer.

15. $\displaystyle\lim_{x\to 3}(5x-4)$ **16.** $\displaystyle\lim_{x\to -2}(5-2x-x^2)$

17. $\displaystyle\lim_{x\to 2}(5x-3)(3x+5)$

18. $\displaystyle\lim_{x\to -3}(x^3-6x^2+3x-1)$

19. $\displaystyle\lim_{t\to 3}\frac{t^2+1}{t}$ **20.** $\displaystyle\lim_{x\to 2}\frac{3x+5}{5x-3}$

21. $\displaystyle\lim_{t\to -2}\frac{t+2}{t^2-4}$ **22.** $\displaystyle\lim_{t\to 3}\frac{t^2-9}{t-3}$

23. $\displaystyle\lim_{x\to 5}\frac{x-5}{x^2+5x-50}$ **24.** $\displaystyle\lim_{x\to -1}\frac{x+1}{x^2-5x-6}$

25. $\displaystyle\lim_{x\to -2}\frac{x^2-4}{x^3+8}$ **26.** $\displaystyle\lim_{x\to 4}\frac{x^3-64}{x^2-16}$

27. $\displaystyle\lim_{x\to -1}\frac{\frac{1}{x+2}-1}{x+1}$ **28.** $\displaystyle\lim_{x\to 0}\frac{\frac{1}{x+1}-1}{x}$

29. $\displaystyle\lim_{u\to 0}\frac{\sqrt{4+u}-2}{u}$ **30.** $\displaystyle\lim_{v\to 0}\frac{\sqrt{v+7}-3}{v}$

31. $\displaystyle\lim_{x\to 5}\frac{\sqrt{x-1}-2}{x-5}$ **32.** $\displaystyle\lim_{x\to 1}\frac{\sqrt{3}-\sqrt{x+2}}{1-x}$

In Exercises 33–38, use a graphing utility to graph the function and use the graph to determine whether the specified limit exists.

33. $f(x)=\begin{cases}3x, & x\le 1\\ 2-x, & x>1\end{cases}$, $\displaystyle\lim_{x\to 1} f(x)$

34. $f(x)=\begin{cases}5-x, & x\le 2\\ x+3, & x>2\end{cases}$, $\displaystyle\lim_{x\to 2} f(x)$

35. $g(x)=e^{-2/x}$, $\displaystyle\lim_{x\to 0} g(x)$

36. $g(x)=e^{-4/x^2}$, $\displaystyle\lim_{x\to 0} g(x)$

37. $g(x) = \dfrac{\sin 4x}{2x}, \quad \lim\limits_{x\to 0} g(x)$

38. $g(x) = \dfrac{\tan 2x}{x}, \quad \lim\limits_{x\to 0} g(x)$

In Exercises 39 and 40, (a) use a graphing utility to complete the table to numerically approximate the limit and (b) rationalize the numerator to find the exact value of the limit algebraically.

x	1.1	1.01	1.001	1.0001
$f(x)$				

39. $\lim\limits_{x\to 1^+} \dfrac{\sqrt{2x+1} - \sqrt{3}}{x-1}$

40. $\lim\limits_{x\to 1^+} \dfrac{1 - \sqrt{x}}{x-1}$

In Exercises 41–44, evaluate the one-sided limit.

41. $\lim\limits_{x\to 5^-} \dfrac{|x-5|}{x-5}$

42. $\lim\limits_{x\to -2^+} \dfrac{|x+2|}{x+2}$

43. $\lim\limits_{x\to 2^+} f(x)$ where $f(x) = \begin{cases} 5 - x, & x \le 2 \\ x^2 - 3, & x > 2 \end{cases}$

44. $\lim\limits_{x\to 0^-} f(x)$ where $f(x) = \begin{cases} x - 6, & x \ge 0 \\ x^2 - 4, & x < 0 \end{cases}$

In Exercises 45–48, use a graphing utility to graph the function. Determine the limit (if it exists) by evaluating the corresponding one-sided limits.

45. $\lim\limits_{x\to 3} \dfrac{|x-3|}{x-3}$

46. $\lim\limits_{x\to 8} \dfrac{8-x}{8-x}$

47. $\lim\limits_{x\to 2} \dfrac{2}{x^2-4}$

48. $\lim\limits_{x\to -3} \dfrac{1}{x^2+9}$

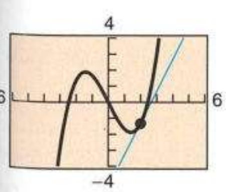
In Exercises 49 and 50, find $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$.

49. $f(x) = 3x - x^2$

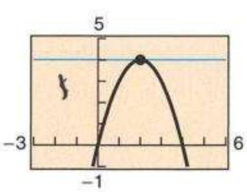
50. $f(x) = x^2 - 5x - 2$

12.3 In Exercises 51 and 52, approximate the slope of the tangent line to the graph at the point.

51.



52.



In Exercises 53–56, use a graphing utility to graph the function and the tangent line at the point $(2, f(2))$. Use the graph to estimate the slope of the tangent line.

53. $f(x) = x^2 - 2x$

54. $f(x) = 6 - x^2$

55. $f(x) = \sqrt{x+2}$

56. $f(x) = \sqrt{x^2+5}$

In Exercises 57–60, find a formula for the slope of the graph. Use the formula to find the slope at the two points.

57. $g(x) = x^2 - 4x$

(a) $(0, 0)$ (b) $(5, 5)$

58. $g(x) = \dfrac{1}{4}x^4$

(a) $(-2, 4)$ (b) $\left(1, \frac{1}{4}\right)$

59. $g(x) = \dfrac{4}{x-6}$

(a) $(7, 4)$ (b) $(8, 2)$

60. $g(x) = \sqrt{x}$

(a) $(1, 1)$ (b) $(4, 2)$

In Exercises 61–70, find the derivative of the function.

61. $f(x) = 5$

62. $g(x) = -3$

63. $g(x) = -4$

64. $f(x) = 3x$

65. $h(x) = 5 - \frac{1}{2}x$

66. $f(x) = \frac{1}{2}x + 3$

67. $f(t) = \sqrt{t+5}$

68. $f(x) = \sqrt{12 - x}$

69. $g(s) = \dfrac{4}{s+5}$

70. $g(t) = \dfrac{6}{5-t}$

12.4 In Exercises 71–76, find the limit (if it exists). Use a graphing utility to verify your answer.

71. $\lim\limits_{x\to\infty} \dfrac{4x}{2x-3}$

72. $\lim\limits_{x\to\infty} \dfrac{7x}{14x+2}$

73. $\lim\limits_{x\to\infty} \dfrac{2x}{x^2-25}$

74. $\lim\limits_{x\to\infty} \dfrac{x^2}{2x+3}$

75. $\lim\limits_{x\to\infty} \left(4 - \dfrac{7}{x^3}\right)$

76. $\lim\limits_{x\to\infty} (x-2)^{-3}$

77. *Average Cost* The cost function for a certain model of calculator is

$C = 22.50x + 12{,}200$

where C is measured in dollars and x is the number of calculators produced.

(a) Find the average cost per unit when $x = 100$ and when $x = 1000$.

(b) Determine the limit of the average cost function as x approaches infinity. Explain the meaning of the limit in the context of the problem.

In Exercises 78–81, write the first five terms of the sequence and find the limit of the sequence (if it exists). Assume n begins with 1.

78. $a_n = \dfrac{3n-1}{2n+1}$ **79.** $a_n = \dfrac{n}{n^2+1}$

80. $a_n = \dfrac{1}{2n^2}[3 - 2n(n+1)]$

81. $a_n = \left(\dfrac{2}{n}\right)\left\{n + \dfrac{2}{n}\left[\dfrac{n(n-1)}{2} - n\right]\right\}$

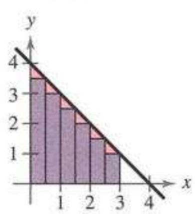
12.5 In Exercises 82 and 83, use the summation formulas and properties to rewrite the sum as a rational function $S(n)$. Use $S(n)$ to complete the table. Then find $\lim_{n\to\infty} S(n)$.

x	10^0	10^1	10^2	10^3
$S(n)$				

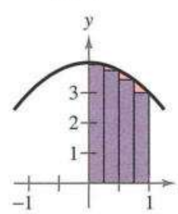
82. $\displaystyle\sum_{i=1}^{n}\left(\frac{4i^2}{n^2} - \frac{i}{n}\right)\left(\frac{1}{n}\right)$ **83.** $\displaystyle\sum_{i=1}^{n}\left[4 - \left(\frac{3i}{n}\right)^2\right]\left(\frac{3i}{n^2}\right)$

In Exercises 84 and 85, approximate the area of the region using the indicated number of rectangles of equal width.

84. $f(x) = 4 - x$



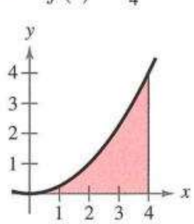
85. $f(x) = 4 - x^2$



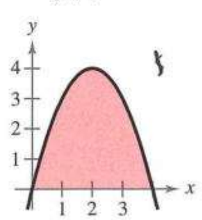
In Exercises 86 and 87, complete the table showing the approximate area of the region in the graph using n rectangles of equal width.

n	4	8	20	50
Approximate Area				

86. $f(x) = \frac{1}{4}x^2$



87. $f(x) = 4x - x^2$



In Exercises 88–93, use the limit process to find the area of the region between the graph of the function and the x-axis over the specified interval.

88. $y = 10 - x,\ [0, 10]$

89. $y = 2x - 6,\ [3, 6]$

90. $y = x^2 + 4,\ [-1, 2]$

91. $y = 8(x - x^2),\ [0, 1]$

92. $y = 2(x^2 - x^3),\ [-1, 1]$

93. $y = 4 - (x - 2)^2,\ [0, 4]$

94. ***Real Estate*** The table lists the measurements of a lot bounded by a stream and two straight roads that meet at right angles, where x and y are measured in feet (see figure).

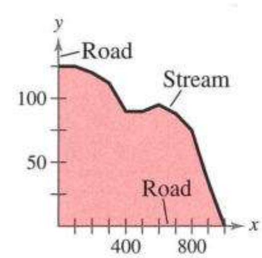
x	0	100	200	300	400	500
y	125	125	120	112	90	90

x	600	700	800	900	1000
y	95	88	75	35	0

(a) Use the regression capabilities of a graphing utility to find a model of the form $y = ax^3 + bx^2 + cx + d$.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the model in part (a) to estimate the area of the lot.



Synthesis

True or False? **In Exercises 95 and 96, determine whether the statement is true or false. Justify your answer.**

95. The limit of the sum of two functions is the sum of the limits of the two functions.

96. If the degree of the numerator $N(x)$ of a rational function $f(x) = N(x)/D(x)$ is greater than the degree of its denominator $D(x)$, then the limit of the rational function as x approaches ∞ is 0.

12 Chapter Test

Take this test as you would take a test in class. After you are done, check your work against the answers given in the back of the book.

The *Interactive* CD-ROM and *Internet* versions of this text provide answers to the Chapter Tests and Cumulative Tests. They also offer Chapter Pre-Tests (which test key skills and concepts covered in previous chapters) and Chapter Post-Tests, both of which have randomly generated exercises with diagnostic capabilities.

In Exercises 1–3, use a graphing utility to graph the function and approximate the limit (if it exists). Then approximate the limit (if it exists) algebraically by the appropriate techniques.

1. $\displaystyle\lim_{x\to -2} \frac{x^2-1}{2x}$ **2.** $\displaystyle\lim_{x\to -1} \frac{2x^2-x-3}{x+1}$ **3.** $\displaystyle\lim_{x\to 5} \frac{\sqrt{x}-2}{x-5}$

In Exercises 4 and 5, use a graphing utility to graph the function and approximate the limit. Write an approximation that you believe is accurate to three decimal places. Then create a table to verify your limit numerically.

4. $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{x}$ **5.** $\displaystyle\lim_{x\to 0} \frac{e^{2x}-1}{x}$

6. Use a graphing utility to graph $f(x) = 3x^2 - 5x - 2$ and the tangent line at the point $(2, f(2))$. Use the graph to approximate the slope of the tangent line.

7. Find a formula for the slope of the graph of $f(x) = 2x^3 + 6x$. Then find the slope at the point $(-1, -8)$.

In Exercises 8–10, find the derivative of the function.

8. $f(x) = 6 - \frac{2}{3}x$ **9.** $f(x) = 2x^2 - 7x + 3$ **10.** $f(x) = \dfrac{1}{x+7}$

In Exercises 11–13, find the limit (if it exists). Use a graphing utility to graphically verify your results.

11. $\displaystyle\lim_{x\to\infty} \frac{6}{5x-1}$ **12.** $\displaystyle\lim_{x\to\infty} \frac{1-3x^2}{x^2-5}$ **13.** $\displaystyle\lim_{x\to -\infty} \frac{3x^3}{x+2}$

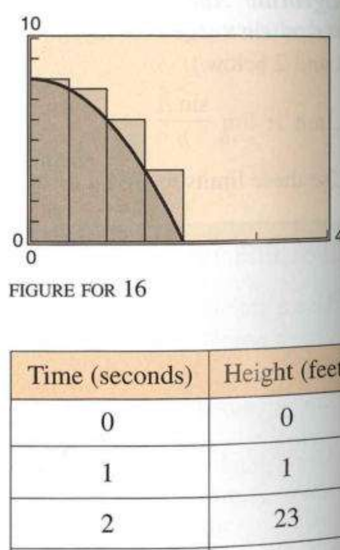


FIGURE FOR 16

In Exercises 14 and 15, write the first five terms of the sequence and find the limit of the sequence (if it exists). Assume n begins with 1.

14. $a_n = \dfrac{n^2+3n-4}{2n^2+n-2}$ **15.** $a_n = \dfrac{1+(-1)^n}{n}$

16. Approximate the area of the region in the graph of $f(x) = 8 - 2x^2$ at the right using the indicated number of rectangles of equal width.

In Exercises 17 and 18, use the limit process to find the area of the region between the graph of the function and the x-axis over the specified interval.

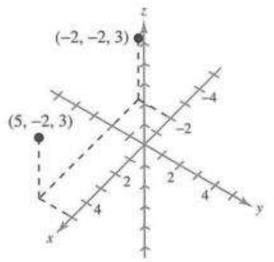
17. $f(x) = x + 2$ Interval: $[-2, 2]$ **18.** $f(x) = 1 - x^3$ Interval: $[0, 1]$

19. The table shows the height of a space shuttle during its first 5 seconds of motion. (a) Use the regression capabilities of a graphing utility to fit the quadratic model $y = ax^2 + bx + c$ to the data. (b) The value of the derivative of the model is the rate of change of height with respect to time, or the velocity, at that instant. Find the velocity of the shuttle after 5 seconds.

Time (seconds)	Height (feet)
0	0
1	1
2	23
3	60
4	115
5	188

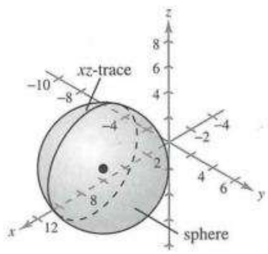
Chapter Test (page 804)

1.



2. No. $\left(\sqrt{76}\right)^2 + \left(\sqrt{102}\right)^2 \neq \left(\sqrt{194}\right)^2$ **3.** $(7, 1, 2)$

4.



$(x - 7)^2 + (y - 1)^2 + (z - 2)^2 = 19$

5. $\mathbf{u} = \langle -2, 6, -6\rangle$, $\mathbf{v} = \langle -12, 5, -5\rangle$

6. (a) $\sqrt{194}$ (b) 84 (c) $\langle 0, 62, 62\rangle$ **7.** $46.23°$

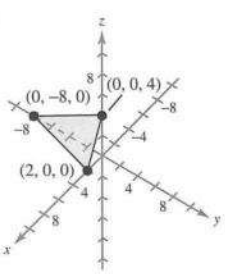
8. (a) $x = 8 - 2t,\ y = -2 + 6t,\ z = 5 - 6t$

(b) $\dfrac{x-8}{-2} = \dfrac{y+2}{6} = \dfrac{z-5}{-6}$

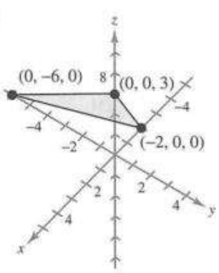
9. $y + z - 3 = 0$ **10.** Neither **11.** Orthogonal

12. 200

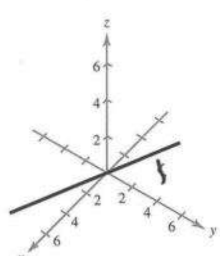
13.



14.



15. $\dfrac{4\sqrt{14}}{7}$ **16.**

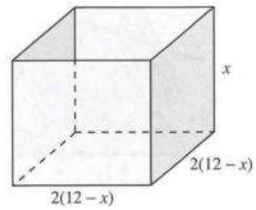


17. $6\sqrt{3}$

Chapter 12

Section 12.1 (page 813)

1. (a)



(b) $V = lwh$

$= 2(12 - x) \cdot 2(12 - x) \cdot x$

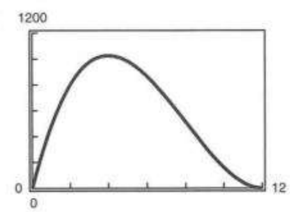
$= 4x(12 - x)^2$

(c)

x	3	3.5	3.9	4	4.1	4.5	5
V	972	1011.5	1023.5	1024	1023.5	1012.5	980

$\lim\limits_{x \to 4} V = 1024$

(d)



3.

x	2.9	2.99	2.999	3
$f(x)$	−4.7	−4.97	−4.997	−5

x	3.001	3.01	3.1
$f(x)$	−5.003	−5.03	−5.3

−5; yes

5.

x	2.9	2.99	2.999	3
$f(x)$	0.1695	0.1669	0.1667	Error

x	3.001	3.01	3.1
$f(x)$	0.1666	0.1664	0.1639

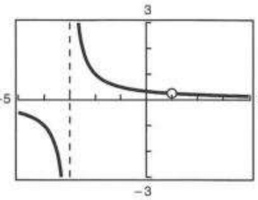
$\frac{1}{6}$; no

7.

x	0.9	0.99	0.999	1
$f(x)$	0.2564	0.2506	0.2501	Error

x	1.001	1.01	1.1
$f(x)$	0.2499	0.2494	0.2439

$\frac{1}{4}$;

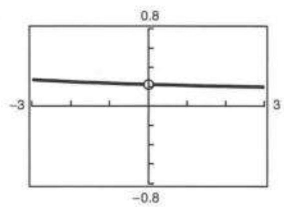


9.

x	-0.1	-0.01	-0.001	0
$f(x)$	0.2247	0.2237	0.2236	Error

x	0.001	0.01	0.1
$f(x)$	0.2236	0.2235	0.2225

0.2236;

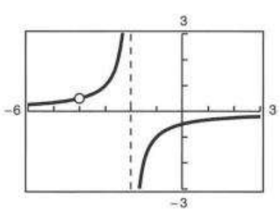


11.

x	-4.1	-4.01	-4.001	-4
$f(x)$	0.4762	0.4975	0.4998	Error

x	-3.999	-3.99	-3.9
$f(x)$	0.5003	0.5025	0.5263

$\frac{1}{2}$;

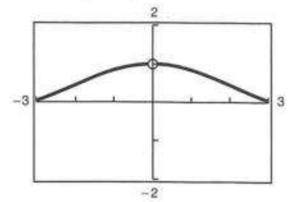


13.

x	-0.1	-0.01	-0.001	0
$f(x)$	0.9983	0.9999	0.9999	Error

x	0.001	0.01	0.1
$f(x)$	0.9999	0.9999	0.9983

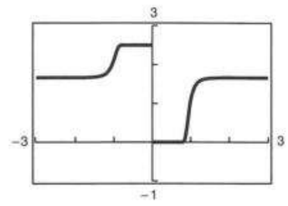
1;



15. 1 **17.** -1 **19.** Does not exist

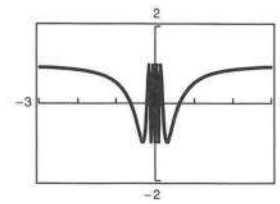
21. Does not exist

23.



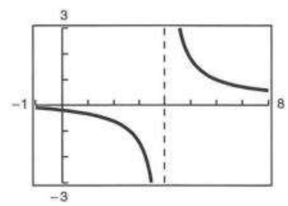
No

25.



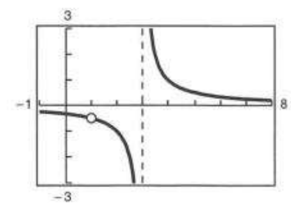
No

27.



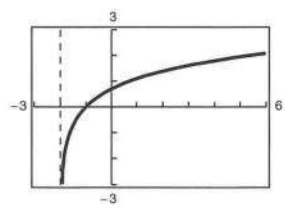
No

29.



Yes

31.



Yes

33. 0 **35.** 3 **37.** 5 **39.** 0 **41.** $e^3 \approx 20.08$

43. 0 **45.** $\frac{\pi}{6}$ **47.** Does not exist

49. (a) -12 (b) 9 (c) $\frac{1}{2}$ (d) $\sqrt{3}$

51. (a) 8 (b) $\frac{3}{8}$ (c) 3 (d) $-\frac{61}{8}$

53. True. This means that no matter how close x gets to c, there will be both positive and negative x-values that yield $f(x) = 3$ and $f(x) = -3$. This implies that the limit does not exist.

55. (a) Answers will vary. (b) Answers will vary.

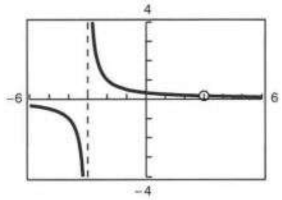
57. No. The function may approach different values from the right and left of 4. For example,

$$f(x) = \begin{cases} 0, & x < 2 \\ 4, & x = 2 \\ 6, & x > 2 \end{cases}$$

has $f(2) = 4$, but $\lim_{x \to 2} f(x) \neq 4$.

59. As a function's x-value approaches 5 from both the right and left sides, its corresponding output values approach 12.

61.



$\frac{1}{6}$; $x \neq \pm 3$; It may not be clear from a graph if a function is not defined at a single point. Examining a function graphically and algebraically ensures that you will find all points where the function is not defined.

63. $-\frac{1}{3}$, $x \neq 5$ **65.** $\frac{5x + 4}{5x + 2}$, $x \neq \frac{1}{3}$

67. $\frac{x^2 - 3x + 9}{x - 2}$, $x \neq -3$

69. 1 **71.** $\sqrt{70}$ **73.** $7\sqrt{2}$

Section 12.2 *(page 824)*

1. -15 **3.** $-\frac{9}{10}$ **5.** $\frac{7}{13}$ **7.** 1

9. (a) 1 (b) 3 (c) 5

$g_2(x) = -2x + 1$

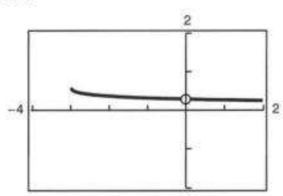
11. (a) 2 (b) 0 (c) 0

$g_2(x) = x(x + 1)$

13. $\frac{1}{12}$ **15.** 4 **17.** $\frac{1}{2\sqrt{5}}$ **19.** $\frac{1}{4}$

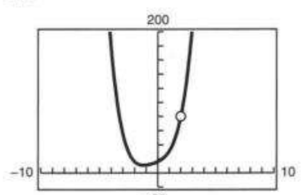
21. -1 **23.** Does not exist

25.



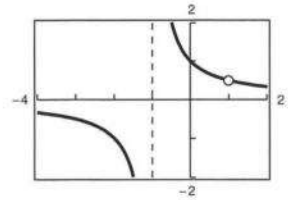
$\lim_{x \to 0} \frac{\sqrt{x + 3} - \sqrt{3}}{x} \approx 0.2887$

27.



$\lim_{x \to 2} \frac{x^5 - 32}{x - 2} = 80$

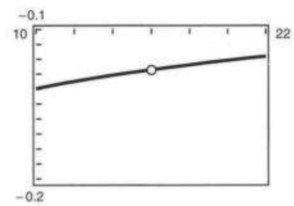
29.



x	0.9	0.99	0.999	0.9999
$f(x)$	0.5263	0.5025	0.5003	0.5000

$\lim_{x \to 1^-} \frac{(x - 1)}{(x^2 - 1)} = 0.5$

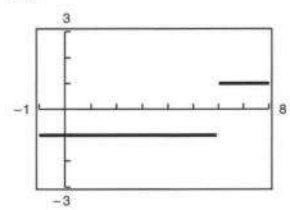
31.



x	16.1	16.01	16.001	16.0001
$f(x)$	-0.12481	-0.12498	-0.124998	-0.1249998

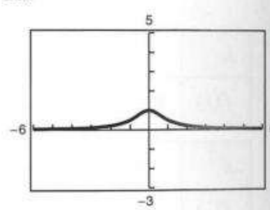
$\lim_{x \to 16^+} \frac{4 - \sqrt{x}}{x - 16} = -0.125$

33.



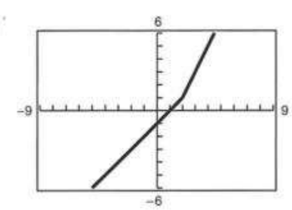
The limit does not exist.

35.



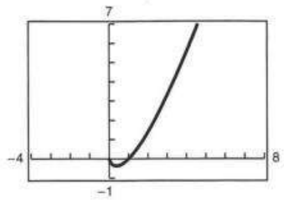
$\lim_{x \to 1} \frac{1}{x^2 + 1} = \frac{1}{2}$

37.



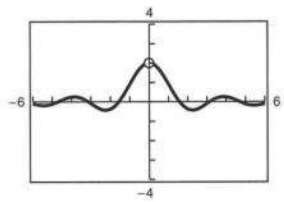
$\lim_{x \to 2} f(x)$ where $f(x) = \begin{cases} x - 1, & x \leq 2 \\ 2x - 3, & x > 2 \end{cases} = 1$

39.



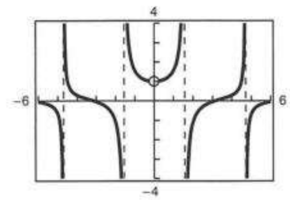
$\lim_{x\to 0^+} (x \ln x) = 0$

41.



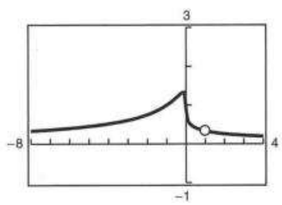
$\lim_{x\to 0} \dfrac{\sin 2x}{x} = 2$

43.



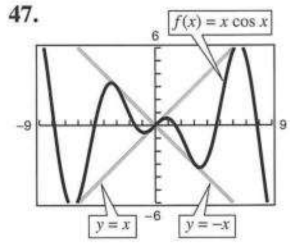
$\lim_{x\to 0} \dfrac{\tan x}{x} = 1$

45.



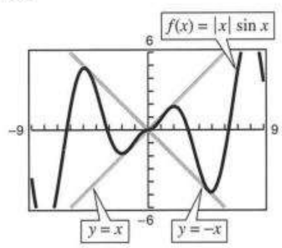
$\lim_{x\to 1} \dfrac{1 - \sqrt[3]{x}}{1 - x} = 0.333$

47.



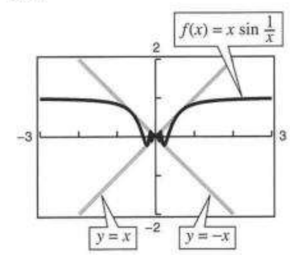
$\lim_{x\to 0} x \cos x = 0$

49.



$\lim_{x\to 0} |x| \sin x = 0$

51.



$\lim_{x\to 0} x \sin \dfrac{1}{x} = 0$

53. (a) 0. Direct substitution (b) 1

55. 3 **57.** $\dfrac{1}{2\sqrt{x}}$ **59.** $2x - 3$

61. -32 feet per second **63.** Answers will vary.

65. True **67. and 69.** Answers will vary.

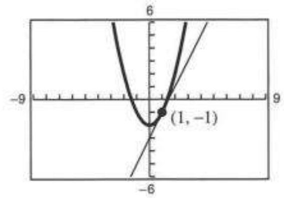
71. $x - 2y - 3 = 0$ **73.** Neither orthogonal nor parallel

75. Neither orthogonal nor parallel

Section 12.3 *(page 833)*

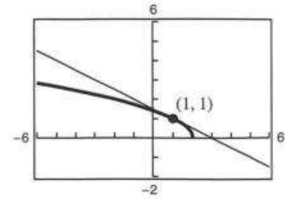
1. 0 **3.** $\frac{1}{2}$

5.



2

7.



$-\frac{1}{2}$

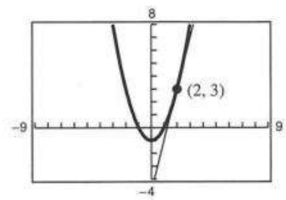
9. -2 **11.** 2 **13.** -1 **15.** $\frac{1}{6}$

17. $-2x$; (a) 0 (b) 2

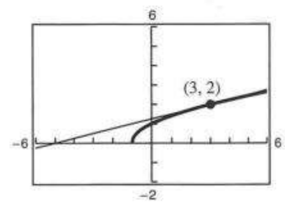
19. $-\dfrac{1}{(x+4)^2}$; (a) $-\dfrac{1}{16}$ (b) $-\dfrac{1}{4}$ **21.** 0

23. $-\dfrac{1}{3}$ **25.** $-\dfrac{2}{x^3}$

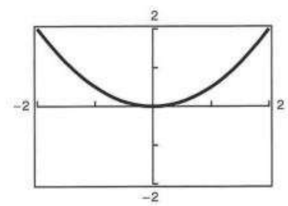
27. 4; $y = 4x - 5$



29. $\frac{1}{4}$; $y = \frac{1}{4}x + \frac{5}{4}$



31.

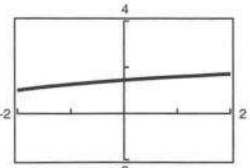


x	-2	-1.5	-1	-0.5	0
$f(x)$	2	1.125	0.5	0.125	0
$f'(x)$	-2	-1.5	-1	-0.5	0

x	0.5	1	1.5	2
$f(x)$	0.125	0.5	1.125	2
$f'(x)$	0.5	1	1.5	2

They appear to be the same.

33.



x	-2	-1.5	-1	-0.5	0
$f(x)$	1	1.225	1.414	1.581	1.732
$f'(x)$	0.5	0.408	0.354	0.316	0.289

x	0.5	1	1.5	2
$f(x)$	1.871	2	2.121	2.236
$f'(x)$	0.267	0.25	0.236	0.224

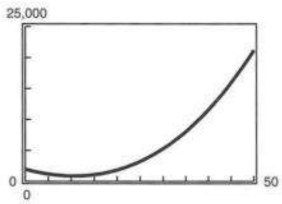
They appear to be the same.

35. $f'(x) = 2x - 4$; Horizontal tangent at $(2, -1)$

37. $f'(x) = 9x^2 - 9$; Horizontal tangents at $(-1, 6)$ and $(1, -6)$

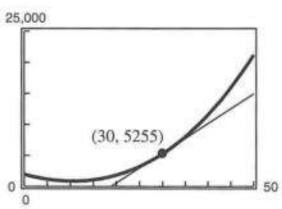
39. (a) $y = 0.073t^3 + 7.89t^2 - 192.4t + 1955$

(b)



500. In 1980, the per capita debt is increasing at the rate of \$500 per year.

(c)



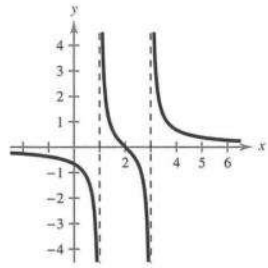
The slope given by the graphing utility (478.1) is close to the estimate (500).

41. True **43.** (b) **45.** (d)

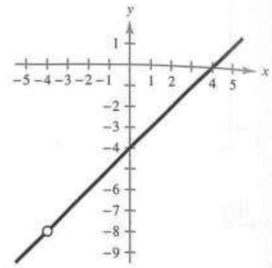
47. Answers will vary. Example: a sketch of any linear function with positive slope

49. Answers will vary. Example: a sketch of any quadratic function of the form $y = a(x - 1)^2 + k$

51.



53.



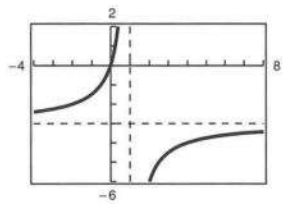
55. $\langle 0, 42, 0\rangle$ **57.** $\langle -140, -46, 57\rangle$

Section 12.4 *(page 841)*

1. (c) **3.** (d) **5.** 0 **7.** -1 **9.** 2

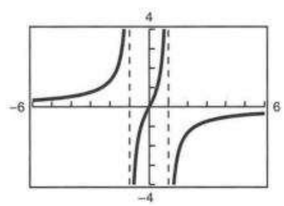
11. Does not exist **13.** -1 **15.** -4 **17.** -5

19.



Horizontal asymptote: $y = -3$

21.

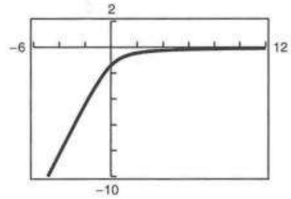


Horizontal asymptote: $y = 0$

23.

x	10^0	10^1	10^2	10^3
$f(x)$	-0.7321	-0.0995	-0.0100	-0.0010

x	10^4	10^5	10^6
$f(x)$	-1.0×10^{-4}	-1.0×10^{-5}	-1.0×10^{-6}

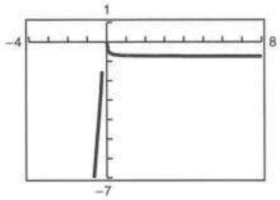


$\lim_{x\to\infty} f(x) = 0$

25.

x	10^0	10^1	10^2	10^3
$f(x)$	-0.7082	-0.7454	-0.7495	-0.74995

x	10^4	10^5	10^6
$f(x)$	-0.749995	-0.7499995	-0.75



$\lim_{x\to\infty} f(x) = -\frac{3}{4}$

27. $1, \frac{3}{5}, \frac{2}{5}, \frac{5}{17}, \frac{3}{13}$
Limit: 0

29. $\frac{1}{5}, \frac{1}{2}, \frac{9}{11}, \frac{8}{7}, \frac{25}{17}$
Limit does not exist.

31. 2, 3, 4, 5, 6
Limit does not exist.

33. $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}$
Limit: 0

35.

n	10^0	10^1	10^2	10^3
a_n	2	1.55	1.505	1.5005

n	10^4	10^5	10^6
a_n	1.50005	1.500005	1.5000005

$\lim_{n\to\infty} a_n = \frac{3}{2}$

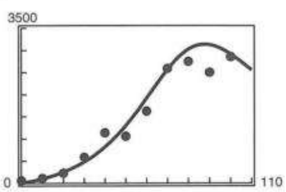
37.

n	10^0	10^1	10^2	10^3
a_n	16	6.16	5.4136	5.3413

n	10^4	10^5	10^6
a_n	5.3341	5.33341	5.333341

$\lim_{n\to\infty} a_n = \frac{16}{3}$

39. (a)



(b) 3022.9 thousand students

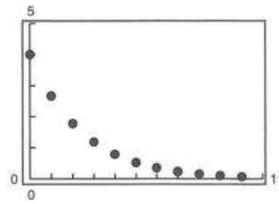
(c) As time approaches infinity, the number of high school graduates approaches zero.

41. False. Graph $y = \dfrac{x^2}{x+1}$.

43. Let $f(x) = \dfrac{1}{x^2}$, $g(x) = \dfrac{1}{x^2}$, and $c = 0$. Now

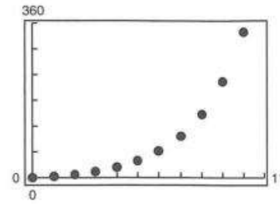
$\lim_{x\to 0} \dfrac{1}{x^2} = \infty$, and $\lim_{x\to 0}[f(x) - g(x)] = 0$.

45.



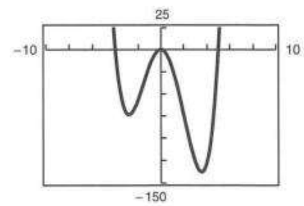
Converges to 0

47.

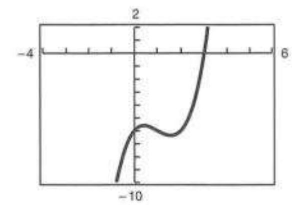


Diverges

49. $-4, 5, 0, 0$



51. 3



53. 60 **55.** 150

Section 12.5 *(page 849)*

1. 1830 **3.** 44,100 **5.** 5850

7. $S(n) = \dfrac{n^2 + 2n + 1}{4n^2}$

n	10^0	10^1	10^2	10^3
$S(n)$	1	0.3025	0.255025	0.25050025

Limit: $\frac{1}{4}$

9. $S(n) = \dfrac{2n^2 + 3n + 7}{2n^2}$

n	10^0	10^1	10^2	10^3
$S(n)$	6	1.185	1.0154	1.0015

Limit: 1

11. $S(n) = \dfrac{14n^2 + 3n + 1}{6n^3}$

n	10^0	10^1	10^2	10^3
$S(n)$	3	0.2385	0.02338	0.00233

Limit: 0

13. $S(n) = \dfrac{4n^2 - 3n - 1}{6n^2}$

n	10^0	10^1	10^2	10^3
$S(n)$	0	0.615	0.66165	0.66617

Limit: $\frac{2}{3}$

15. 14.25 **17.** 1.2656

19.

n	4	8	20	50
Approximate Area	18	21	22.8	23.52

21.

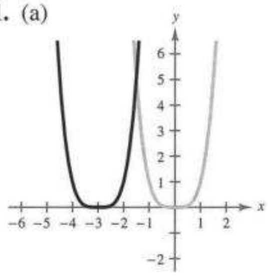
n	4	8	20	50
Approximate Area	3.52	2.85	2.48	2.34

23. 2 **25.** $\frac{10}{3}$ **27.** $\frac{17}{4}$ **29.** $\frac{51}{4}$ **31.** $\frac{1}{6}$ **33.** 1

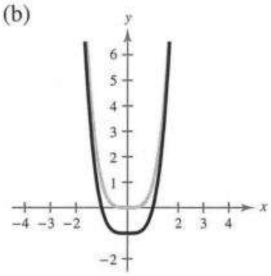
35. Area is 105,208.33 square feet $\approx$ 2.4153 acres

37. True **39.** Answers will vary.

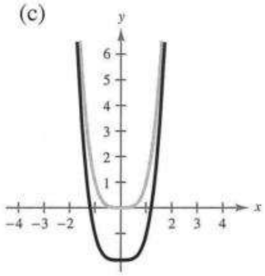
41. (a)



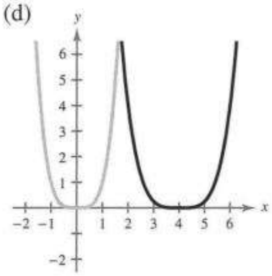
(b)



(c)



(d)



43. $\langle 24, -30 \rangle$ **45.** 18

Review Exercises *(page 852)*

1.

x	1.9	1.99	1.999	2
$f(x)$	0.1299	0.1255	0.1250	?

x	2.001	2.01	2.1
$f(x)$	0.1250	0.1245	0.1205

$\lim_{x \to 2} \dfrac{x - 2}{3x^2 - 4x - 4} = \dfrac{1}{8}$

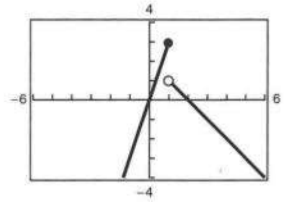
3. Yes **5.** No **7.** 5 **9.** $\frac{3}{10}$ **11.** 0

13. (a) 64 (b) 7 (c) 20 (d) $\frac{4}{5}$

15. 11 **17.** 77 **19.** $\frac{10}{3}$ **21.** $-\frac{1}{4}$ **23.** $\frac{1}{15}$

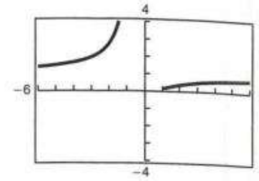
25. $-\frac{1}{3}$ **27.** -1 **29.** $\frac{1}{4}$ **31.** $\frac{1}{4}$

33.



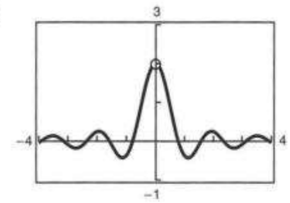
Does not exist

35.



Does not exist

37.



$\lim_{x \to 0} g(x) = 2$

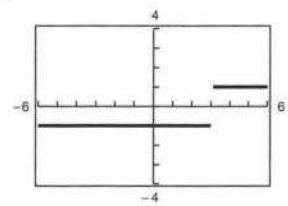
39.

x	1.1	1.01	1.001	1.0001
$f(x)$	0.5680	0.5764	0.5772	0.5773

$\lim_{x \to 1^+} \dfrac{\sqrt{2x + 1} - \sqrt{3}}{x - 1} \approx 0.577.$ $\left(\text{Actual limit is } \dfrac{\sqrt{3}}{3}\right).$

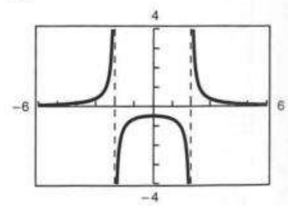
41. -1 **43.** 1

45.



Limit does not exist.

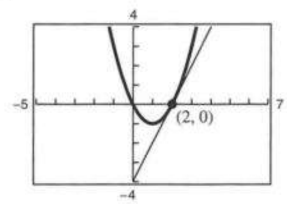
47.



Limit does not exist.

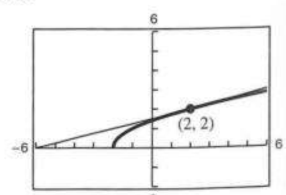
49. $3 - 2x$ **51.** Slope: 2

53.



2

55.



$\frac{1}{4}$

57. $2x - 4$; (a) -4 (b) 6

59. $-\dfrac{4}{(x-6)^2}$; (a) -4 (b) -1

61. $f'(x) = 0$ **63.** $g'(x) = 0$ **65.** $h'(x) = -\frac{1}{2}$

67. $f'(t) = \dfrac{1}{2\sqrt{t+5}}$ **69.** $g'(s) = -\dfrac{4}{(s+5)^2}$

71. 2 **73.** 0 **75.** 4

77. (a) $C = 22.50(100) + 12{,}200 = \$144.50$

$C = 22.50(1000) + 12{,}200 = \$34.70$

(b) \$22.50. As the number of calculators gets very large, the average cost approaches \$22.50.

79. $\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \frac{5}{26}$

Limit: 0

81. $-2, 1, 2, \frac{5}{2}, \frac{14}{5}$

Limit: 4

83. $S(n) = -\dfrac{3(n+1)(n+9)}{4n^2}$

n	10^0	10^1	10^2	10^3
$S(n)$	-15	-1.5675	-0.8257	-0.7575

Limit: $-\frac{3}{4}$

85. $\frac{113}{32} \approx 3.5313$

87.

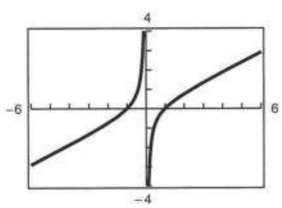
n	4	8	20	50
Approximate Area	10	10.5	10.64	10.6624

89. 9 **91.** $\frac{4}{3}$ **93.** 9

95. True. $\lim\limits_{x\to c}[f(x) + g(x)] = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x)$

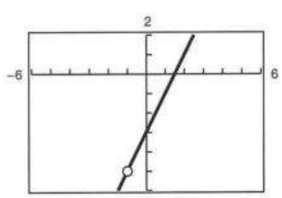
Chapter Test *(page 856)*

1.



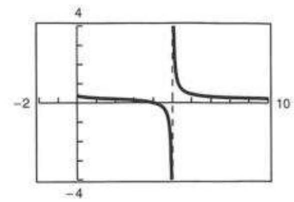
$$\lim_{x\to -2} \frac{x^2-1}{2x} = -\frac{3}{4}$$

2.



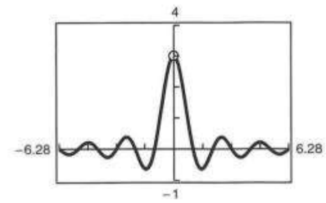
$$\lim_{x\to -1} \frac{2x^2-x-3}{x+1} = -5$$

3.



The limit does not exist.

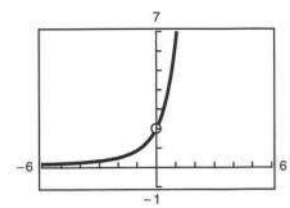
4.



$$\lim_{x\to 0} \frac{\sin 3x}{x} \approx 3$$

x	-0.02	-0.01	0	0.01	0.02
$f(x)$	2.998	2.999	3	2.999	2.998

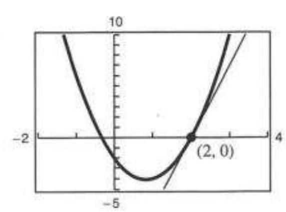
5.



$$\lim_{x\to 0} \frac{e^{2x}-1}{x} \approx 2$$

x	-0.004	-0.003	-0.002	-0.001	0
$f(x)$	1.992	1.994	1.996	1.998	2

6.



$m = 7$

7. $6x^2 + 6$; 12 **8.** $-\frac{2}{3}$ **9.** $4x - 7$

10. $-\dfrac{1}{(x+7)^2}$ **11.** 0 **12.** -3

13. Does not exist **14.** $0, \frac{3}{4}, \frac{14}{19}, \frac{12}{17}, \frac{36}{53}$; Limit: $\frac{1}{2}$

15. $0, 1, 0, \frac{1}{2}, 0$; Limit: 0

16. $\frac{25}{2}$ **17.** 8 **18.** $\frac{3}{4}$

19. (a) $y = 8.79x^2 - 6.2x - 0.4$ (b) 81.7