AP Review Packet: 27 Essential AP Calculus Concepts

1. Limits

A. Finite Limits - *Try direct substitutions first, if that fails, try* to factor or use L'Hopital's Rule.

1. $\lim\limits_{x\to -2} x^2+5x+4$

2. $\lim\limits_{x\to 2} \dfrac{8}{x-2}$

3. $\lim\limits_{x\to -2} \dfrac{x^2-4}{x+2}$

4. $\lim\limits_{x\to 0} \dfrac{3\sin x}{x}$

5. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x}$

B. Limits to Infinity - *These are also called horizontal asymptotes.*

- If the degree of the numerator is smaller than that of the denominator, then the $\lim\limits_{x\to\infty} = 0$.
- If the degree of the numerator is equal to that of the denominator, then the $\lim\limits_{x\to\infty} = \dfrac{\text{lead coefficient}}{\text{lead coefficient}}$.
- If the degree of the numerator is larger than that of the denominator, then the $\lim\limits_{x\to\infty}$ Does Not Exist
- Be careful of limits going to $-\infty$. Sometimes they don't follow the patterns above!
- When you have e^x, remember that it "dominates" any polynomial and you need to look at find a simple left-end and right-end model to evaluate.

6. $\lim\limits_{x\to\infty} \dfrac{x^2-3x+24}{x^3+2}$

7. $\lim\limits_{x\to\infty} \dfrac{5-3x^2+8x^3}{3x^3+2}$

8. $\lim\limits_{x\to\infty} \dfrac{x^4-3x^2+24}{x^3+2}$

9. $\lim\limits_{x\to -\infty} \dfrac{x^3-3x^2+2x-4}{4x^5-2x+4}$

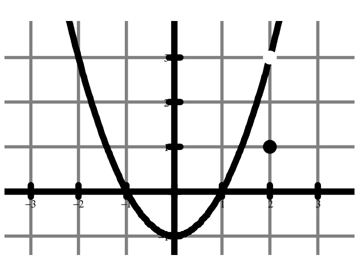
10. $\lim\limits_{x\to\infty} e^x+3x^2-4x^3$

11. $\lim\limits_{x\to -\infty} e^x+3x^2-4x^3$

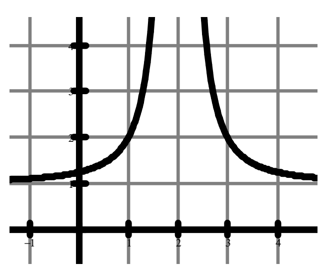
C. Limits from a Picture – *Look at the graph and find the "intended height" of the function. Remember, left and right hand limits must agree for the limit to exist.*

12. $\lim\limits_{x\to 2} f(x) =$

$f(2) =$

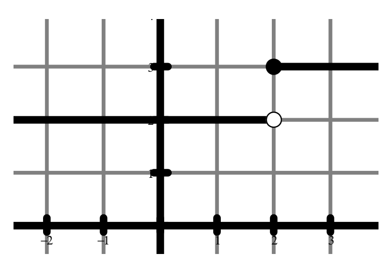


13. $\lim\limits_{x\to 2} f(x) =$

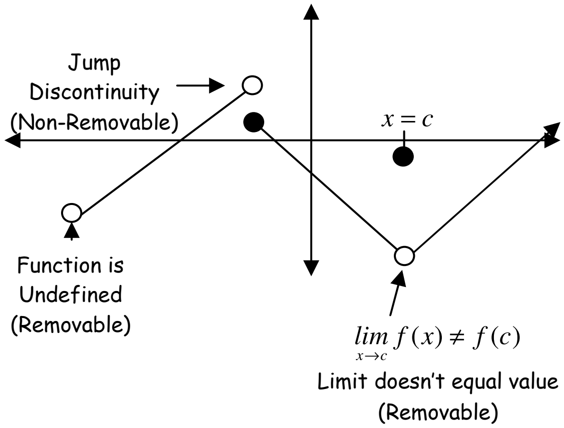


14. $\lim\limits_{x\to 2^+} f(x) =$ $\quad\lim\limits_{x\to 2^-} f(x) =$

$\lim\limits_{x\to 2} f(x) =$ $\quad f(2) =$

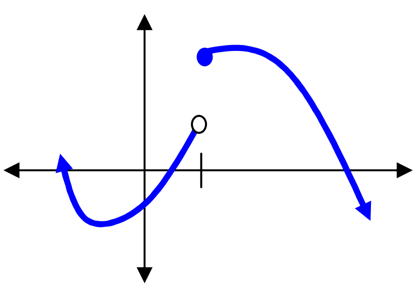


2. Types of Discontinuity

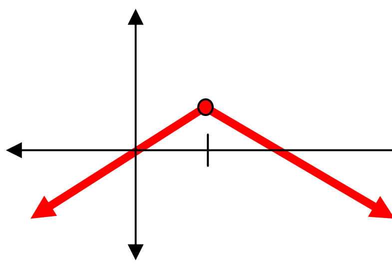


3. Types of Non-Differentiability

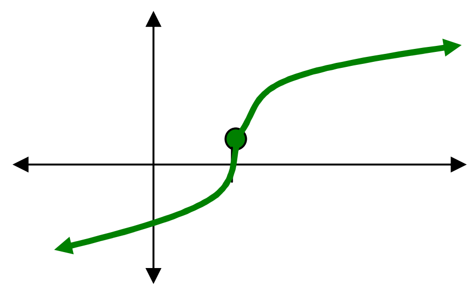
The slope on left approach must agree with slope on right approach.



Not continuous



Corner/cusp



Vertical tangent

15. For what values of a and b is the function $g(x) = \begin{cases} x^2 - a, & x \le 2 \\ bx - 8, & x > 2 \end{cases}$ continuous and differentiable?

4. Intermediate Value Theorem

16. $f(x)$ is continuous on [2, 4] and differentiable on (2,4). If $f(2) = 6$ and $f(3) = -2$ and $f(4) = 6$:

a. How many solutions are guaranteed?

b. Does $f(x) = 2$ for a value of x in the interval [2, 4]?

c. Can $f(x) = 8$ for some value of x in the interval [2, 4]?

5. Basic Differentiation Rules

Differentiation Rules

$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$ $\frac{d}{dx}[\sin x] = \cos x$ $\frac{d}{dx}[\cos x] = -\sin x$

$\frac{d}{dx}[\tan x] = \sec^2 x$ $\frac{d}{dx}[\sec x] = \sec x \tan x$ $\frac{d}{dx}[\cot x] = -\csc^2 x$

$\frac{d}{dx}[\csc x] = -\csc x \cot x$ $\frac{d}{dx}(\ln u) = \frac{1}{u} du$ $\frac{d}{dx}\left(e^u\right) = e^u du$

The Product Rule

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

17. $f(x) = (x+3)^2(2x-1), \quad f'(x) =$

The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{\left[g(x)\right]^2}$$

18. $f(x) = \frac{(x+3)^2}{2x-1}, \quad f'(x) =$

The Chain Rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

19. $f(x) = \sin(3x^2), \quad f'(x) =$

Derivative of an Inverse

$$\left(f^{-1}\right)'(a) = \frac{1}{f'(b)}$$

$f:(a,b)$

$f^{-1}:(b,a)$

20. If $f(x) = x^3 + x$, and a = 10 find $\left(f^{-1}\right)'(a) =$

21. Find $(f^{-1})'(2)$ if $f(x) = 2x^2 - 3x$ for $x > 0$.

6. Derivative Rules From Tabular Data

x	3	2	5
$f(x)$	3	5	3
$f'(x)$	5	2	2
$g(x)$	5	3	2
$g'(x)$	2	5	3

22. If $h(x) = f(x)g(x)$, find $h'(2)$.

23. If $h(x) = f(g(x))$, find $h'(3)$.

24. If $h(x) = \dfrac{f(x)}{g(x)}$ find $h'(3)$.

7. Equation of a Tangent Line

- The derivative indicates the *instantaneous rate of change*, or the *slope* of a line *tangent* to a curve.
- The equation of a line tangent to $f(x)$ at the point (a, b) is given by: $y - b = f'(a)(x - a)$.
- A "normal line" is perpendicular to a tangent line through the same point.

25. Find the equation of the line tangent to $f(x) = 3x^2 - 2x$ at (2, 8).

26. Find the equation of the line normal to $f(x) = 3x^2 - 2x$ at (2, 8).

27. For what x value does the graph $f(x) = x^2 + 4x$ have a tangent with a slope of 6?

28. Find the equation of a line tangent to $f(x) = -x^2 - 5x$ that is parallel to $6x - 2y = 8$.

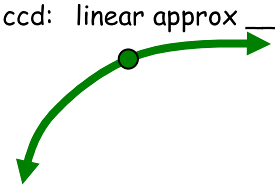
29. At what point does $f(x) = 3x^2 - 2x$ have a slope of 4?

30. Find the equation of a line tangent to $f(x) = x^2 + 3x + 2$ through the point (1,6) and use your local linear approximation to estimate $f(1.1)$.

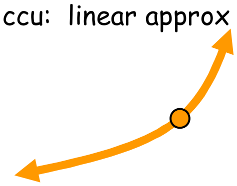
8. Slope and Relative Error

Draw in a tangent line at the indicated point and fill in the blank for each picture.

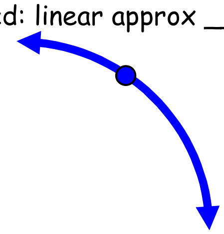
31. inc & ccd: linear approx _____ actual value



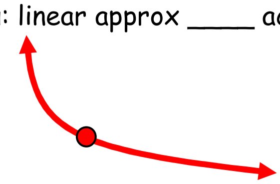
32. inc & ccu: linear approx _____ actual value



33. dec & ccd: linear approx _____ actual value



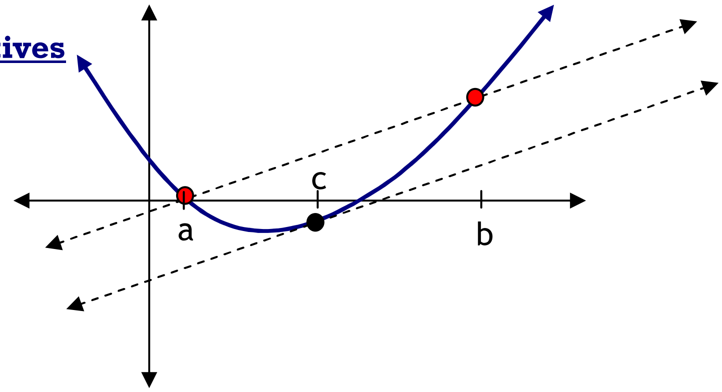
34. dec & ccu: linear approx _____ actual value



9. The Mean Value Theorem for Derivatives

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The slope of the tangent at c equals the slope of the secant through a and b.



35. If f is continuous and differentiable for all values on (2,5) and $f(2) = -3,\ f(3) = 4$ and $f(5) = -3$, then on the interval (2,5)…

a. What can be said about the slope?

b. How many extrema?

36. Find the average rate of change for $f(x) = -x^2 - 4x + 3$ on [-2, 0]. (slope of the secant line)

37. For what value c, such that 0 $\leq$ c $\leq$ 3, is the instantaneous rate of change for $f(x) = x^2 - 2x$ equal to the average rate of change over the interval [0, 3]?

38. Find the average velocity of a particle on the interval [1, 3] if the particle's position is given by $s(t) = -t^2 + 5t$, where t is measured in seconds and s(t) in feet.

b. At what time, c, does the particle reach its average velocity?

10. Extreme Value Theorem

Extrema on a closed interval [a, b]: Test $f(a), f(b)$ and $f(c\#)$ between a and b.

39. Find the absolute extrema for $f(x) = -2x^2 - 8x + 2$ on [-3, 0].

40. Find the any absolute extrema for $f(x) = x^2 - x - 6$ on [0, 4].

11. Slope and Concavity

- *Critical Points* occur where $f'(x)=0$ or is undefined.
- *Increasing Intervals* occur where $f'(x)>0$.
- *Decreasing Intervals* occur where $f'(x)<0$.
- *Minimums* occur where f' changes from neg to pos.
- *Maximums* occur where f' changes from pos to neg.
- Points of inflection occur where $f''(x)=0$ or is undefined AND has a sign change.
- Intervals that are *concave up* occur where $f''(x)>0$ (or when f' is inc).
- Intervals that are *concave down* occur where $f''(x)<0$ (or when f' is dec).

41. Find all critical points, indicate increasing/decreasing intervals and all min's and max's for $f(x)=x^3-x^2-5x-3$.

42. Find the x location for all local extrema for $f(x)=\frac{1}{3}x^3-\frac{1}{2}x^2-2x+1$.

43. Find all points of inflection and indicate the concavity over each interval for $f(x)=\frac{1}{20}x^5+\frac{1}{6}x^3+2x$.

2nd Derivative Test:
For any critical point c,
If $f''(c)>0$ the function is concave up, indicating a min.
If $f''(c)<0$ the function is concave down, indicating a max.

44. If $f''(2)=-3$ and $f'(2)=0$, what can you conclude about $f(2)$?

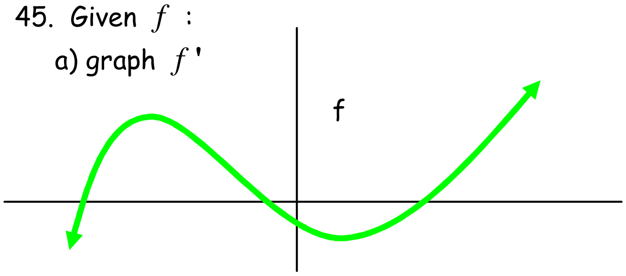
12. Curve Sketching

Slope become y values (focus on slopes) $\downarrow$	$f(a)$ is the height of f at a. $f'(a)$ is the slope of the f at a. $f''(a)$ is the concavity of f at a.	$\uparrow$ y values become slopes (focus on y-values)

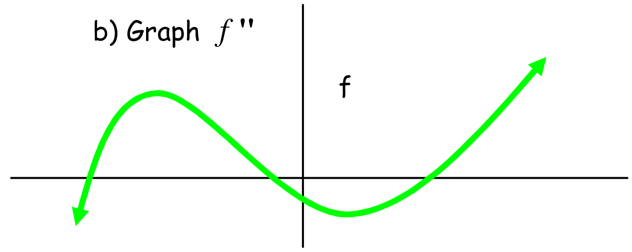
Pay attention to what the given graph represents!!!!!!!!

45. Given f :

a) graph f'

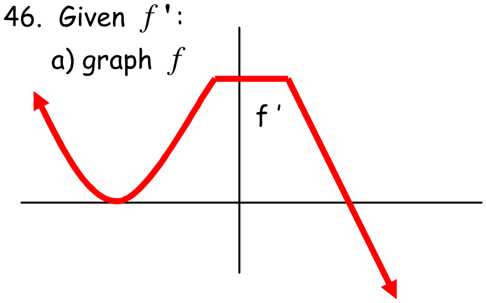


b) Graph f''

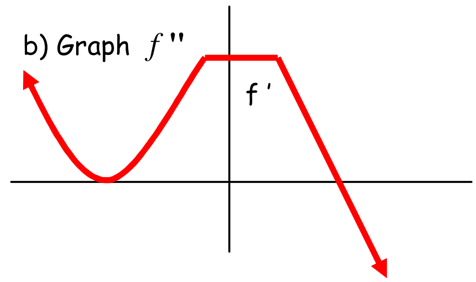


46. Given f':

a) graph f

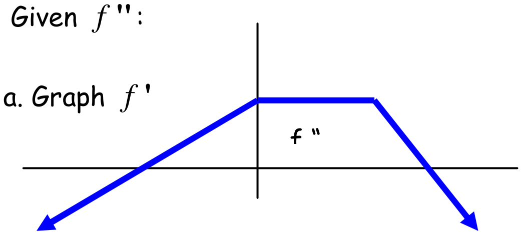


b) Graph f''



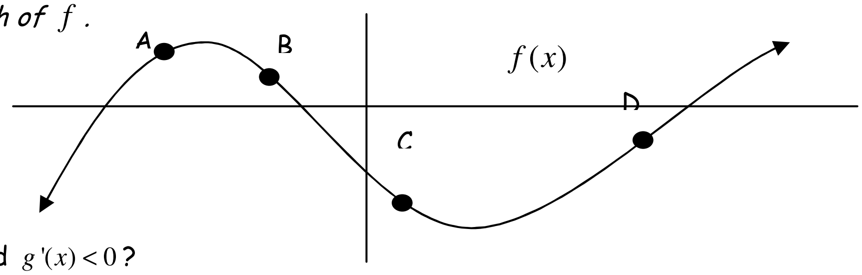
47. Given f'':

a. Graph f'



13. Slope and Concavity From Graph

Answer the following based on the given graph of f .

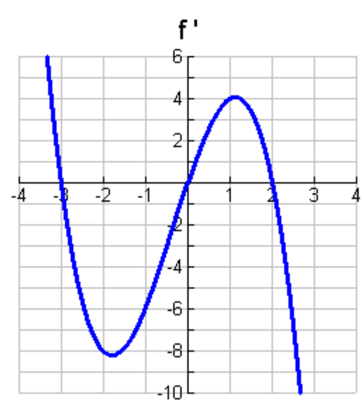


48. At what point is $f'(x) < 0$ and $f''(x) > 0$?

49. At what point is $f(x) < 0$ and $f'(x) > 0$?

50. If $g(x) = f'(x)$, at what point is $g(x) < 0$ and $g'(x) < 0$?

Answer the following based on the given graph of f'.

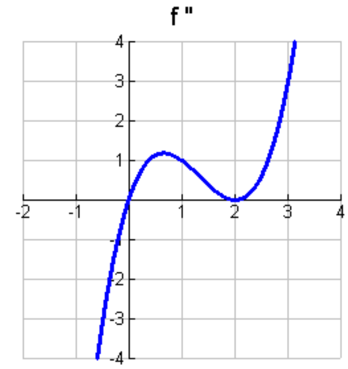


51. What are the critical numbers for f ?

52. Where does f have relative max's?

53. What are the locations for points of inflection on f ?

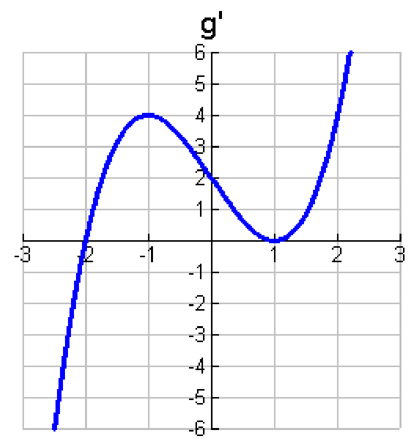
Answer the following based on the given graph of f''.



54. What are the locations for points of inflection on f ?

55. What are the critical numbers for f ?

Use the given graph of g' to answer the following on the interval [-3, 3]:



56. Find the values for which g has local extrema.

57. State the intervals on which g is increasing.

58. Find the values for which g has a point of inflection.

59. State the location and classify the extrema on g''.

60. Sketch a graph of g.

14. Curve Sketching

61. Let f be the function that is even and continuous on the closed interval [-6, 6]. The function f and its derivatives have the properties indicated on the table below.

x	0	0 < x < 2	2	2 < x < 4	4	4 < x < 6
$f(x)$	-2	Negative	0	Positive	2	Positive
$f'(x)$	Undefined	Positive	1	Positive	Undefined	Negative
$f''(x)$	Undefined	Negative	0	Positive	Undefined	Positive

a. Identify all x-coordinates at which f attains local extrema and classify extrema as minimum or maximum. Justify your answer.

b. Identify all x-coordinates of points of inflection. Justify your answer.

c. Sketch a graph of f.

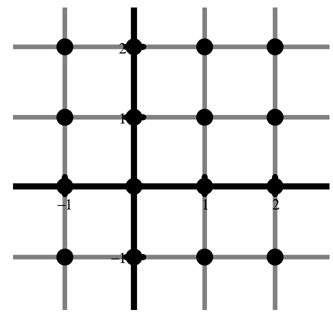
15. Implicit Differentiation

62. If $x^2 + xy = 10$, then when $x = 2$, $\frac{dy}{dx} = ?$

63. Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = 2xy$ with the initial condition $f(0) = 2$.

64. Consider the differential equation $\frac{dy}{dx} = x - 1$.

a. Sketch a slope field for the given differential equation at the twelve indicated points.



b. Sketch the solution curve through the point (1, 0).

c. Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = x - 1$ with the initial condition $f(1) = 0$.

16. Indefinite Integrals

- $\int$ *means find the antiderivative.*
- *Remember your "+ C"*

65. $\int \frac{5}{x^3} dx$

66. $\int \frac{x^3 - 3x^2 + x}{x^2} dx$

67. $\int x\sqrt{x^2 + 5}\,dx$

68. $\int \cos x\, dx$

69. $\int 9x^2 \sec^2\left(3x^3\right) dx$

70. $\int (\csc x \cot x) e^{\cot x}\, dx$

17. Riemann Sums

- *Find a cumulative area of rectangles.*

71. Use a right-hand Riemann sum with four equal subdivisions to estimate the integral $\int_0^2 (-x^2 + 2x) dx$.

72. Use a left-hand Riemann sum with four equal subdivisions to estimate the integral $\int_0^2 (-x^2 + 2x) dx$.

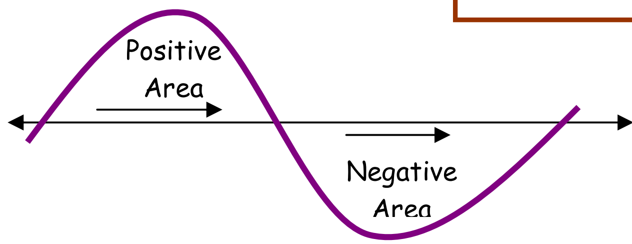
73. Use a mid-point Riemann sum with four equal subdivisions to estimate the integral $\int_0^4 |x - 2| dx$.

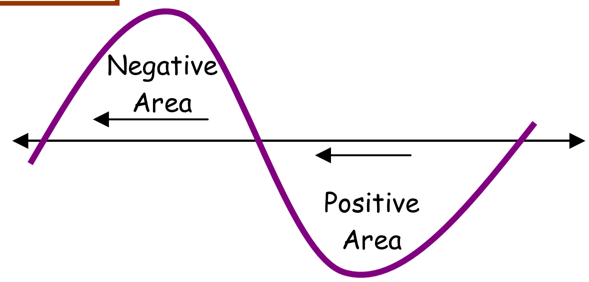
74. Use a trapezoidal Riemann sum with three subdivisions of length 2, 3 and 1 respectively to estimate the integral $\int_0^6 (-x^2 + 6x) dx$.

18. The Fundamental Theorem of Calculus

- *Definite Integrals represent the area under the graph up to the x-axis.*

$$\int_a^b f(x)dx = F(b) - F(a)$$





75. Evaluate $\int_{-1}^{3}(x^2-4)dx$

76. Evaluate $\int_{3}^{-2}(2x-1)dx$

77. Find the value of x and k that divides the area between the x-axis, $x=4$, and $y=\sqrt{x}$ into two regions of equal area.

19. Properties of Definite Integrals

78. If $\int_2^5 g(x)dx = 10$, then find:

a. $\int_2^5 [g(x)+4]dx$

b. $\int_2^5 7g(x)dx$

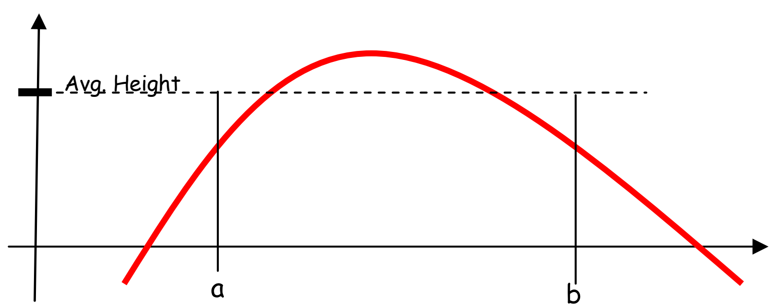
c. $\int_0^3 g(x+2)dx$

20. The Mean Value Theorem for Integrals

$\int_a^b f(x)dx = (b-a)$(Average Height)

Area = width x length

Average Height (Value) $= \dfrac{1}{(b-a)}\int_a^b f(x)dx$



79. Find the average value of $f(x) = x^3 + 8x$ on [1, 3].

80. Find the average velocity of a projectile on the interval [1, 4] for $v(t) = -t^2 + 8t$, where t is measured in seconds and $v(t)$ in cm per second.

b. At what time on [1, 4] does the projectile reach its average velocity on [1, 4]?
(A calculator may be used for question 4b.)

21. The Second Fundamental Theorem of Calculus

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x) * dx$$

Taking the derivative of an integral cancel each other out.

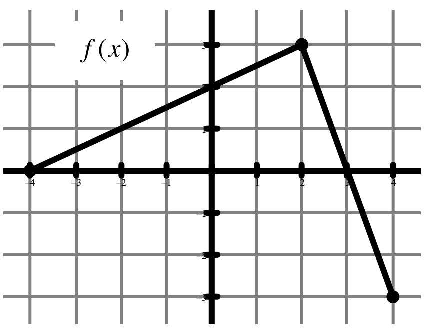
81. Evaluate $\frac{d}{dx}\left[\int_0^{x^3} \sqrt{t^2+1}dt\right]$

82. Evaluate the derivative of $F(x) = \int_{2x}^{x^3} (t+2)dt$

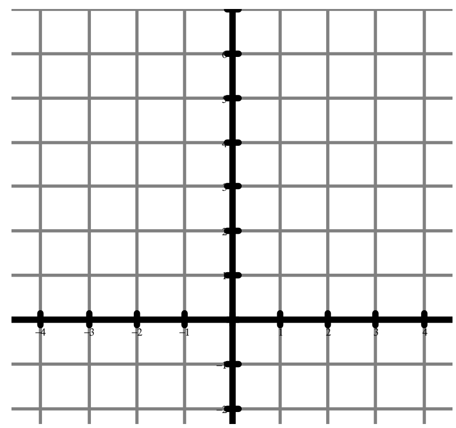
83. Find $f'(2)$ if $f(x) = \int_3^{x^2} t^3 dt$.

22. Functions Defined as Integrals

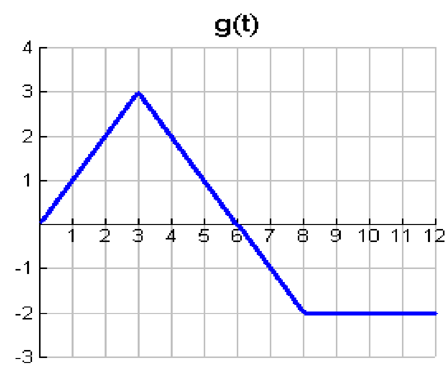
84. Fill in the table and sketch a graph of $g(x)$ if $g(x) = \int_0^x f(x)dx$



x	$g(x)$	$g'(x)$	$g''(x)$
-4			
-2			
0			
2			
3			
4			



85. The graph of function g shown to the right consists of three line segments. Let h be defined as $h(x) = \int_0^x g(t)dt.$



a. Find $h(6)$, $h'(6)$ and $h''(6)$.

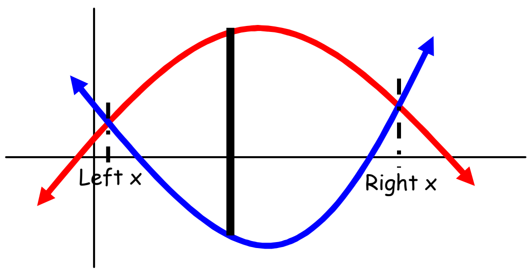
b. For what values of x in the open interval (0, 12) is h increasing?

c. For what values of x in the open interval (0, 12) is h concave down?

d. Sketch a graph of h.

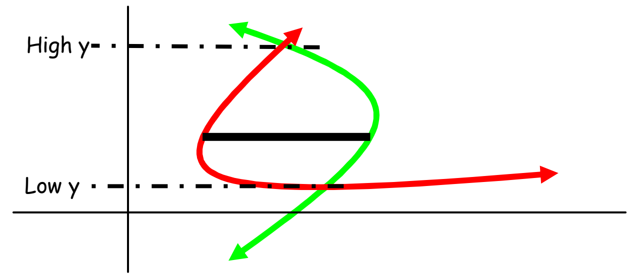
23. Area Between Two Curves

- **BE CAREFUL WHEN DEALING WITH AREA!** When finding area between two curves by $A=\int(Top - Bottom)\ \mathrm{dx}$ the answer is always positive.
- For Accumulated Rates and Definite Integrals - If the AP Exam asks for "the area between a function and the x-axis, determine the areas of regions above and below the axis separately and add them together.



If one function is above the other function on the entire interval, then

$$\int_{\text{left x}}^{\text{right x}}(Top - Bottom)dx$$



If one function is to the right of the other function on the entire interval, then

$$\int_{\text{low y}}^{\text{high y}}(Right - Left)dy$$

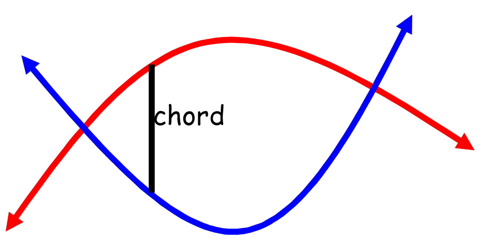
86. Find a value of x, that divides the area bounded by the x-axis and the function $y=x^3+x^2-6x$ into two sectors of equal area.

87. Set up an integral expression and use it to find the area between the x-axis and $y=\cos x$ on $\left[\frac{\pi}{4},\frac{5\pi}{6}\right]$.

24. Finding Volumes

SLABS

Look for the word "cross sections" in the problem.



Chord = side = diameter

$$V=\int_a^b(Area\ formula)dx$$

Isos rt. Tri (sitting on hyp): $\frac{1}{2}\left(\frac{chord}{\sqrt{2}}\right)^2$

Isos rt. Tri (sitting on side): $\frac{1}{2}(chord)^2$

Semicircle: $\frac{1}{2}\pi\left(\frac{chord}{2}\right)^2$

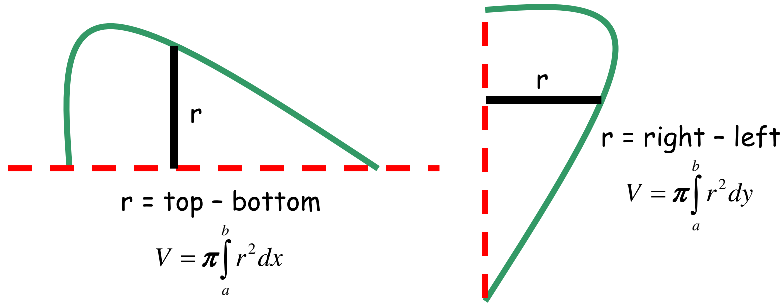
Square: $(chord)^2$

Rectangle: $(chord)(given\,height)$

Eq. Tri: $\frac{\sqrt{3}}{4}(chord)^2$

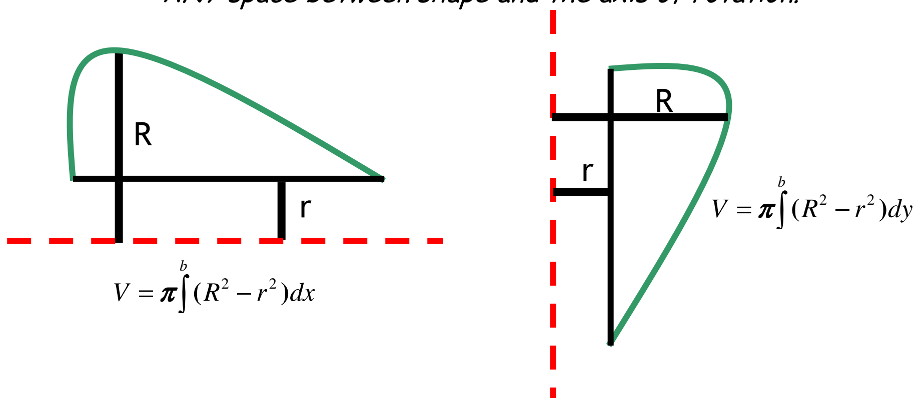
DISCS

NO space between shape and the axis of rotation.



WASHERS

ANY space between shape and the axis of rotation.



88. Find the volume of the solid whose base is bounded by $f(x) = x^2$, $x = 0$, $x = 2$, and $y = 4$ and whose cross sections are semicircles.

89. Find the volume of the solid formed by $f(x) = x^2$, $x = 0$, $x = 2$, and $y = 4$ about:

a. the y-axis b. the x-axis c. the line $y = 4$ d. the line $x = 3$

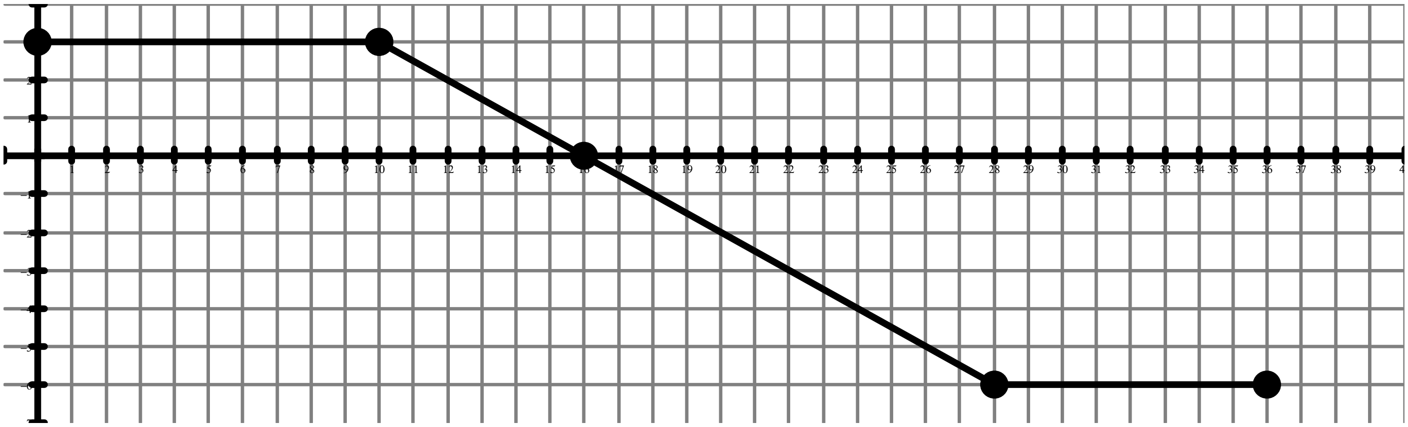
90. Set up an integral expression and use it to find the volume of a three dimensional solid whose base is bounded by the functions $y = e^x$, $y = 2$ and $x = 0$ if every cross section perpendicular to the x-axis is an equilateral triangle.

91. Set up an integral expression and use it to find the volume if the region bounded by the functions $y = e^x$, $y = 2$ and $x = 0$ is rotated about the y-axis.

92. Set up an integral expression and use it to find the volume if the region bounded by the functions $y = e^x$, $y = 2$ and $x = 0$ is rotated about the x-axis.

25. Position, Velocity, Acceleration

The graph above represents the velocity of a particle traveling on the x-axis, where t is time in seconds and $v(t)$ is velocity in ft/sec. The position at $t=0$ is 5 units to the right of the origin.



93. What is the acceleration at $t=12$? *Justify your answer & include appropriate units.*

94. What is the displacement of the particle from $t=0$ to $t=28$? *Justify your answer & include appropriate units.*

95. What is the particle's position at $t=28$? *Justify your answer & include appropriate units.*

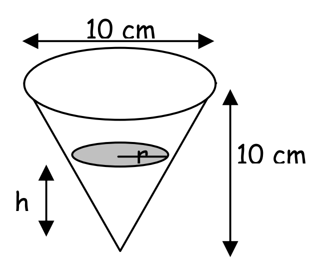
96. What is the total distance traveled from $t=0$ to $t=28$? *Justify your answer & include appropriate units.*

97. When does the particle return to its original position? *Justify your answer & include appropriate units.*

98. On what interval is the particle decelerating? *Justify your answer & include appropriate units.*

26. Related Rates

99. A container has the shape of an open right cone, as shown in the figure. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $-\frac{3}{10} cm/hr$.



a. Find the volume V of water in the container when $h=5$ cm. *Indicate units of measure.*

b. Find the rate of change of the volume of water in the container, with respect to time, when $h=5$ cm. *Indicate units of measure.*

c. Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

27. Adding and Removing Free Response Problems

100. A tank contains 125 gallons of heating oil at time $t=0$. During the time interval $0 \le t \le 12$ hours, heating oil is pumped into the tank at a rate

$$H(t) = 2 + \frac{10}{\left(1+\ln\left(t+1\right)\right)} \text{ gallons per hour.}$$

During the same time interval, heating oil is removed from the tank at the rate

$$R(t) = 12\sin\left(\frac{t^2}{47}\right) \text{ gallons per hour.}$$

a. How many gallons of heating oil are pumped into the tank during the time interval $0 \le t \le 12$ hours?

b. Is the level of heating oil in the tank rising or falling at time $t=6$ hours? Give a reason for your answer.

c. How many gallons of heating oil are in the tank at time $t=12$ hours?

d. At what time t, for $0 \le t \le 12$, is the volume of heating oil in the tank the least? Show the analysis that leads to your conclusion.