Lesson 1A

Warm-up

Find $\frac{d}{dx}$ of y = $3x^2 + 9y^2$

$$\frac{dy}{dx} = 6x\frac{dx}{dx} + 18y\frac{dy}{dx}$$

or $y' = 6x + 18yy'$

Find $\frac{d}{dt}$ of $x^2 + y^2 = 7$

change in y

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

change in t

Find $\frac{d}{dt}$ of V = $\frac{1}{3}\pi r^2 h$ Prod Rule

$$\frac{dV}{dt} = \frac{2}{3}\pi r\frac{dr}{dt}h + \frac{1}{3}\pi r^2\frac{dh}{dt}$$

Find $\frac{d}{dt}$ of $V = \frac{4}{3}\pi r^3$

change in radius

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

change in time

Find $\frac{d}{dt}$ of $SA = 2lw + 2lh + 2wh$

$$\frac{dSA}{dt} = 2\frac{d\ell}{dt}w + 2\ell\frac{dw}{dt} + 2\frac{d\ell}{dt}h + 2\ell\frac{dh}{dt} + 2\frac{dw}{dt}h + 2w\frac{dh}{dt}$$

Find $\frac{d}{dt}$ of A = lw

$$\frac{dA}{dt} = \frac{d\ell}{dt}w + \ell\frac{dw}{dt}$$

Find $\frac{d}{dt}$ of $c^2 = a^2 + b^2$

$$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt}$$

Position

If $s(t) = 4t^2 + 7t + 8$ is the position function of a particle measured in feet per year, then find the instantaneous velocity at t = 3 and include the units.

ft/yr

$s'(t) = v(t)$

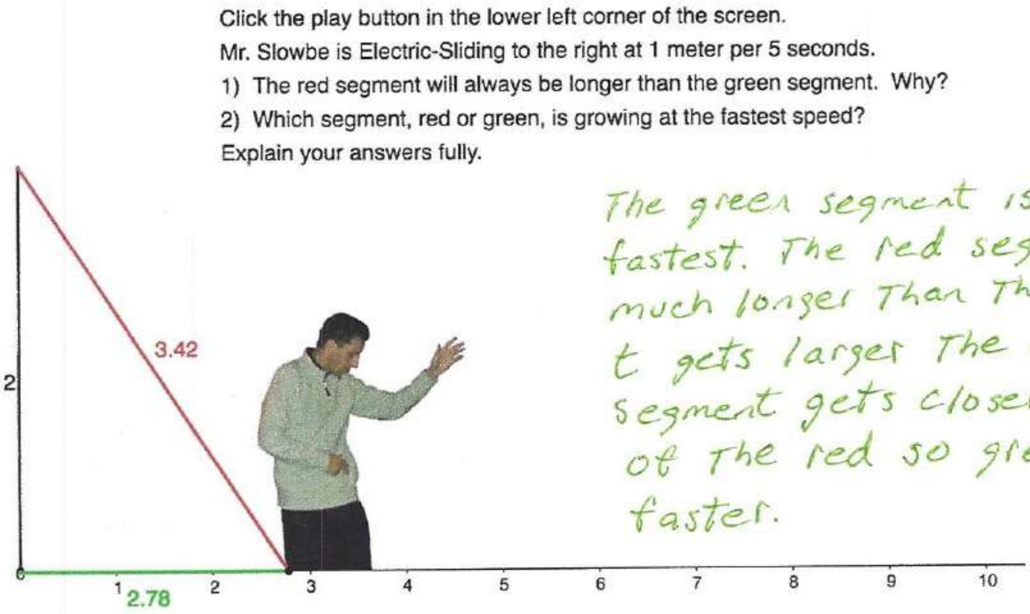
$v(t) = 8t + 7$ $v(3) = 8(3) + 7$

$= 31$ ft/yr

Topic: Related Rates

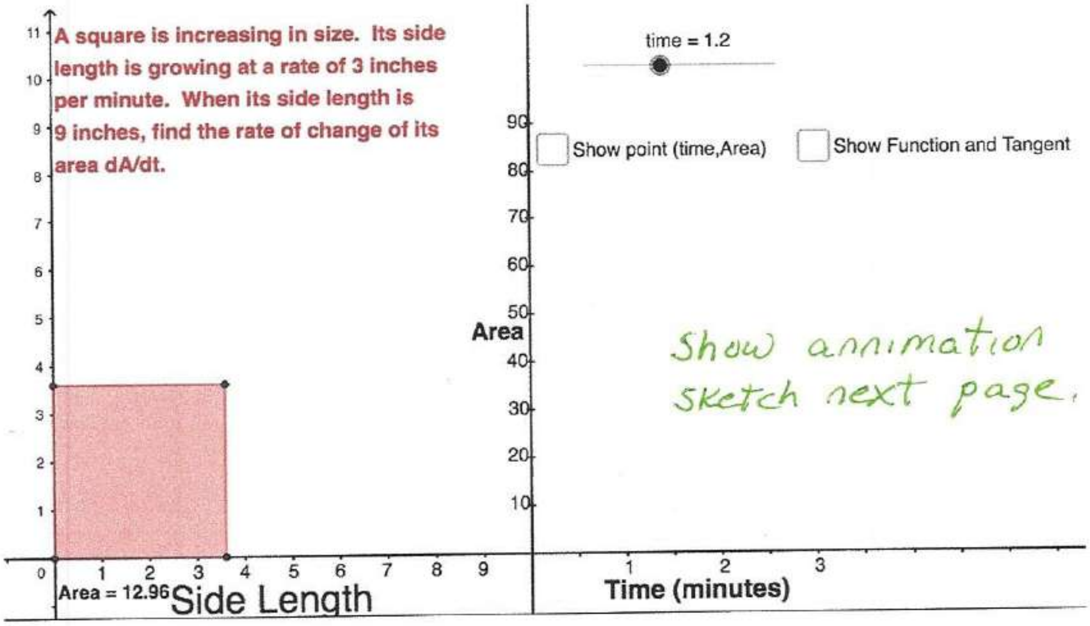
Goal: To solve problems that involve the rate at which a rate is changing.

GeoGebra applets at https://www.geogebra.org/m/CHDeZPzY



Click the play button in the lower left corner of the screen.
Mr. Slowbe is Electric-Sliding to the right at 1 meter per 5 seconds.
1) The red segment will always be longer than the green segment. Why?
2) Which segment, red or green, is growing at the fastest speed?
Explain your answers fully.

The green segment is growing the fastest. The red segment starts out much longer than the green but as t gets larger the length of the green segment gets closer to the length of the red so green is growing faster.



A square is increasing in size. Its side length is growing at a rate of 3 inches per minute. When its side length is 9 inches, find the rate of change of its area dA/dt.

time = 1.2

☐ Show point (time,Area) ☐ Show Function and Tangent

Area = 12.96

Show annimation sketch next page.

Watch the animation several times to develop intuition about this growing square. Then sketch what you think the Time vs Area graph would look like.

A square is increasing in size. Its side length is growing at a rate of 3 inches per minute. When its side length is 9 inches, find the rate of change of its area, $\frac{dA}{dt}$.

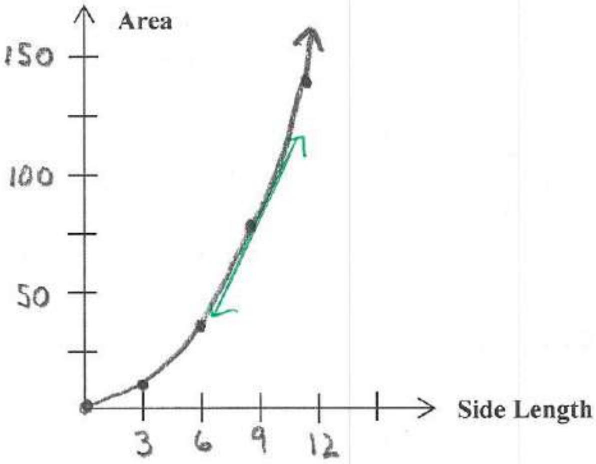
$\frac{ds}{dt} = 3$

$s = 9$

1) There are actually *three* varying quantities in this problem: Time, Side Length, and Area.
First, let's organize these variables in a table for specific values of time.

Time (minutes)	Side Length (inches)	Area (in^2)
0	0	0
1	3	9
2	6	36
3	9	81
4	12	144

2) Sketch the graph of side length vs. Area.



3) Write an equation for the function in #2. What does the slope of the tangent represent in each?

a) In Side Length vs. Area, the equation of the function is $A = s^2$, and the slope of the tangent lines tell us change in area with respect to side length

4) Use your knowledge of implicit differentiation to answer the question at the top of the page.

$A = s^2$

$\frac{d}{dt}(A = s^2)$

$\frac{dA}{dt} = 2s\,\frac{ds}{dt}$

From given info

$\frac{ds}{dt} = 3 \quad s = 9$

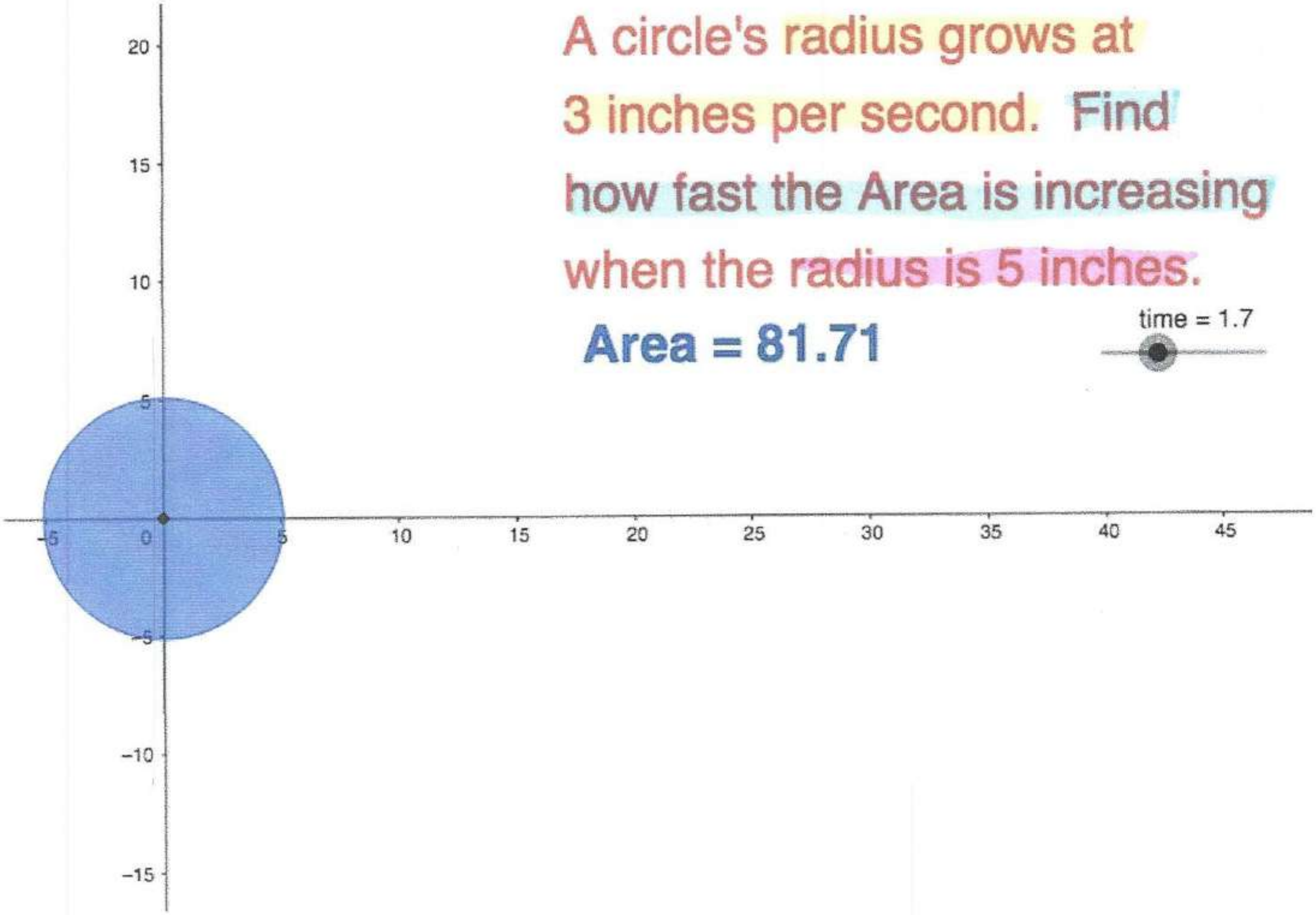
$\frac{dA}{dt} = 2 \cdot 9 \cdot 3$

$= 54$

$\frac{dA}{dt} = 54\ \frac{in^2}{min}$

when the side length is 9 in
the area is changing at
a rate of 54 in²/min

Identify what is given and what you are asked to find by annotating the problem. The answers to the questions below will assist you in solving these problems.



A circle's radius grows at 3 inches per second. Find how fast the Area is increasing when the radius is 5 inches.

Area = 81.71

time = 1.7

1. What rate are you given?

   $\frac{dr}{dt} = 3$ in/sec

2. What rate are you asked to find?

   $\frac{dA}{dt}$

3. What other info are you given?

   $r = 5$ in

4. What info can you get from the problem?

   $A = \pi r^2$ cannot sub 5 for r b/c r is changing

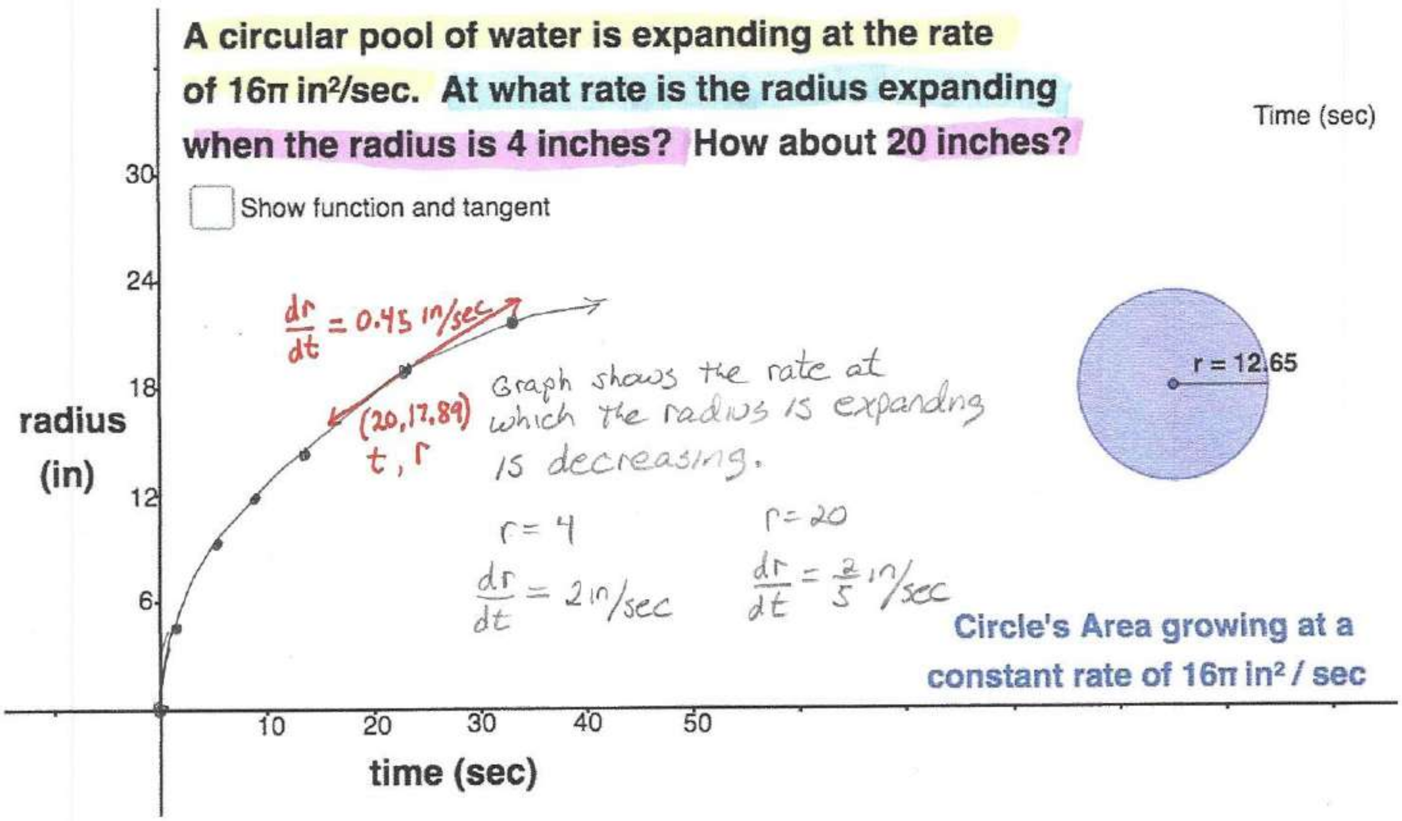
   $\frac{d}{dt}(A = \pi r^2)$

   $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ $r=5$ $\frac{dr}{dt} = 3$

   $\frac{dA}{dt} = 2\pi(5)(3)$

   $\frac{dA}{dt} = 30\pi$ in²/sec

when the radius is 5 in,
the area of the circle
is increasing at a rate
of 30π in²/sec



A circular pool of water is expanding at the rate of 16π in²/sec. At what rate is the radius expanding when the radius is 4 inches? How about 20 inches?

☐ Show function and tangent

Graph shows the rate at which the radius is expanding is decreasing.

r = 4

$\frac{dr}{dt} = 2$ in/sec

r = 20

$\frac{dr}{dt} = \frac{2}{5}$ in/sec

1. Given rate: $\frac{dA}{dt} = 16\pi$ in²/sec   2. Rate to find: $\frac{dr}{dt}$

3. Other info: r = 4 in, r = 20 in

$$\text{Area} = \pi r^2$$

$$\frac{d}{dt}(A = \pi r^2)$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

r = 4 in

$$16\pi = 2\pi(4)\frac{dr}{dt}$$

$$16\pi = 8\pi \frac{dr}{dt}$$

$$\frac{dr}{dt} = 2 \text{ in/sec}$$

r = 20 in

$$16\pi = 2\pi(20)\frac{dr}{dt}$$

$$16\pi = 40\pi \frac{dr}{dt}$$

$$\frac{16\pi}{40\pi} = \frac{dr}{dt} \quad \frac{dr}{dt} = \frac{2}{5} \text{ in/sec}$$

When the radius is 4 in, the radius is expanding at a rate of 2 in/sec

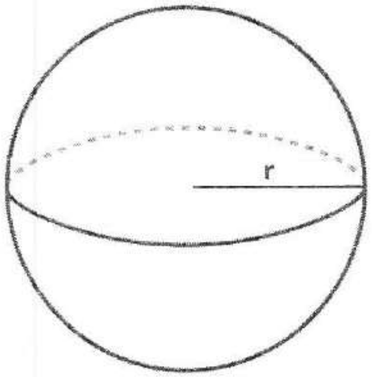
When the radius is 20 in, the radius is expanding at a rate of $\frac{2}{5}$ in/sec.

SPHERES

What's the Trick? Focus on what you are trying to find and what you need in order to find that variable. Often the problem will give you the information you need, but you have to solve one thing in order to find another.

A spherical balloon is being inflated at a constant rate of 5 cubic inches per minute. When the radius of the balloon is 4 inches, how fast is the surface area of the balloon changing? $\frac{dv}{dt}$

$\frac{ds}{dt}$



1. What rate are you given?

   $\frac{dv}{dt} = 5$

2. What rate are you asked to find?

   $\frac{ds}{dt} = ?$

3. What other info are you given?

   $r = 4$

4. What info can you get from the problem and what formulas will you need?

   $\text{Vol Sphere} = \frac{4}{3}\pi r^3$

   $\text{SA Sphere} = 4\pi r^2$

Use Vol to get $\frac{dr}{dt}$

$\frac{d}{dt}\left(V = \frac{4}{3}\pi r^3\right)$

$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$

$5 = 4\pi \cdot 4^2 \frac{dr}{dt}$

$\frac{5}{64\pi} = \frac{dr}{dt}$

* start here

$\frac{d}{dt}\left(SA = 4\pi r^2\right)$

$\frac{ds}{dt} = 8\pi r \frac{dr}{dt}$

$\frac{ds}{dt} = 8\pi \cdot 4 \cdot \frac{5}{64\pi}$

$\frac{ds}{dt} = \frac{20}{8} = \frac{5}{2}\ in^2/min$

when the radius of the balloon is 4 inches the surface area of the balloon is changing at a rate of $\frac{5}{2}\ in^2/min$.

Assignment: #1-4 next 2 pages

Related Rates Practice Problems

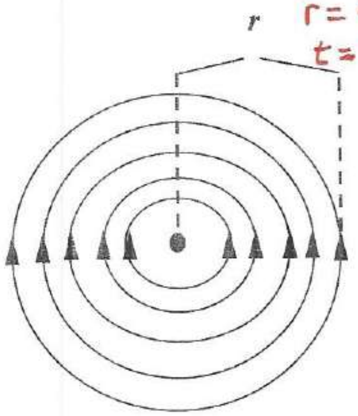
1) A pebble is dropped into a calm pool of water, causing ripples in the form of concentric circles. If each ripple moves out from the center at a rate of 1 ft/s, at what rate is the total area of disturbed water increasing after 4 seconds?

$r = 4$ ft

$t = 4$

$\frac{dr}{dt} = 1$

$\frac{dA}{dt} = ?$



$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi \cdot 4 \cdot 1$$

$$\frac{dA}{dt} = 8\pi \text{ ft}^2/\text{sec}$$

2) A balloon is being inflated at a constant rate of 4.5 cubic feet per minute. Find the rate of change of the radius when the radius is 2 feet.

$\frac{dr}{dt} = ?$

$r = 2$

$\frac{dv}{dt} = 4.5$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$4.5 = 4\pi \cdot 2^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{4.5}{16\pi} \text{ ft}/\text{min}$$

3) A balloon is being filled with helium at the rate of 4 cubic ft/min. Find the rate, in square feet per minute, at which the surface area is increasing when the volume is $\frac{32\pi}{3}$ cubic feet.

$\frac{dV}{dt} = 4 \text{ ft}^3/\text{min}$ Find $\frac{dA}{dt}$ when Vol $= \frac{32\pi}{3}$ ft^3 and $r = 2$

$V = \frac{4}{3}\pi r^3 \leftarrow$ Need r $\quad A = 4\pi r^2$

$\frac{32\pi}{3} = \frac{4}{3}\pi r^3$

$8 = r^3$

$* \; r = 2$

$\therefore \frac{dA}{dt} = 8\pi r \frac{dr}{dt}$

Need $\frac{dr}{dt}$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$4 = 4\pi \cdot 2^2 \frac{dr}{dt}$

$* \; \frac{dr}{dt} = \frac{1}{4\pi}$

$\frac{dA}{dt} = 8\pi \cdot 2 \cdot \frac{1}{4\pi}$

$\frac{dA}{dt} = 4 \text{ ft}^2/\text{min}$

4) The radius of a sphere is increasing at the uniform rate of 0.3 inches per second. At the instant when the surface area, S, becomes 100π square inches, what is the rate of increase, in cubic inches per second, in the volume, V?

$\frac{dr}{dt} = 0.3 \text{ in/sec}$ $\quad S = 100\pi \text{ in}^2$ $\quad \frac{dV}{dt} = ?$

so $r = 5$

$A = 4\pi r^2$

$100\pi = 4\pi r^2$

$25 = r^2$

$r = 5$

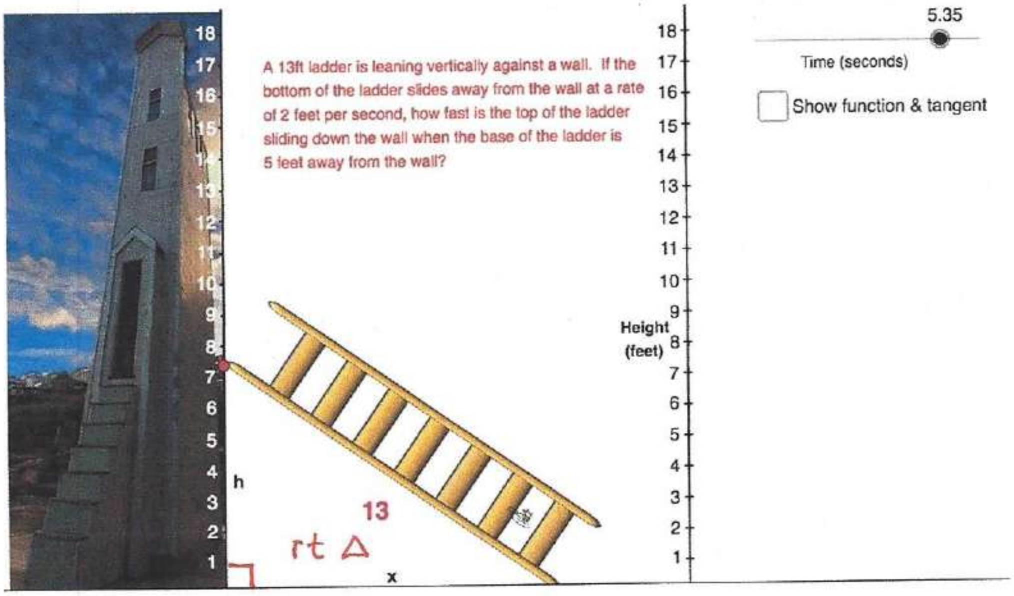
$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\frac{dV}{dt} = 4\pi \cdot 5^2 (0.3)$

$\frac{dV}{dt} = 100\pi \cdot \frac{3}{10}$

$\frac{dV}{dt} = 30\pi \text{ in}^3/\text{sec}$



A 13ft ladder is leaning vertically against a wall. If the bottom of the ladder slides away from the wall at a rate of 2 feet per second, how fast is the top of the ladder sliding down the wall when the base of the ladder is 5 feet away from the wall?

Questions:

Does the top of the ladder fall faster near $t = 0$ or as t gets bigger?

As t gets bigger

Why is $\frac{dh}{dt}$ negative?

b/c The height of The ladder is dec.

Does the ladder change length? No

1. What rate are you given?

$\frac{dx}{dt} = 2$ ft/sec

2. What rate are you asked to find?

$\frac{dh}{dt} = ?$ when $x = 5$

3. What other info are you given?

$C = 13$

4. What info can you get from the problem and what formulas will you need?

$a^2 + b^2 = c^2$

$h^2 + 5^2 = 13^2$

$h^2 = 144$

$h = 12$

$\frac{d}{dt}\left(x^2 + h^2 = 13^2\right)$

$2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0 \rightarrow$ Need h

$2 \cdot 5 \cdot 2 + 2 \cdot 12 \frac{dh}{dt} = 0$

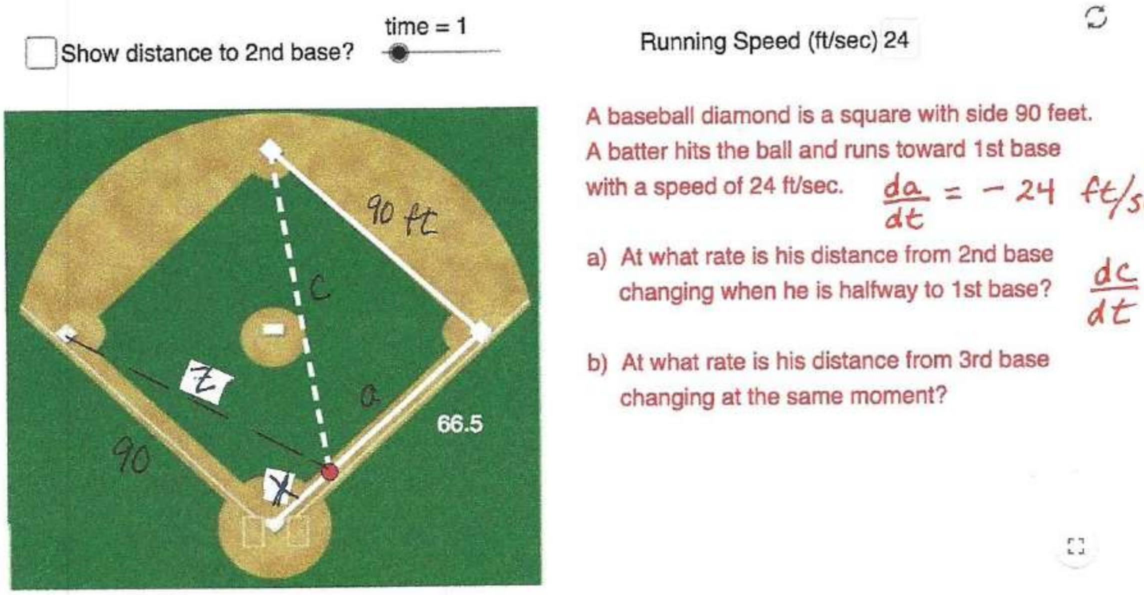
$20 + 24\frac{dh}{dt} = 0$

$24\frac{dh}{dt} = -20$

$\frac{dh}{dt} = \frac{-20}{24} = \frac{-5}{6}$ ft/sec

h changes and x changes so you cannot sub these into your original equation.

The top of the Ladder is falling at a rate of $\frac{5}{6}$ ft per sec when the base of the ladder is 5 ft from the wall.



☐ Show distance to 2nd base? time = 1

Running Speed (ft/sec) 24

A baseball diamond is a square with side 90 feet.
A batter hits the ball and runs toward 1st base
with a speed of 24 ft/sec. $\frac{da}{dt} = -24$ ft/sec decreasing

a) At what rate is his distance from 2nd base changing when he is halfway to 1st base? $\frac{dc}{dt} = ?$ when $a = 45$

b) At what rate is his distance from 3rd base changing at the same moment?

a) $(a^2 + 90^2 = c^2)\frac{d}{dt}$

$2a\frac{da}{dt} = 2c\frac{dc}{dt}$

Need c when $a = 24$

$c^2 = 24^2 + 90^2$

$c = 100.623$ ft

Hey!

$\frac{2(24)(-24)}{2(100.623)} = \frac{2(100.623)}{2(100.623)}\frac{dc}{dt}$

$\frac{dc}{dt} = -10.733$ ft/sec

The batter's distance from 2nd base is decreasing at a rate of 10.733 ft/sec when he is halfway to 1st base.

b) $(x^2 + 90^2 = z^2)\frac{d}{dt}$

$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$

Need z when $x = 24$

$z^2 = 90^2 + 24^2$

$z = 100.623$ ft

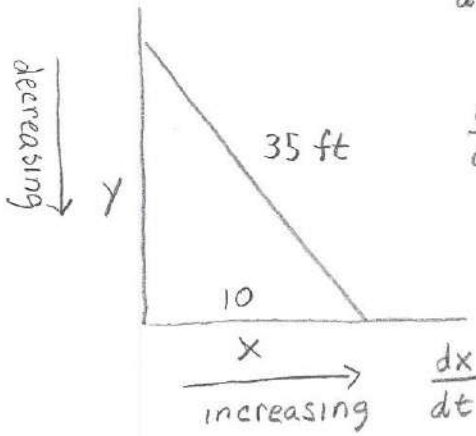
$2(24)(-24) = 2(100.623)\frac{dz}{dt}$

$\frac{dz}{dt} = 10.733$ ft/sec

The batter's distance from 3rd base is increasing at a rate of 10.733 ft/sec when he is halfway to 1st base.

Assignment: #1 + 2 next page

1) A ladder 35 ft. long is leaning against a house. If the base of the ladder is pulled away from the wall at a rate of 3 ft/s, how fast is the top of the ladder moving down the wall when the base of the ladder is 10 ft from the wall?



$\frac{dy}{dt} = ?$ when $x = 10$

$x^2 + y^2 = 35^2$

$\frac{d}{dt}(x^2+y^2=35)^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$

$\frac{dx}{dt} = 3 \text{ ft/sec}$

Need y when $x = 10$

$10^2 + y^2 = 35^2$

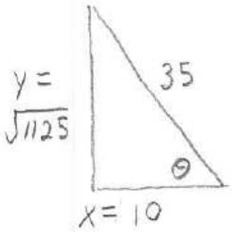
$y^2 = 11.25$

$y = \sqrt{1125}$

$10(3) + \sqrt{1125}\frac{dy}{dt} = 0$

$\frac{dy}{dt} = \frac{-30}{\sqrt{1125}}$ ft/sec

How fast is the angle between the ground and the base of the ladder changing in the problem above?



$\frac{d\theta}{dt} = ?$ decreasing

$\cos\theta = \frac{x}{35}$

$\frac{d}{dt}\left(\cos\theta = \frac{x}{35}\right)$

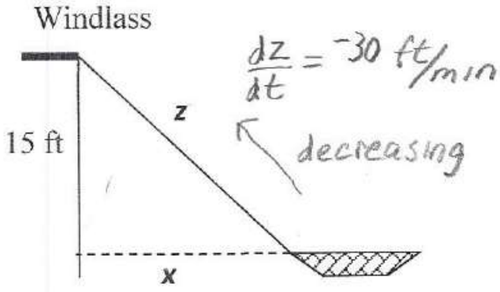
$(-\sin\theta)\frac{d\theta}{dt} = \frac{1}{35}\frac{dx}{dt}$

$-\frac{\sqrt{1125}}{35}\frac{d\theta}{dt} = \frac{1}{35}(3)$

$\frac{d\theta}{dt} = \frac{3}{35}\cdot\frac{-35}{\sqrt{1125}}$

$\frac{d\theta}{dt} = \frac{-3}{\sqrt{1125}}$ rad/sec

2) A windlass is used to tow a boat to the dock. The rope is attached to the boat at a point 15 ft below the level of the windlass. If the windlass pulls in the rope at a rate of 30 ft/min, at what rate is the boat approaching the dock when there is 75 ft of rope out? When there is 25 ft of rope out?



$\frac{dz}{dt} = -30$ ft/min

$\frac{dx}{dt} = ?$ when $z = 75$

decreasing

$15^2 + x^2 = z^2$

$\frac{d}{dt}(15^2 + x^2 = z^2)$

$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$

$x\frac{dx}{dt} = z\frac{dz}{dt}$

Need x

$15^2 + x^2 = 75^2$

$x^2 = 5400$

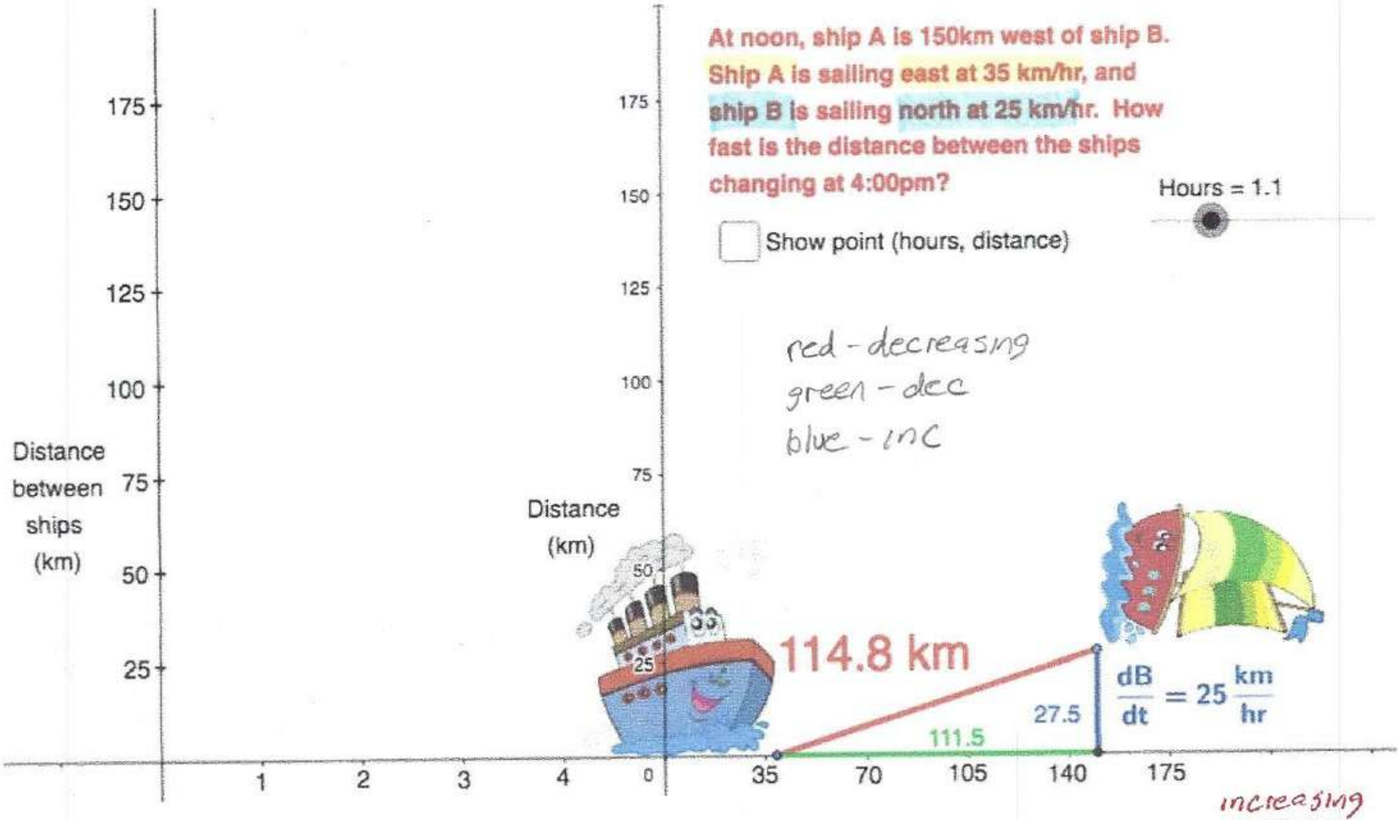
$x = \sqrt{5400}$

$x = 30\sqrt{6}$

$30\sqrt{6}\frac{dx}{dt} = 75(-30)$

$\frac{dx}{dt} = \frac{75(-30)}{30\sqrt{6}}$

$\frac{dx}{dt} = \frac{-75}{\sqrt{6}}$ ft/min



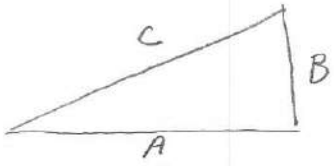
At noon, ship A is 150km west of ship B. Ship A is sailing east at 35 km/hr, and ship B is sailing north at 25 km/hr. How fast is the distance between the ships changing at 4:00pm?

Hours = 1.1

☐ Show point (hours, distance)

red - decreasing
green - dec
blue - inc

increasing distance



$\frac{dB}{dt} = 25\ KM/hr$ when $B=100$

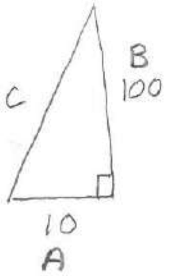
At 4PM, each ship has sailed for 4 hours.

Ship A $4(35) = 140$ KM
Ship B $4(25) = 100$ KM

$\frac{dA}{dt} = -35\ Km/hr$ when $A=10$

decreasing distance

Since ship A is sailing east, it is 10 miles To where ship B started



$c^2 = 10^2 + 100^2$

$c \approx 100.499$ KM

$A^2 + B^2 = C^2$

All 3 are changing so you cannot sub for any variable

$\frac{d}{dt}(A^2+B^2=C^2)$

$2A\frac{dA}{dt} + 2B\frac{db}{dt} = 2C\frac{dc}{dt}$

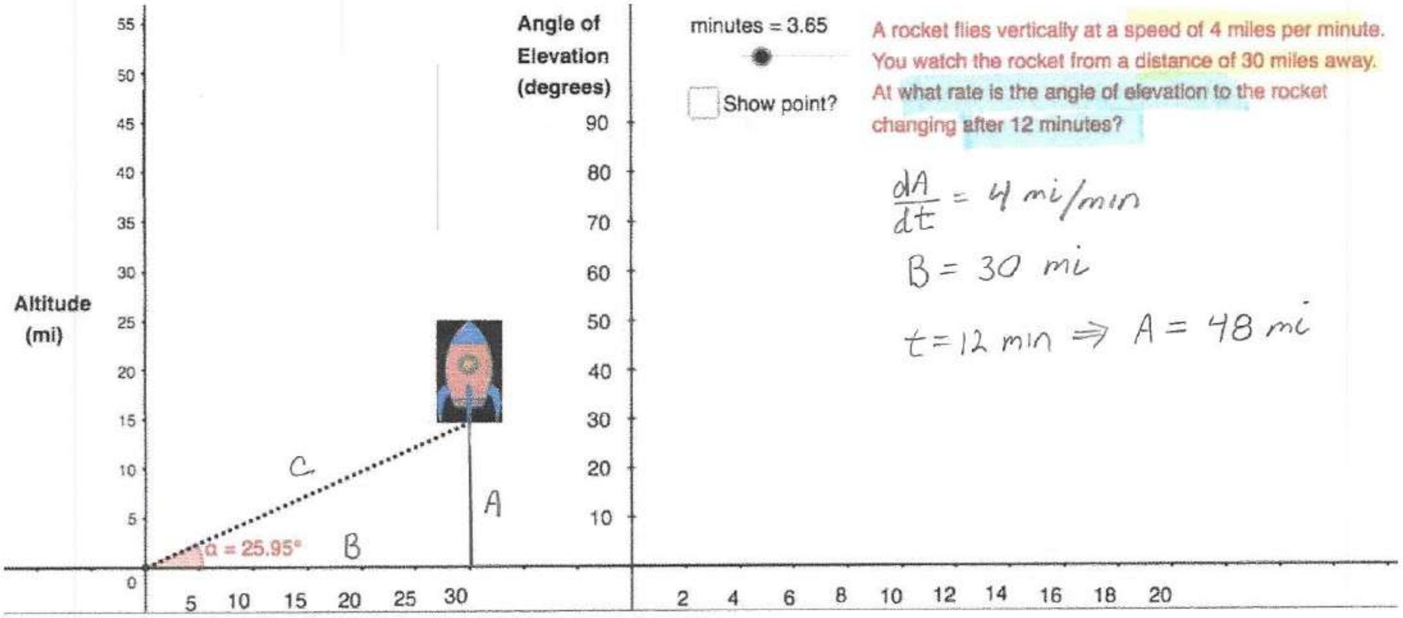
$2(10)(-35) + 2(100)(25) = 2(100.499)\frac{dc}{dt}$

$-700 + 5000 = 200.998\frac{dc}{dt}$

$4300 =$

$\frac{dc}{dt} = 21.393\ KM/hr$

At 4 pm the distance between the ships is increasing at a rate of 21.393 Km/hr



minutes = 3.65

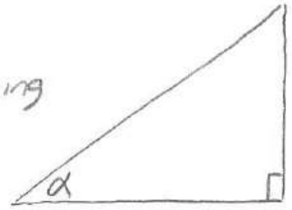
☐ Show point?

A rocket flies vertically at a speed of 4 miles per minute. You watch the rocket from a distance of 30 miles away. At what rate is the angle of elevation to the rocket changing after 12 minutes?

$\frac{dA}{dt} = 4$ mi/min

$B = 30$ mi

$t = 12$ min $\Rightarrow A = 48$ mi



* α is changing

* changing

B=30

* not changing

$\sin\alpha = \frac{A}{C}$

$\cos\alpha = \frac{B}{C}$

$\tan\alpha = \frac{A}{B}$

$\tan\alpha = \frac{A}{30}$

$\tan\alpha = \frac{1}{30}A$

$\frac{d}{dt}\left(\tan\alpha = \frac{1}{30}A\right)$

$\sec^2\alpha \frac{d\alpha}{dt} = \frac{1}{30}\frac{dA}{dt}$

when $A = 48$ and $B = 30$

$C^2 = 48^2 + 30^2$

$C = 56.604$

$\sec\alpha = \frac{56.604}{30}$

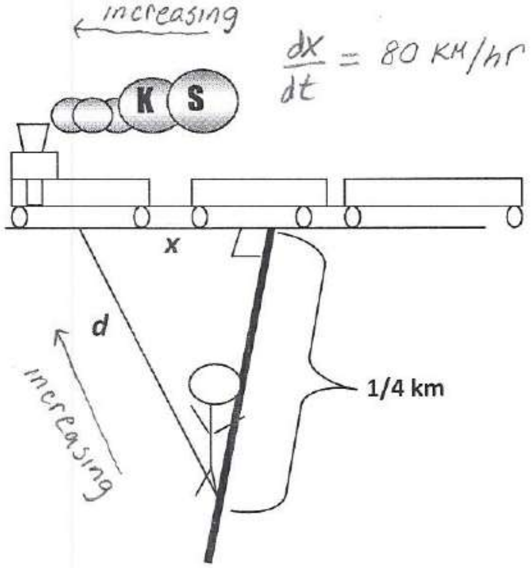
$\left(\frac{56.604}{30}\right)^2 \frac{d\alpha}{dt} = \frac{1}{30}(4)$

$\frac{d\alpha}{dt} = \frac{4}{30}\left(\frac{30}{56.604}\right)^2$

$\frac{d\alpha}{dt} = 0.037$ rad/min

Assignment: #1+2 next page

1) Suppose a person is standing along a straight road 1/4 km from a railroad that crosses the road at right angles. If a train is moving at a constant speed of 80 km/h, at what rate is the distance between the person and the engine changing when the engine is 1/2 km past the crossing?



$\frac{dx}{dt} = 80$ KM/hr

$$x^2 + \left(\tfrac{1}{4}\right)^2 = d^2$$

$$\frac{d}{dt}\left(x^2 + \left(\tfrac{1}{4}\right)^2 = d^2\right)$$

$$2x\frac{dx}{dt} = 2d\frac{dd}{dt}$$

$$x\frac{dx}{dt} = d\frac{dd}{dt}$$

Need d

$$\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{4}\right)^2 = d^2$$

$$\frac{1}{4} + \frac{1}{16} = d^2$$

$$\frac{5}{16} = d^2$$

$$d = \frac{\sqrt{5}}{4}$$

$$\frac{1}{2}(80) = \frac{\sqrt{5}}{4}\frac{dd}{dt}$$

$$40 = \frac{\sqrt{5}}{4}\frac{dd}{dt}$$

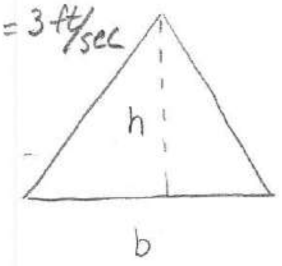
$$\frac{160}{\sqrt{5}} \text{ km/hr} = \frac{dd}{dt}$$

$\frac{dd}{dt} = ?$

when $x = \frac{1}{2}$

2) The base of a triangle is increasing at the rate of 2 ft per sec. and the height is increasing at the rate of 3 ft per sec. What is the rate of change of the area of the triangle changing when the base is 4 ft and the height is 9 ft?

$\frac{dh}{dt} = 3$ ft/sec



$\frac{db}{dt} = 2$ ft/sec

$\frac{dA}{dt} = ?$ when
$b = 4$
$h = 9$

$$\text{Area} = \frac{1}{2}bh$$

$$\frac{d}{dt}\left(A = \frac{1}{2}bh\right)$$

$$\frac{dA}{dt} = \frac{1}{2}\frac{db}{dt}h + \frac{1}{2}b\frac{dh}{dt}$$

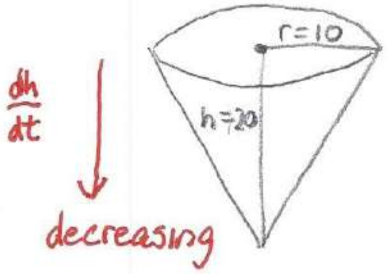
$$\frac{dA}{dt} = \frac{1}{2}(2)(9) + \frac{1}{2}(4)(3)$$

$$\frac{dA}{dt} = 15 \text{ ft}^2/\text{sec}$$

CONES

What's the Trick? To take the derivative of the volume of a cone, you need to use a proportion. The radius and height will always be related in some way and you will need to write an equation to show that.

Example 1: A circular conical reservoir has depth 20 feet and radius of the top 10 feet. Water is leaking out so that the surface is falling at the rate of $\frac{1}{2}\frac{ft}{hr}$. Find the rate, in cubic feet per hour, at which the water is leaving the reservoir when the water is 8 feet deep.



$\frac{dh}{dt} = -\frac{1}{2}$ ft/hr

Find $\frac{dV}{dt}$ when $h = 8$ ft

$V = \frac{1}{3}\pi r^2 h$

Too many variables

Relationship between radius + height

$\frac{r}{h} = \frac{10}{20}$ Solve for r b/c we

$20r = 10h$ Know $\frac{dh}{dt}$.

$r = \frac{1}{2}h$ Now sub $\frac{1}{2}h$ for r.

$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$

$V = \frac{1}{12}\pi h^3$

$\frac{d}{dt}\left(V = \frac{1}{12}\pi h^3\right)$

$\frac{dV}{dt} = \frac{3}{12}\pi h^2 \frac{dh}{dt}$ Sub $\frac{dh}{dt} = \frac{1}{2}$ and $h = 8$

$\frac{dV}{dt} = \frac{1}{4}\pi(8)^2\left(-\frac{1}{2}\right)$

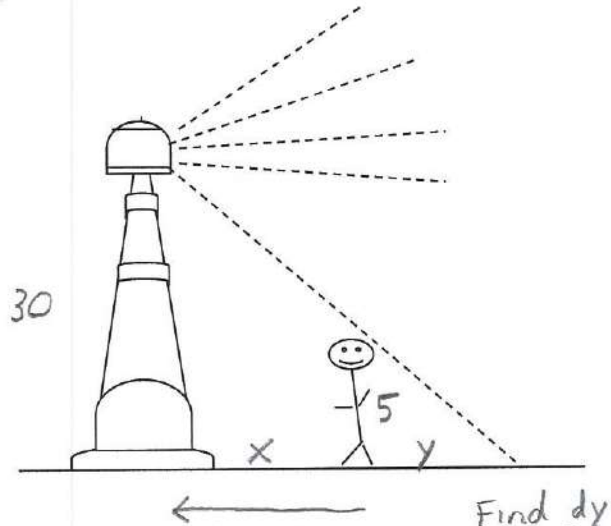
$\frac{dv}{dt} = -8\pi$ ft^3/hr

when the water is 8 ft high, the volume of water in the tank is decreasing at a rate of 8π ft^3/hr

THE SHADOW PROBLEM

What's the Trick? You have to define your variables in a very specific way to make this problem manageable AND you have to use proportions (remember the cones earlier?)

Example 1: A young boy is out at night running toward a street lamp at 6 feet per second. If the streetlamp is 30 feet tall and the boy is 5 feet tall in his running stance, how fast is his shadow length changing when he is 4 feet from the base of the lamppost?



$\frac{dx}{dt} = -6 \text{ ft/sec}$

Find $\frac{dy}{dt}$ when $x = 4$

$$\frac{30}{5} = \frac{x+y}{y}$$

$$30y = 5x + 5y$$

$$\frac{d}{dt}\left(25y = 5x\right)$$

$$25\frac{dy}{dt} = 5\frac{dx}{dt}$$

$$25\frac{dy}{dt} = 5(-6)$$

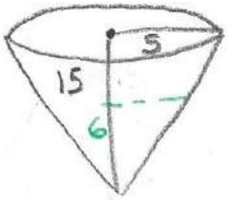
$$\frac{dy}{dt} = \frac{-30}{25} \text{ ft/sec}$$

Assignment: #1-3 next 2 pages

Related Rates Practice Problems

1) A water tank has the shape of an inverted cone with altitude 15 ft and base radius 5 ft. If water is pumped in at 13 ft^3 per minute, how fast is the water level rising when the water is 6 ft deep?

$\frac{dV}{dt} = 13$ $\frac{dh}{dt}$ $h = 6$



$\frac{5}{15} = \frac{r}{h}$

$\frac{1}{3} = \frac{r}{h}$

$\frac{h}{3} = r$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h$

$V = \frac{\pi}{3} \cdot \frac{h^3}{9}$

$V = \frac{\pi}{27} h^3$

$\frac{dv}{dt} = \frac{\pi}{9} h^2 \frac{dh}{dt}$

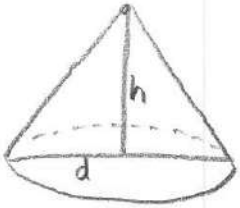
$13 = \frac{\pi}{9} \cdot 6^2 \frac{dh}{dt}$

$\frac{13}{4\pi}$ ft/min $= \frac{dh}{dt}$

$d = 2h$

2) Sand is being poured onto a beach creating a cone whose base diameter is always twice its height. The sand is being poured at the rate of 20 cubic inches per second. When the height of the conical pile is 6 inches, how fast is the radius changing? $\frac{dv}{dt} = 20$ $h = 6$

$\frac{dr}{dt}$



$d = 2h$

$2r = 2h$

$r = h$

So when $h=6$

Then $r=6$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi r^2 \cdot r$

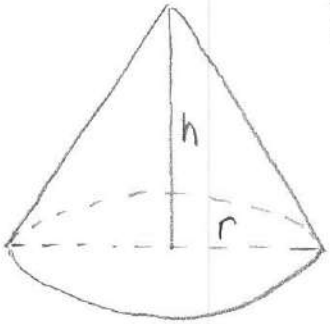
$V = \frac{1}{3}\pi r^3$

$\frac{dV}{dt} = \pi r^2 \frac{dr}{dt}$

$20 = \pi \cdot 6^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{20}{36\pi}$ in/sec

7) Sand is falling into a conical pile at the rate of 40 cubic feet per minute. The diameter of the base is always four times the altitude. At what rate is the height of the pile changing when it is 20 ft high?



$\frac{dv}{dt} = 40$ cu ft/min altitude = height

$d = 4h$ $\frac{dh}{dt} = ?$ when $h = 20$

$2r = 4h$

$r = 2h$

$$\text{Volume} = \frac{1}{3}\pi r^2 h$$

$$\boxed{\frac{d}{dt}\left(v = \frac{1}{3}\pi r^2 h\right) \quad \text{don't want to do } \frac{dr}{dt} \text{ b/c we don't have a value to plug in.}}$$

sub 2h for r

$$\frac{d}{dt}\left(v = \frac{1}{3}\pi 4h^3\right)$$

$$\frac{dv}{dt} = 4\pi h^2 \frac{dh}{dt}$$

$$40 = 4\pi(20)^2 \frac{dh}{dt}$$

$$\frac{40}{4\pi(400)} = \frac{dh}{dt}$$

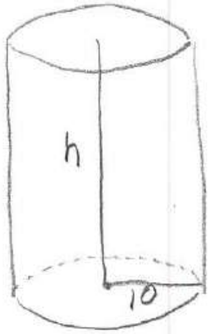
$$\frac{dh}{dt} = \frac{1}{40\pi} \text{ ft/min}$$

Unit #6 Assignment #1 Handout

Related Rates: Change/Time

Purpose: To develop a method for calculating the rates of change of two or more related variables, each of which changes with respect to time.

1) The volume of a cylindrical can is given by $V = \pi r^2 h$. The radius of the can is 10 feet. If water is being poured into the can at the rate of 4 ft^3/min, what is the rate of change of the height? ***(1/25πft/min)***



$\frac{dV}{dt} = 4\,ft^3/min$ $\qquad$ $\frac{dh}{dt}$

$V = \pi(10^2)h$

$V = 100\pi h$

$\frac{d}{dt}(V = 100\pi h)$

$\frac{dV}{dt} = 100\pi\frac{dh}{dt}$

$4 = 100\pi\frac{dh}{dt}$

$\frac{4}{100\pi} = \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{25\pi}\ ft/min$

2) The volume of a cylindrical can is given by $V = \pi r^2 h$. The radius of the can is 6 feet. If the height of the water is changing at the rate 2 ft/min, what is the rate of change of the volume? ***(72πft^3/min)***

$\frac{dh}{dt} = 2\,ft/min$ $\qquad$ $\frac{dV}{dt} = ?$

$V = \pi(6)^2h$

$\frac{d}{dt}(V = 36\pi h)$

$\frac{dV}{dt} = 36\pi\frac{dh}{dt}$

$\frac{dV}{dt} = 36\pi(2)$

$\frac{dV}{dt} = 72\pi\ ft^3/min$

3) The length of a rectangle is increasing at the rate of 4 yards per minute and the width is decreasing at the rate of 2 yd per minute.

A. What is the rate of change of the area, when the length is 8 yd and the width is 5 yd? ***(4 yd²/min)***

$L = 8$ $\quad w = 5$

dec ↓

$\frac{dw}{dt} = -2 \text{ yd/min}$

inc →

$\frac{dL}{dt} = 4 \text{ yd/min}$

$\text{Area} = LW$

$\frac{d}{dt}(A = LW)$

$\frac{dA}{dt} = \frac{dL}{dt} \cdot W + L \cdot \frac{dw}{dt}$

$\frac{dA}{dt} = 4(5) + 8(-2)$

$\frac{dA}{dt} = 4 \text{ yd}^2/\text{min}$

B. What is the rate of change of the perimeter, when the length is 8 yd and the width is 5 yd? ***(4 yd/min)***

$P = 2L + 2W$

$\frac{d}{dt}(P = 2L + 2W)$

$\frac{dP}{dt} = 2\frac{dL}{dt} + 2\frac{dw}{dt}$

$\frac{dP}{dt} = 2(4) + 2(-2)$

$\frac{dP}{dt} = 4 \text{ yd/min}$

C. If the length is always twice the width and the rate of change of the length is now 7 yd/min, what is the rate of change of the area when the width is 20 yd? ***(280 yd²/min)***

$L = 2W$ or $\frac{L}{2} = W$

$\frac{dL}{dt} = 7 \text{ yd/min}$

$\frac{dA}{dt} = ?$

when $W = 20$

Then $L = 40$

$A = LW$

$A = L \cdot \frac{L}{2}$

$A = \frac{1}{2}L^2$

$\frac{d}{dt}\left(A = \frac{1}{2}L^2\right)$

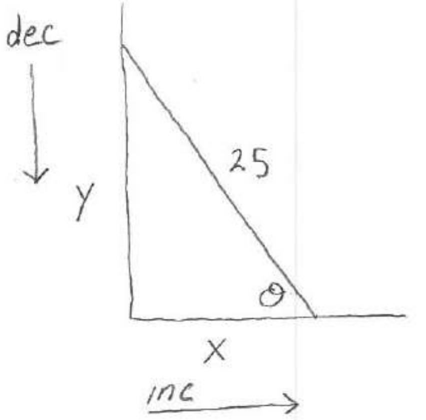
$\frac{dA}{dt} = L\frac{dL}{dt}$

$\frac{dA}{dt} = 40(7)$

$\frac{dA}{dt} = 280 \text{ yd}^2/\text{min}$

4) A ladder 25 ft. long is leaning against a house. If the base of the ladder is pulled away from the wall at a rate of 2 ft/s, how fast is the top of the ladder moving down the wall when the base of the ladder is:

a) 7 ft. from the wall? How fast is the angle the base of the ladder makes with the floor changing? ***(−7/12 ft/sec, −1/12 rad/sec)***



$\frac{dx}{dt} = 2$ ft/sec

$\frac{dy}{dt} = ?$ when $x = 7$

$x^2 + y^2 = 25^2$

$7^2 + y^2 = 25^2$

$y^2 = 576$

$y = 24$

$\frac{d}{dt}(x^2 + y^2 = 25^2)$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$

$7(2) + 24\frac{dy}{dt} = 0$

$\frac{dy}{dt} = \frac{-7}{12}$ ft/sec

$\sin\theta = \frac{y}{25}$

$\frac{d}{dt}\left(\sin\theta = \frac{y}{25}\right)$

$\cos\theta\,\frac{d\theta}{dt} = \frac{1}{25}\frac{dy}{dt}$

$\frac{7}{25}\,\frac{d\theta}{dt} = \frac{1}{25}\cdot\frac{-7}{12}$

$\frac{d\theta}{dt} = \frac{1}{25}\cdot\frac{-7}{12}\cdot\frac{25}{7}$

$\frac{d\theta}{dt} = -\frac{1}{12}$ rad/sec

b) 15 ft from the wall? How fast is the angle the base of the ladder makes with the floor changing? ***(−3/2 ft/sec, −1/10 rad/sec)***

$\frac{dx}{dt} = 2$ $\frac{dy}{dt} = ?$ when $x = 15$

$x^2 + y^2 = 25^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$15^2 + y^2 = 25^2$

$y^2 = 400$

$y = 20$

$2(15)(2) + 2(20)\frac{dy}{dt} = 0$

$\frac{dy}{dt} = \frac{-60}{40} = -\frac{3}{2}$ ft/sec

$\cos\theta = \frac{x}{25}$

$-\sin\theta\,\frac{d\theta}{dt} = \frac{1}{25}\frac{dx}{dt}$

$\sin\theta = \frac{y}{25}$ $y = 20$ when $x = 15$

$\sin\theta = \frac{20}{25}$

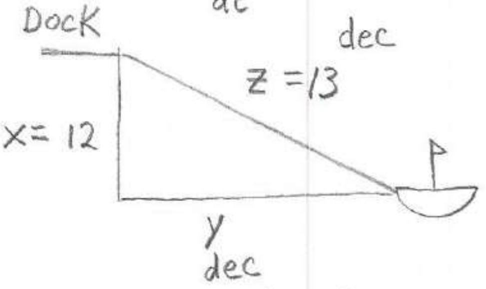
$-\frac{20}{25}\,\frac{d\theta}{dt} = \frac{1}{25}\cdot 2$

$\frac{d\theta}{dt} = \frac{-2}{20} = -\frac{1}{10}$ rad/sec

5) A boat is pulled in by means of a winch on the dock 12 ft above the deck of the boat. If the winch pulls in rope at a rate of 4 ft/s, determine the speed of the boat when there is 13 ft of rope out. What happens to the speed of the boat as it gets closer to the dock? ***(–10.4 ft/s; increases)***

$\frac{dz}{dt} = -4$ ft/sec

$\frac{dy}{dz} = ?$ when $z = 13$



$13^2 = 12^2 + y^2$
$25 = y^2$
$y = 5$

$12^2 + y^2 = z^2$

$\frac{d}{dt}(12^2 + y^2 = z^2)$

$2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$y\frac{dy}{dt} = z\frac{dz}{dt}$

$5\frac{dy}{dt} = 13(-4)$

$\frac{dy}{dt} = \frac{-52}{5}$ ft/sec

6) An air traffic controller spots two planes at the same altitude converging on a point as they fly at right angles to one another. (Fig 1) One plane is 150 mi from the point and is moving at 450 mi/h. The other plane is 200 mi from the point and has a speed of 600 mi/h.

a) At what rate is the distance between the planes decreasing? ***(–750 mi/h)***

$\frac{ds}{dt} = ?$

$\frac{dy}{dt} = -600$ mph

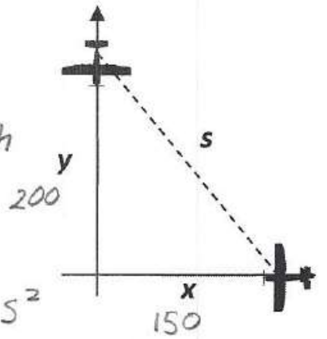


Figure 1

$200^2 + 150^2 = s^2$
$40000 + 22500 = s^2$
$s = 250$

$\frac{dx}{dt} = -450$ mph

$x^2 + y^2 = s^2$

$\frac{d}{dt}(x^2 + y^2 = s^2)$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2s\frac{ds}{dt}$

$x\frac{dx}{dt} + y\frac{dy}{dt} = s\frac{ds}{dt}$

$150(-450) + 200(-600) = 250\frac{ds}{dt}$

$\frac{ds}{dt} = -750$ mph

b) How much time does the traffic controller have to get one of the planes on a different flight path? ***(20 min)***

$200 \text{ mi} \div 600 \frac{\text{mi}}{\text{hr}}$

$200 \text{ mi} \cdot \frac{\text{hr}}{600 \text{ mi}}$

$\frac{1}{3}$ hr

$150 \text{ mi} \cdot \frac{\text{hr}}{450 \text{ mi}}$

$\frac{1}{3}$ hr

20 min

7) A baseball diamond (fig 2) has the shape of a square with sides 90 ft long. If a player is running from second to third base at a speed of 28 ft/s, at what rate is his distance from home plate changing when he is 30 ft from third base? ***(–8.854 ft/s)***

$\frac{dx}{dt} = -28$ ft/sec

$x^2 + 90^2 = s^2$

$\frac{ds}{dt} = ?$ when $x=30$

$\frac{d}{dt}(x^2+90^2=s^2)$

$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$

$30^2+90^2=s^2$

$9000 = s^2$

$x\frac{dx}{dt} = s\frac{ds}{dt}$

$30(-28) = 94.868\frac{ds}{dt}$ $\frac{ds}{dt} = -8.854$ ft/sec

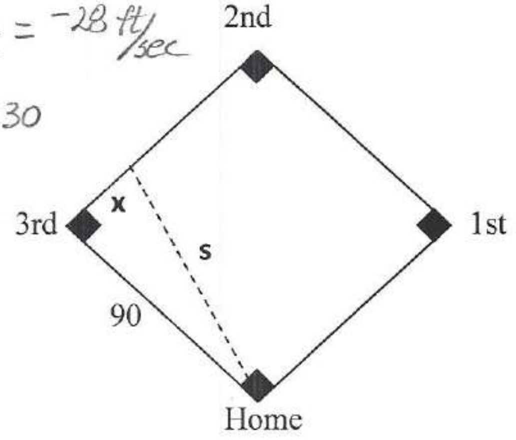
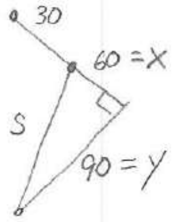


Figure 2

8) For the baseball diamond in Problem 7, suppose the player is running from first to second base at a speed of 28 ft/s. Find the rate at which the distance from home plate is changing when the player is 30 ft from second base? ***(15.532 ft/sec)***



$\frac{dx}{dt} = 28$ ft/sec

$\frac{ds}{dt} = ?$

when $x=60$

$60^2+90^2=s^2$

$s = 108.167$

$x^2+90^2=s^2$

$\frac{d}{dt}(x^2+90^2=s^2)$

$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$

$x\frac{dx}{dt} = s\frac{ds}{dt}$

$60(28) = 108.167\frac{ds}{dt}$

$\frac{ds}{dt} = 15.532$ ft/sec

9) A spherical balloon is inflated with gas at the rate of 20ft^3/min. How fast is the radius of the balloon increasing at the instant the radius is:

a) 1 ft? ***(1.592 ft/m)*** b) 2 ft? ***(0.398 ft/m)***

$\frac{dV}{dt} = 20$ ft^3/min

$\frac{dr}{dt} = ?$

$r = 1$ ft

$V = \frac{4}{3}\pi r^3$

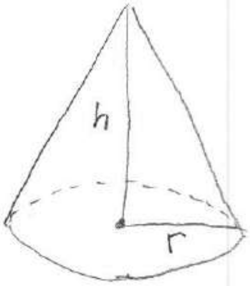
$\frac{d}{dt}(V = \frac{4}{3}\pi r^3)$

$\frac{dV}{dt} = 4\pi r\frac{dr}{dt}$

$20 = 4\pi(1)\frac{dr}{dt}$

$\frac{dr}{dt} = 1.592$ ft/min

10) At a sand and gravel plant, sand is falling off a conveyer and onto a conical pile at the rate of 10 ft^3/min. The diameter of the base of the cone is approximately three times the $\underbrace{\text{altitude}}_{h}$. At what rate is the height of the pile changing when it is 15 ft high? ***(0.006 ft/m)***



$\frac{dv}{dt} = 10\ ft^3/min$

$\frac{dh}{dt} = ?$

when $h = 15$

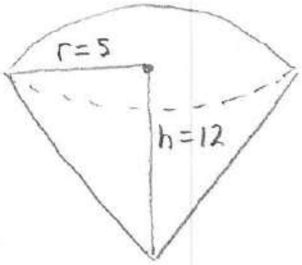
$d = 3h$

$2r = 3h$

$r = \frac{3}{2}h$

$$V = \frac{1}{3}\pi r^2 h$$
$$V = \frac{1}{3}\pi\left(\frac{3}{2}h\right)^2 h$$
$$V = \frac{1}{3}\pi \cdot \frac{9}{4}h^3$$
$$\frac{d}{dt}\left(V = \frac{1}{3}\pi \cdot \frac{9}{4}h^3\right)$$
$$\frac{dV}{dt} = \frac{9}{4}\pi h^2 \frac{dh}{dt}$$
$$10 = \frac{9}{4}\pi (15)^2 \frac{dh}{dt}$$
$$\frac{10 \cdot 4}{9 \cdot 15^2 \pi} = \frac{dh}{dt} \qquad \frac{dh}{dt} = 0.006\ ft/min$$

11) A conical tank (with the vertex down) is 10 ft across the top and 12 ft deep. If water is flowing onto the tank at a rate of 10 ft^3/min, find the rate of change of the depth of the water at the instant it is 8 ft deep. ***(0.286 ft/m)***



$\frac{dv}{dt} = 10\ ft^3/min$

$\frac{dh}{dt} = ?$

when $h = 8$

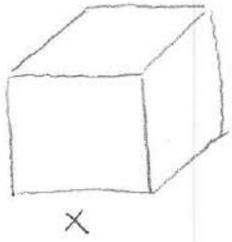
$\frac{r}{h} = \frac{5}{12}$

$r = \frac{5h}{12}$

$$V = \frac{1}{3}\pi r^2 h$$
$$V = \frac{1}{3}\pi\left(\frac{5h}{12}\right)^2 h$$
$$V = \frac{25\pi}{3 \cdot 144}h^3$$
$$\frac{d}{dt}\left(V = \frac{25\pi}{144}h^3\right)$$
$$\frac{dV}{dt} = \frac{3 \cdot 25\pi}{144}h^2 \frac{dh}{dt}$$
$$10 = \frac{75\pi}{144}(8)^2 \frac{dh}{dt}$$
$$\frac{10 \cdot 144}{75\pi \cdot 64} = \frac{dh}{dt} \qquad \frac{dh}{dt} = 0.095\ \frac{ft}{min}$$

12) The edge of a cube is expanding at the rate of 3 cm/s. How fast is the volume changing when each edge is:

a) 1 cm? ***(9 cm^3/s)*** b) 10 cm? ***(900 cm^3/s)***



$\frac{dx}{dt} = 3\ cm/s$

$\frac{dv}{dt} = ?$

when $x = 1$

$V = x^3$

$\frac{d}{dt}(V = x^3)$

$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

$\frac{dV}{dt} = 3(1)^2(3)$ $\frac{dv}{dt} = 9\ cm^3/sec$

13) The conditions are the same as in problem 12. Now measure how fast the $\underbrace{\text{surface area}}$ is changing when each edge is: $\frac{dA}{dt}$

a) 1 cm ***(36 cm^2/s)*** $\frac{dx}{dt} = 3\ cm/sec$ b) 10 cm ***(360 cm^2/s)***

$SA = 6x^2$

$\frac{dA}{dt} = ?$

$\frac{d}{dt}(A = 6x^2)$

when $x = 1$

$\frac{dA}{dt} = 12x \frac{dx}{dt}$

$\frac{dA}{dt} = 12(1)(3)$

$\frac{dA}{dt} = 36\ cm^2/sec$

14) A point is moving along the curve $y = x^2$ so that dx/dt is 2 cm/min. Find dy/dt when:

a) $x = 0$ ***(0 cm/m)***

$$\frac{d}{dt}(y = x^2)$$

$$\frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dy}{dt} = 2(0)(2)$$

$$\frac{dy}{dt} = 0 \text{ cm/min}$$

b) $x = 3$ ***(12 cm/m)***

$$\frac{dy}{dt} = 2(3)(2)$$

$$\frac{dy}{dt} = 12 \text{ cm/min}$$

15) An airplane is flying at 31,680 ft above ground level and the flight path passes directly over a radar antenna. The radar picks up the plane and determines that the distance s from the unit to the plane is changing at the rate of 4 mi/min when the distance is 10 mi. Compute the speed of the plane in miles per hour. (Fig 3) ***(–300 mph)***

$$\frac{ds}{dt} = -4 \text{ mi/min}$$

$$\frac{31{,}680}{5{,}280} = 6 \text{ mi}$$

$$\frac{dx}{dt} = ? \quad \text{when } s = 10$$

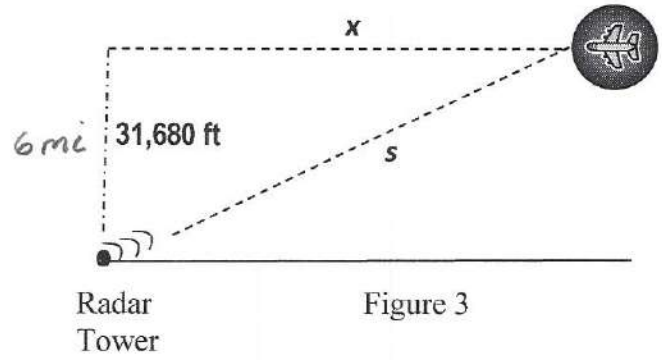
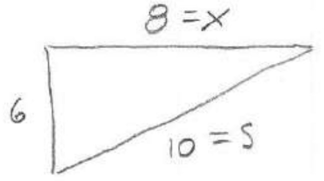


Figure 3



$$6^2 + x^2 = s^2$$

$$\frac{d}{dt}(6^2 + x^2 = s^2)$$

$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$

$$x\frac{dx}{dt} = s\frac{ds}{dt}$$

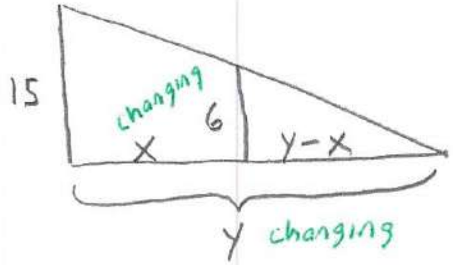
$$8\frac{dx}{dt} = 10(-4)$$

$$\frac{dx}{dt} = -5 \text{ mi/min} \Rightarrow \frac{-5 \text{ mi}}{\text{min}} \cdot \frac{60 \text{ min}}{\text{hr}} \Rightarrow -300 \text{ mi/hr}$$

16) A man 6 ft tall walks at a rate of 5 ft/ s away from a light that is 15 ft above the ground. (See Fig 4) When the man is 10 ft from the base of the light:

a) At what rate is the tip of his shadow moving? *(25/3 ft/s)*

from base of light



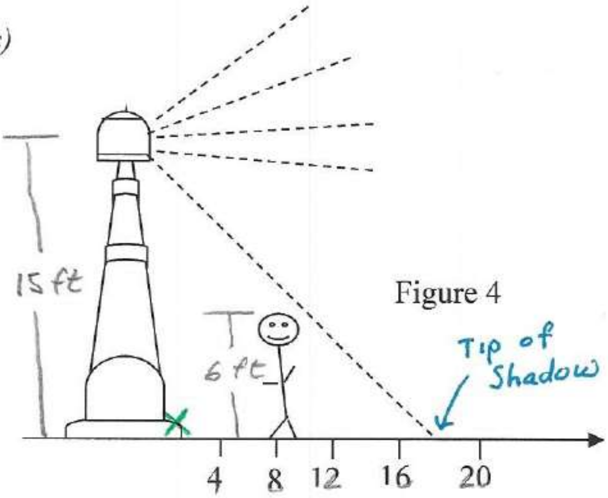


Figure 4

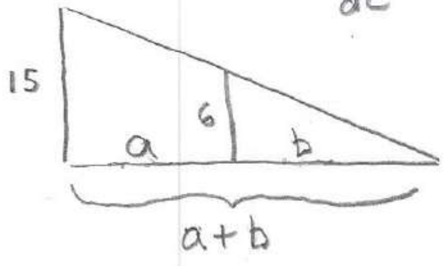
$$\frac{15}{y} = \frac{6}{y-x}$$
$$15y - 15x = 6y$$
$$-15x = -9y$$
$$15x = 9y$$

$$15\frac{dx}{dt} = 9\frac{dy}{dt}$$
$$15(5) = 9\frac{dy}{dt}$$
$$\frac{75}{9} =$$
$$\frac{25}{3}\ ft/sec = \frac{dy}{dt}$$

$$\frac{dx}{dt} = 5\ ft/sec$$

b) At what rate is the length of his shadow changing? *(10/3 ft/s)* given $\frac{da}{dt} = 5$

$\frac{db}{dt}$



$$\frac{15}{a+b} = \frac{6}{b}$$
$$15b = 6a + 6b$$
$$9b = 6a$$
$$9\frac{db}{dt} = 6\frac{da}{dt}$$
$$9\frac{db}{dt} = 6(5)$$
$$\frac{db}{dt} = \frac{10}{3}\ ft/sec$$

17) As a spherical raindrop falls, it reaches a layer of dryer air at the lower levels of the atmosphere and begins to evaporate. If this evaporation occurs at a rate proportional to the surface area ($S=4\pi r^2$) of the droplet, show that the radius shrinks at a constant rate. *($dr/dt = k$)*

Rate Proportional

$$\frac{dV}{dt} = \frac{SA \cdot K}{1}$$

$$\frac{dV}{dt} = 4\pi r^2 \cdot K$$

Volume of Sphere

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Substitute

$$4\pi r^2 K = 4\pi r^2 \frac{dr}{dt}$$

$$K = \frac{dr}{dt}$$

18) Given the revenue and cost function $R(x) = 32x - 0.3x^2$ and $C(x) = 5x + 13$, where x is the daily unit production, find the rate of change of profit (profit = revenue – cost) with respect to time when 10 units are produced and the rate of change of production is 8 units per day.

Find $\frac{dP}{dt}$ $x = 10$ un $\frac{dx}{dt} = 8$ un/day

$$P = 32x - 0.3x^2 - (5x + 13)$$

$$P = -0.3x^2 + 27x - 13$$

$$\frac{dP}{dt} = -0.6x\frac{dx}{dt} + 27\frac{dx}{dt}$$

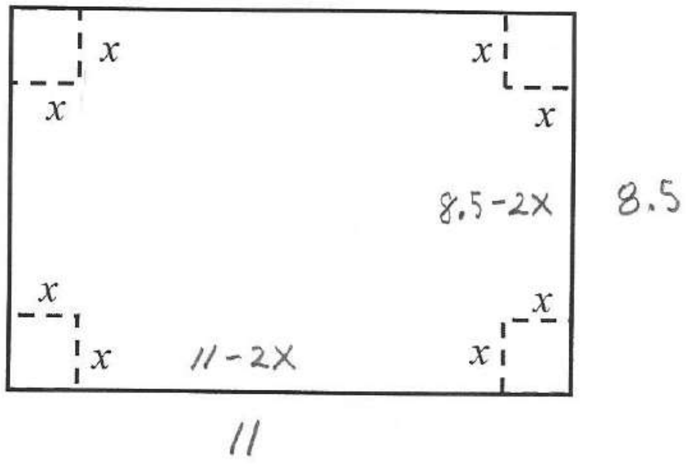
$$\frac{dP}{dt} = -0.6(10)(8) + 27(8)$$

$$\frac{dP}{dt} = \$168/\text{day}$$

Topic: Optimization

Goal: Solve problems in context that involve finding the maximum or minimum using the derivative.

Congruent squares are cut from a standard sheet of paper, and then folded into a box. What size squares will maximize the volume of the box?

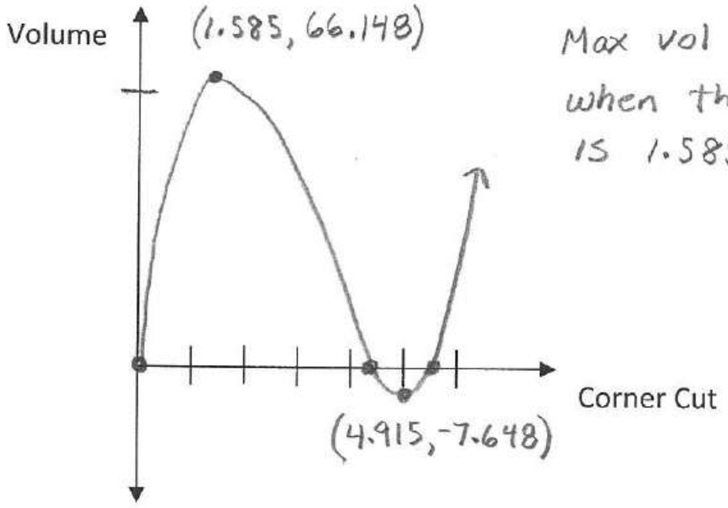


x	Length	Width	Height	Volume
1	9	6.5	1	58.5 in^3
2	7	4.5	2	63 in^3
3	5	2.5	3	37.5 in^3
4	3	0.5	4	6 in^3
x	$11-2X$	$8.5-2X$	X	$(11-2X)(8.5-2X)X$

Calculus

$f' = 0$

when f' goes from pos to neg.

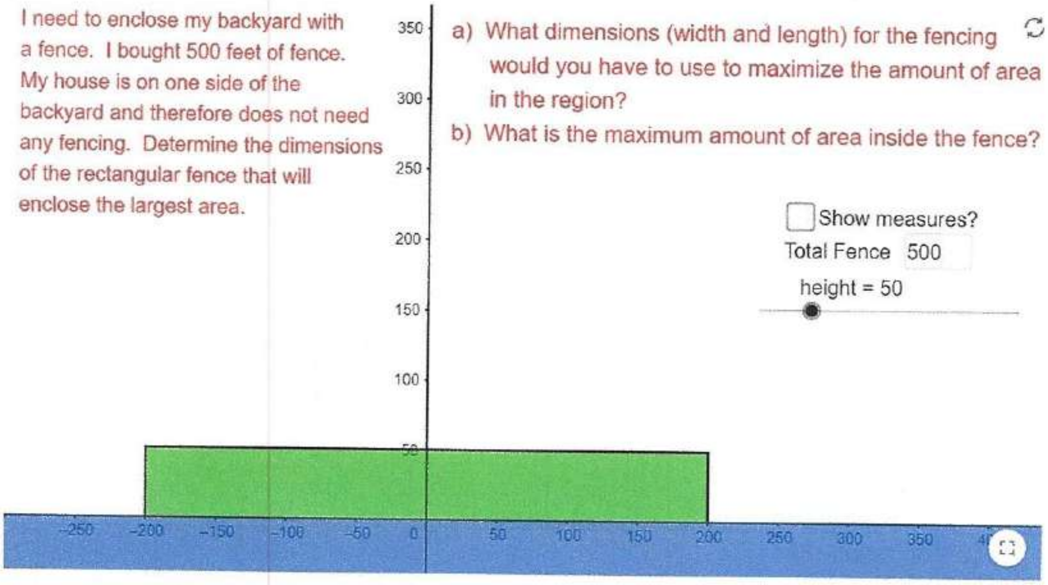


Max vol 66.148 in^3 when the corner cut is 1.585 in.

Three easy steps. (Note: pictures help)

1. Form an equation from what is given. Determine any restrictions on the domain or range.
2. Find the first derivative and then find the extrema.
3. Answer the question.

GeoGebra applets at https://www.geogebra.org/m/acrtjtjg



I need to enclose my backyard with a fence. I bought 500 feet of fence. My house is on one side of the backyard and therefore does not need any fencing. Determine the dimensions of the rectangular fence that will enclose the largest area.

a) What dimensions (width and length) for the fencing would you have to use to maximize the amount of area in the region?

b) What is the maximum amount of area inside the fence?

☐ Show measures?

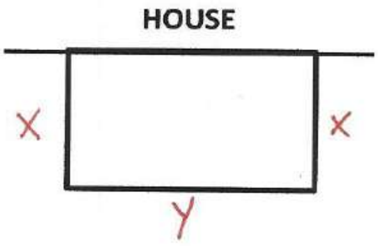
Total Fence 500

height = 50

Solution:

1) Draw a picture, using variables for unknown quantities.

2) Write an equation for the constraint (relationship between the variables).



I have 500 feet of fence. $2x + y = 500$

3) Write the equation to be maximized or minimized.

We are maximizing <u>Area</u>. $A = xy$ $y = 500 - 2x$

4) If necessary, substitute to write the equation in #3 with only one variable.

$A = x(500 - 2x)$ $A = 500x - 2x^2$

5) Find and use the derivative to find the possible maxima and minima.

$\left(A = 500x - 2x^2\right)\frac{d}{dx}$ $\frac{dA}{dx} = 0$

$\frac{dA}{dx} = 500 - 4x$

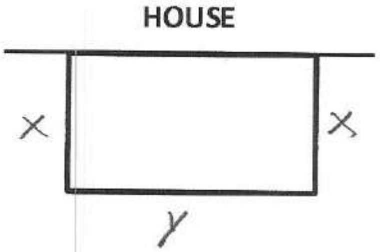
$0 = 500 - 4x$

$4x = 500$

$x = 125$ ft

cont' next page

Problem: I need to enclose my backyard with a fence. I bought 500 feet of fence. My house is on one side of the backyard and therefore does not need any fencing. Determine the dimensions of the rectangular fence that will enclose the largest area.



critical value $x = 125$

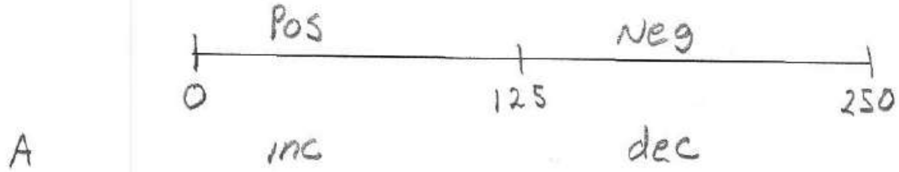
6) Determine whether your solutions are maxima or minima – also consider whether the endpoints of the interval may be absolute maxima or minima.

The smallest and largest values for y are 0 and 250 feet, respectively (why?).

Therefore, the intervals are from (0, 125) and (125, 250)
Use a number line to check the intervals.

$x = 1$ $x = 150$

$\frac{dA}{dx} = 500 - 4x$



so $x = 125$ is a max

Check the endpoints since this is a closed interval. $A = 500x - 2x^2$

$A(0) = 0$ $A(250) = 0$

7) Go back, reread the question, and make sure you answer the question that was asked.

The question asked for the dimensions of the fence.

$x = 125$

$2x + y = 500$

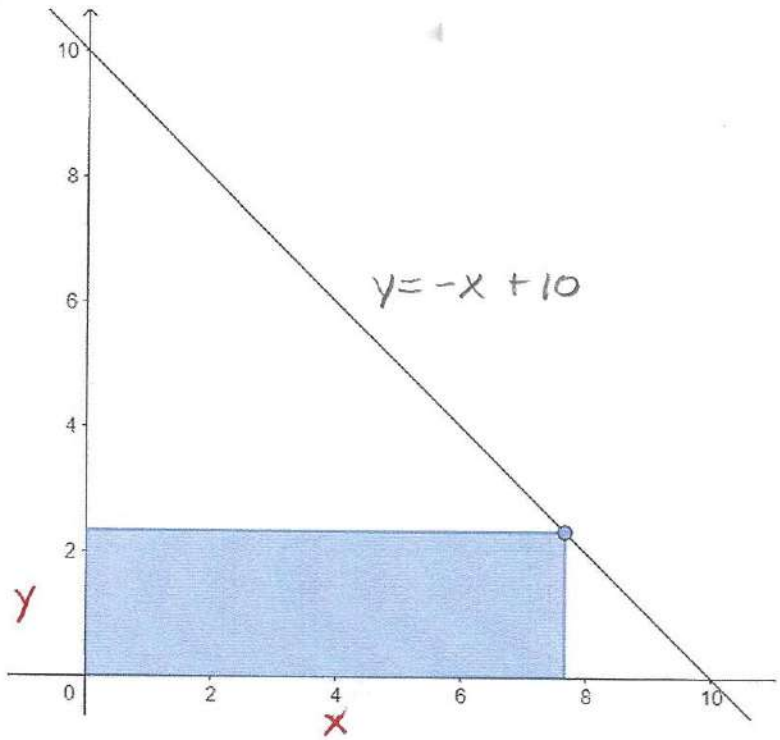
$250 + y = 500$

$y = 250$

125 ft × 250 ft

Draw + Label
Box. Give Formulas

Consider the rectangle whose base and height lie on the positive x-axis and y-axis, respectively, and whose fourth vertex lies on the line pictured. Find the coordinates of the point on the line that maximizes the area of the rectangle.



Area = xy

If you know x
Then $y = -x + 10$

$A = x(-x + 10)$

$A = -x^2 + 10x$

$\frac{dA}{dx} = -2x + 10$

$0 = -2x + 10$

$x = 5$ $\quad y = -5 + 10$

$y = 5$

$(5,5)$

Visually endpoints are not necessary b/c x or y would be zero.

Verify

A' $\quad x = 1 \quad x = 6$

$$\overset{0}{\vdash}\!\!\xrightarrow{\text{Pos}}\!\!\overset{5}{+}\!\!\xrightarrow{\text{Neg}}\!\!\overset{10}{\dashv}$$

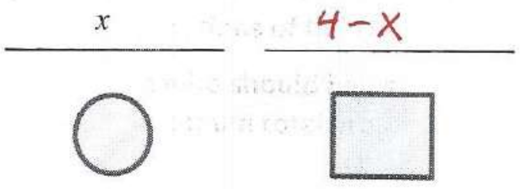
0	5	10

A' goes from pos to neg
⇒ A has a max at $x = 5$

Assignment:

Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much wire should be used for the circle to enclose the minimum total area of the two figures?

Chelsea started by drawing the diagram at the right, she then created the following formula:
$A(x)=\pi\left(\frac{x}{2\pi}\right)^2+\frac{1}{16}(4-x)^2$. How did she create this formula? Show all of the steps she took.



Circle

$A_○ = \pi r^2$

$P_○ = 2\pi r$

$X = 2\pi r$

$\frac{X}{2\pi} = r$

Square

$A_□ = S^2$

$P_□ = 4S$

$4-X = 4S$

$\frac{4-X}{4} = S$

Area Both

$A = \pi r^2 + S^2$

Sub r and S

$A = \pi\left(\frac{X}{2\pi}\right)^2 + \left(\frac{4-X}{4}\right)^2$

$A = \pi\left(\frac{X}{2\pi}\right)^2 + \frac{1}{16}(4-X)^2$

Using Chelsea's formula, find the amount of wire used for each to give the minimum enclosed area.

$A = \frac{\pi X^2}{4\pi^2} + \frac{1}{16}(4-x)^2$

$\frac{d}{dx}\left[A = \frac{1}{4\pi}X^2 + \frac{1}{16}(4-x)^2\right]$

$\frac{dA}{dx} = 2\cdot\frac{1}{4\pi}X + 2\cdot\frac{1}{16}(4-x)(-1)$

$\frac{dA}{dx} = \frac{1}{2\pi}X + \frac{1}{8}(4-x)$

$\left(0 = \frac{X}{2\pi} - \frac{1}{2} + \frac{X}{8}\right)\frac{8\pi}{1}$

$0 = 4X - 4\pi + \pi X$

$4\pi = 4X + \pi X$

$4\pi = X(4+\pi)$

$\frac{4\pi}{4+\pi}$ ft $= X$

for the circle

$4 - \frac{4\pi}{4+\pi}$ ft

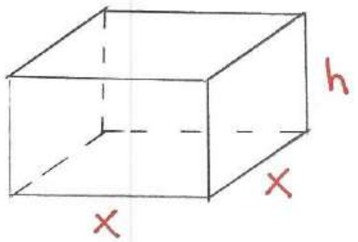
for the square

To check with a number use decimal values.

P. 35, 36, 38, 39, 42

Assign p. 40, 41, 43, 44

2) A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?



SA = Area bottom + 4 sides

$108 = x^2 + 4xh$

$V = LWH$

$V = x^2 \cdot h$

$108 - x^2 = 4xh$

elim h

$\frac{108-x^2}{4x} = h$

$V = x^2\left(\frac{108-x^2}{4x}\right)$

$V = \frac{108x}{4} - \frac{x^3}{4}$

$\frac{d}{dx}\left[V = 27x - \frac{1}{4}x^3\right]$

$\frac{dv}{dx} = 27 - \frac{3}{4}x^2$

$0 = 27 - \frac{3}{4}x^2$

$\frac{3}{4}x^2 = 27$

$x^2 = \frac{108}{3}$

$x = \sqrt{\frac{108}{3}} = \sqrt{36} = 6$

$108 = 6^2 + 4 \cdot 6h$

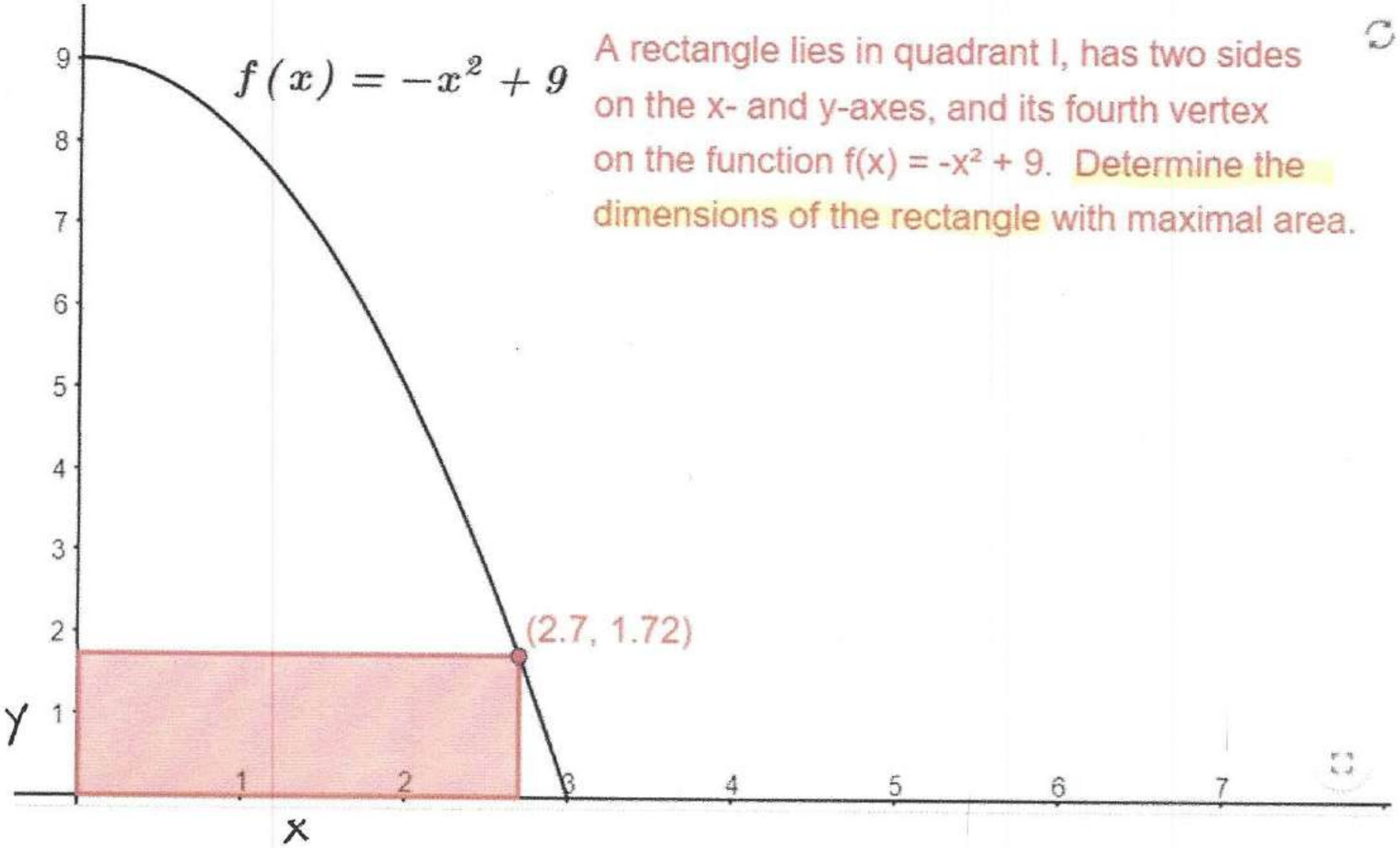
$72 = 24h$

$h = 3$

dimensions

6 in x 6 in x 3 in

A rectangle lies in quadrant I, has two sides on the x- and y-axes, and its fourth vertex on the function f(x) = -x² + 9. Determine the dimensions of the rectangle with maximal area.



$A = x\,y$ where $y = -x^2+9$

$A = x(-x^2+9)$

$A = -x^3+9x$

$\frac{dA}{dx} = -3x^2+9$

$0 = -3x^2+9$

$3x^2 = 9$

$x^2 = 3$

$x = \pm\sqrt{3}$

$x = \sqrt{3}$

$x = \sqrt{3}$

$y = -\sqrt{3}^2+9$

$y = -3+9$

$y = 6$

Dimensions

$\sqrt{3}$ un × 6 un

verify

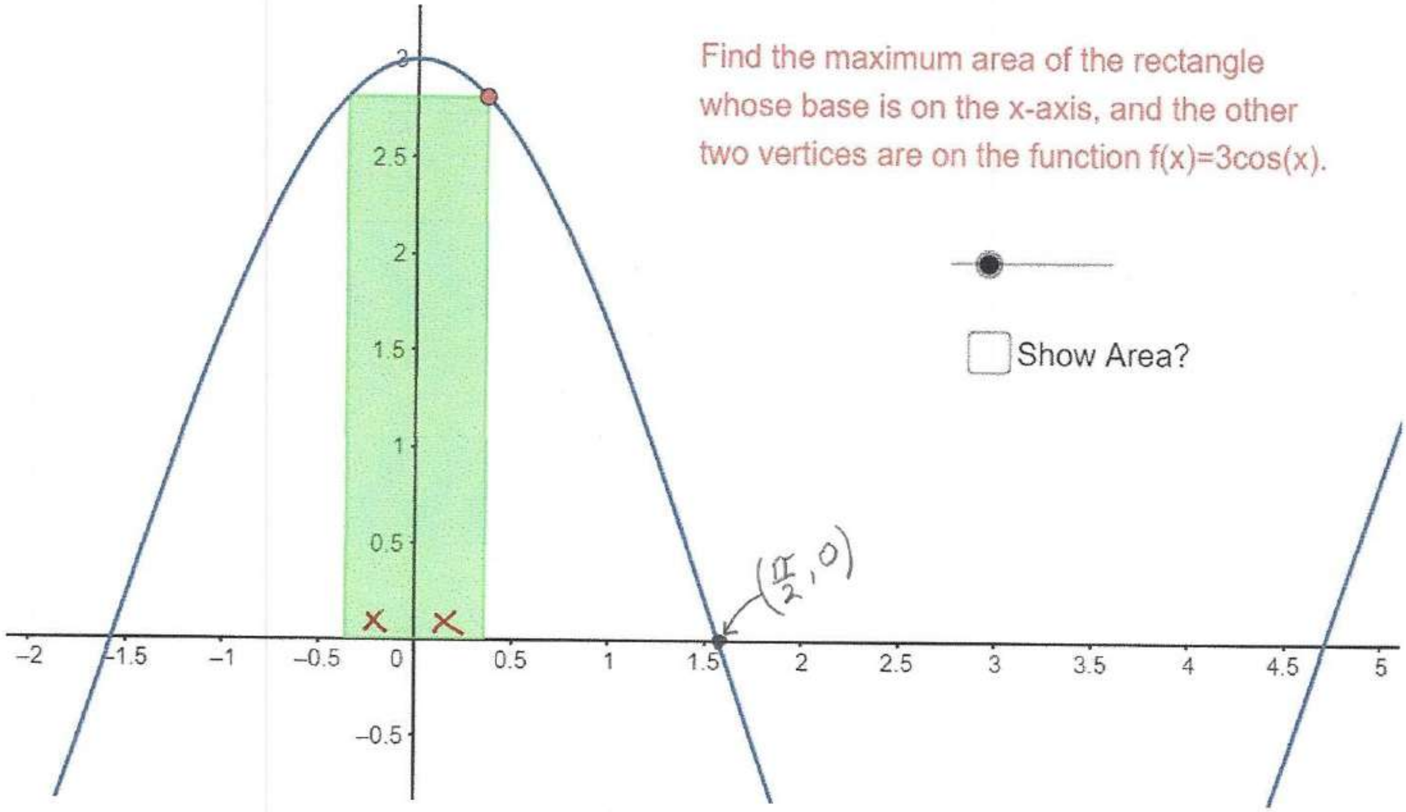
A' $x=1$ $x=2$

Pos Neg

0 $\sqrt{3}$ 3

$x = \sqrt{3}$ is a max

Discuss page 34 + 35.
Assign for HW.



Find the maximum area of the rectangle whose base is on the x-axis, and the other two vertices are on the function f(x)=3cos(x).

After finding derivative use graphing calc.

$A = x\,y \qquad y = 3\cos x$

$A = 2x(3\cos x)$

$\frac{dA}{dx} = 2 \cdot 3\cos x + 2x(-3\sin x)$

$\frac{dA}{dx} = 6\cos x - 6x\sin x$

$0 = 6\cos x - 6x\sin x$

$0 = \cos x - x\sin x$

* Graphing Calculator

$x = 0.860$

$y = 3\cos(0.860)$

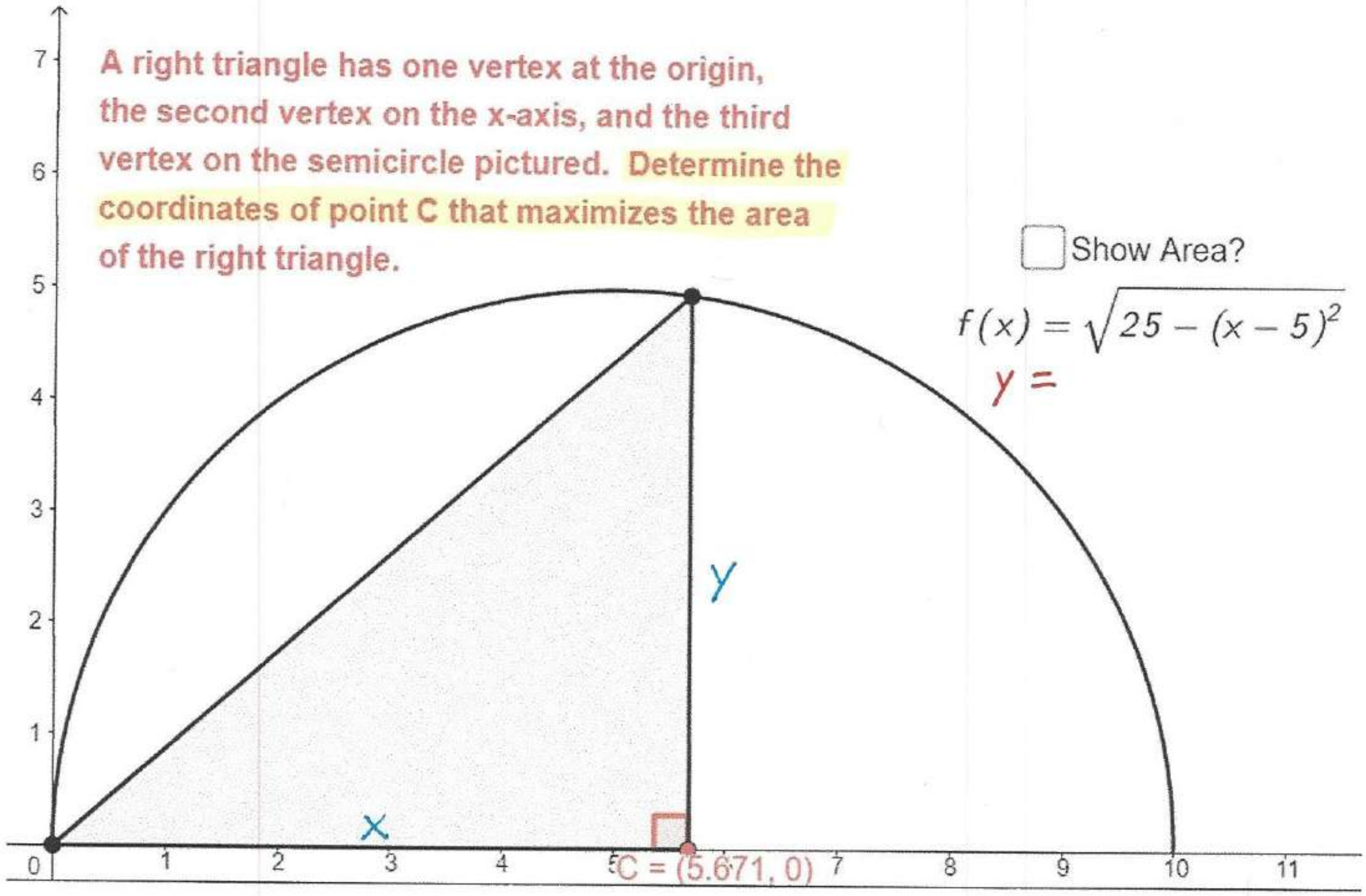
$y = 1.957$

Dimensions

$2x$ by y

$2(0.860)$

1.720_{un} by 1.957_{un}



A right triangle has one vertex at the origin, the second vertex on the x-axis, and the third vertex on the semicircle pictured. Determine the coordinates of point C that maximizes the area of the right triangle.

$A = \frac{1}{2}bh$

$A = \frac{1}{2}x\,y$ where $y = \sqrt{25-(x-5)^2}$

$A = \frac{1}{2}x\sqrt{25-(x-5)^2}$

$A = \frac{1}{2}x\left[25-(x-5)^2\right]^{1/2}$

Graphing Calc after derivative

$$\frac{dA}{dx} = \frac{1}{2}\left[25-(x-5)^2\right]^{1/2} + \frac{1}{2}x\cdot\frac{1}{2}\left[25-(x-5)^2\right]^{-1/2}\left[-2(x-5)\right] = 0$$

Graphing Calc

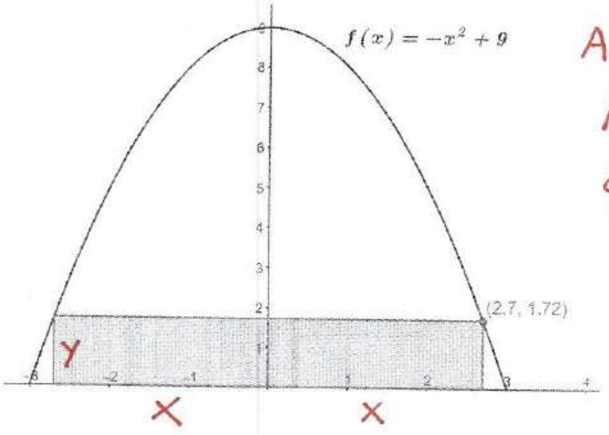
$x = 7.5 \quad f(7.5) = 4.330 = y$

$(7.5, 4.330)$

Assignment: #1-3 on next 2 pages

Optimization Practice

1) Find the value of x and y that maximizes the area of the rectangle shown in the figure below. Then find the maximum area.



$A = 2xy$

$A = 2x(-x^2+9)$

$A = -2x^3 + 18x$

$\frac{dA}{dx} = -6x^2 + 18$

$0 = -6x^2 + 18$

$6x^2 = 18$

$x^2 = 3$

$x = \sqrt{3}$

width $= 2\sqrt{3}$

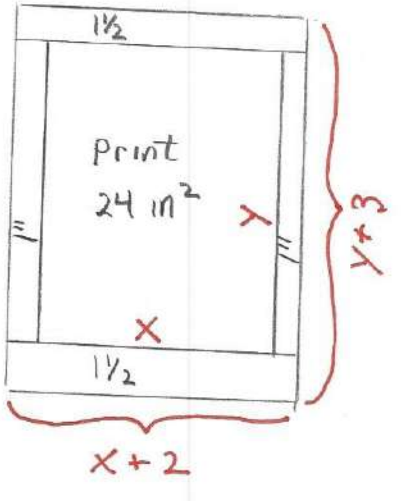
$y = -\sqrt{3}^2 + 9$

$y = 6$ height

$A = 2\sqrt{3} \cdot 6$

$A = 12\sqrt{3}$

2) A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be 1 1/2 inches, and the margins on the left and right are to be 1 inch. What should the dimensions of the page be so that the least amount of paper is used?



$xy = 24$

$A = (x+2)(y+3)$

$y = \frac{24}{x}$

$A = (x+2)\left(\frac{24}{x} + 3\right)$

$A = (x+2)(24x^{-1} + 3)$

$\frac{dA}{dx} = 1(24x^{-1} + 3) + (x+2)(-24x^{-2})$

$0 = \left(\frac{24}{x} + 3\right) + (x+2)\left(-\frac{24}{x^2}\right)$

$0 = \frac{24}{x} + 3 - \frac{24}{x} - \frac{48}{x^2}$

$\frac{48}{x^2} = 3$

$16 = x^2$

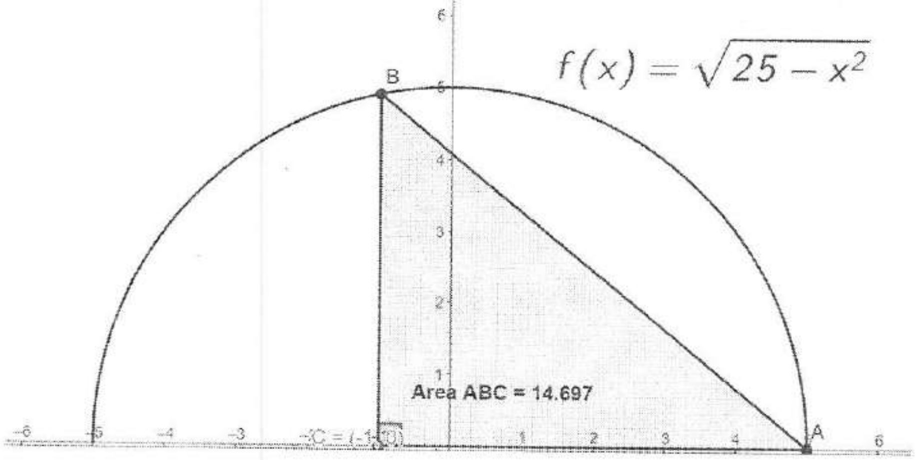
$x = 4$

$y = 6$

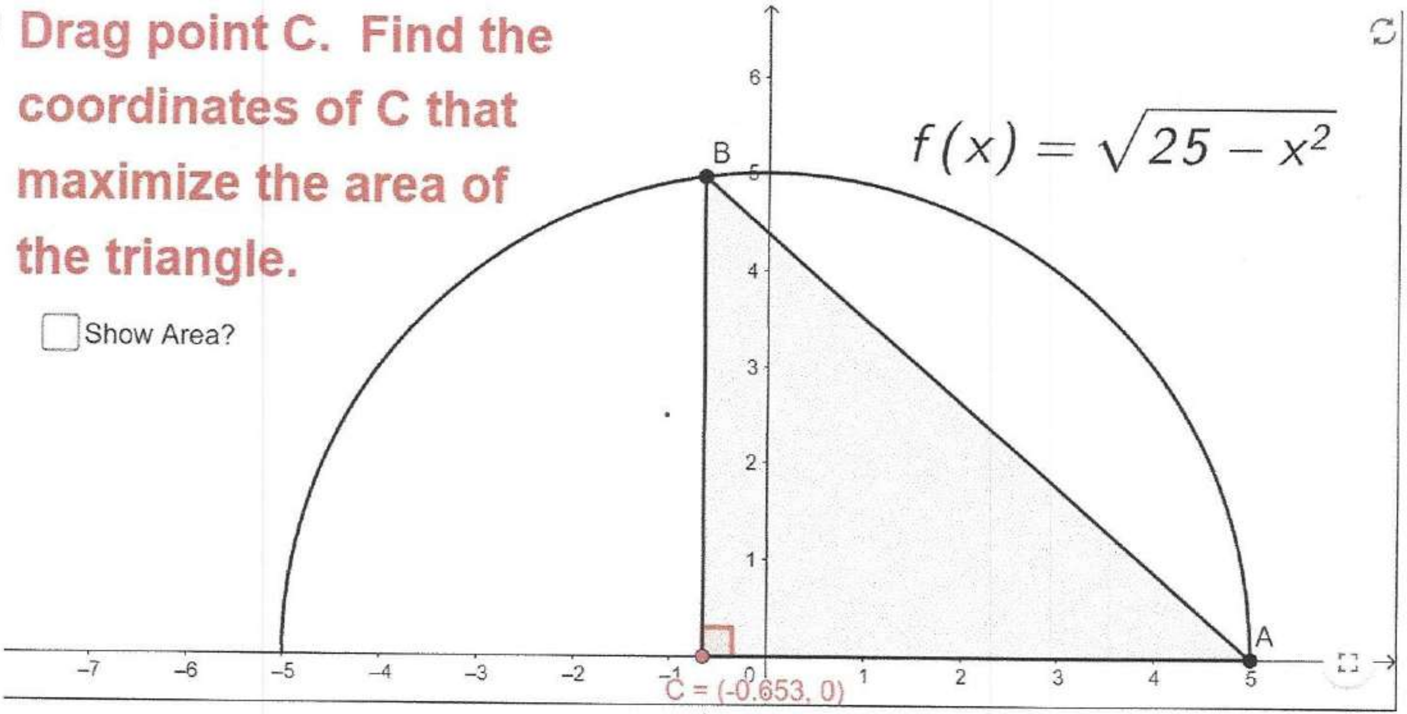
Dimensions

6×9

3) Find the coordinates of point C that maximize the area of the triangle.
Let the base of the triangle be 5 + x.



3)



Drag point C. Find the coordinates of C that maximize the area of the triangle.

☐ Show Area?

cannot use x for base b/c in $y=\sqrt{25-x^2}$, x is defined to be in [0, 5]

$A = \frac{1}{2} bh$

$A = \frac{1}{2}(5+x)(25-x^2)^{1/2}$

$\frac{dA}{dx} = \frac{1}{2}(25-x^2)^{1/2} + \frac{1}{2}(5+x)\frac{1}{2}(25-x^2)^{-1/2}(-2x) = 0$

$\frac{\sqrt{25-x^2}}{2} + \frac{(5+x)(-2x)}{4\sqrt{25-x^2}} = 0$

$\frac{\sqrt{25-x^2}}{2} - \frac{10x+2x^2}{4\sqrt{25-x^2}} = 0$

$\frac{\sqrt{25-x^2}}{2} = \frac{10x+2x^2}{4\sqrt{25-x^2}}$

$4(25-x^2) = 20x + 4x^2$

$100 - 4x^2 = 20x + 4x^2$

$0 = 8x^2 + 20x - 100$

$0 = 2x^2 + 5x - 25$

$0 = (2x-5)(x+5)$

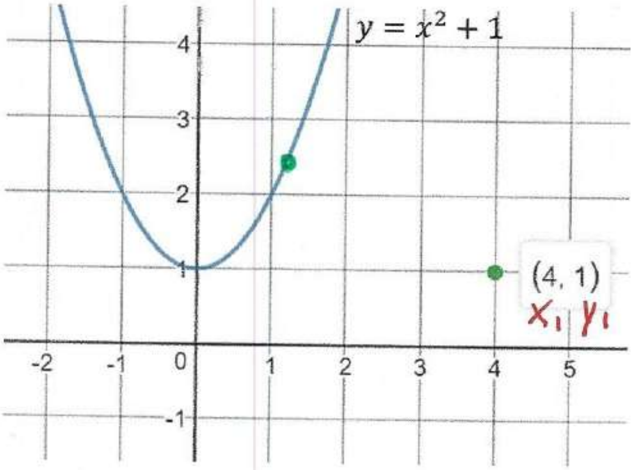
$\swarrow \qquad \downarrow$

$x = \frac{5}{2} \qquad \cancel{x = -5}$

Point C
(-2.5, 0)

Find the point on the graph of the curve y = x^2 + 1 that is closest to the fixed point (4,1).

The distance formula is $d = \sqrt{(x - x_1)^2 + (y - y_1)^2}$ minimize the distance



$d = \sqrt{(x-4)^2 + (y-1)^2}$ $y = x^2 + 1$

$d = \sqrt{(x-4)^2 + (x^2+1-1)^2}$

$d = \sqrt{(x-4)^2 + (x^2)^2}$

$d = \underbrace{\sqrt{(x-4)^2 + x^4}}$

To minimize the distance, you need To minimize what is in the root.

$m(x) = (x-4)^2 + x^4$

$m'(x) = 2(x-4) + 4x^3$

$m'(x) = 2x - 8 + 4x^3$

$0 = 4x^3 + 2x - 8$

Graphing Calc

$x = 1.128$

$y = (1.128)^2 + 1$

$y = 2.272$

$(1.128, 2.272)$

Assignment: #1-3 on next two pages