# *Unit #9—Area/Volume*

You are responsible for doing all of the homework and checking your work. If you get stuck, the solutions are worked out at the end of the unit and the odd numbered exercises are also available online through the textbook publisher. If you still have questions on the homework problems after going over the solutions, then come in at lunch by appointment, afterschool, or during intervention as class time will not be devoted to going over the homework.

**Assignment #1:** Area Under a Curve
Handout

**Assignment #2:** Area Between Curves
Page 361: 1, 3, 4, 10, 13, 14, 22, 23, 27, 28, 29, 35, 36

**Assignment #3:** Average Value
Page 375: 39–46 set it up then use your calculator to find the value.

**Assignment #4:** Volume–Cross Sections
Handout

**Assignment #5:** Volume–Cross Sections Day 2
Page 373: 9, 10, 11, 13, 14, and Handout

**Assignment #6:** Volume–Disks and Washers
Handout

**Assignment #7:** Volume–Disks and Washers–Non–Adjacent Axis of Revolution
Handout

**Assignment #8:** Review

**Test**

Lesson 1

Topic: Area

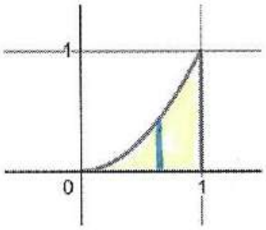
***Goal: To find the area under a curve using definite integrals.***

The area under a curve is defined to be the integral below:

$$\int_a^b (\text{upper -lower})\,dx$$

Note: Need an accurate drawing of the function and be able to identify the *bounded* region.

Find the area ***bounded*** by $y = x^2$ and the $x$-axis ($y = 0$) between $x = 0$ and $x = 1$.




The vertical line in the shaded region represents a skinny rectangle

height of rectangle =

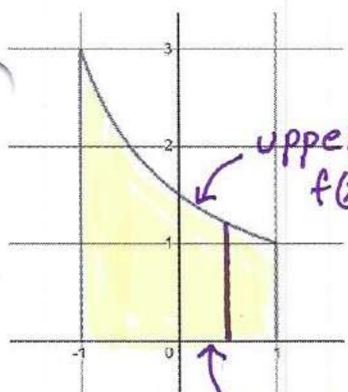
upper − lower

$x^2 - 0$

$$\int_0^1 x^2\,dx$$

h · w

Set up an integral and find the area under the curve of $f(x) = \frac{3}{x+2}$, for $-1 \le x \le 1$.




$$\int_{-1}^{1} \frac{3}{x+2}\,dx = 3\ln|x+2|\Big|_{-1}^{1}$$

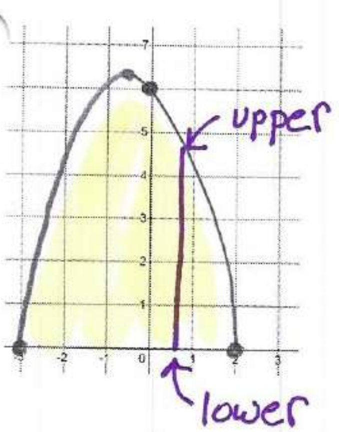
$$= 3\ln 3 - 3\ln 1$$

$$= 3\ln 3$$

or

$\ln 3^3$ log rules

Find the area bounded by $x$-axis ($y = 0$) and $y = 6 - x - x^2$.




$6-x-x^2-0$

y-int $(0,6)$

$0 = -x^2-x+6$

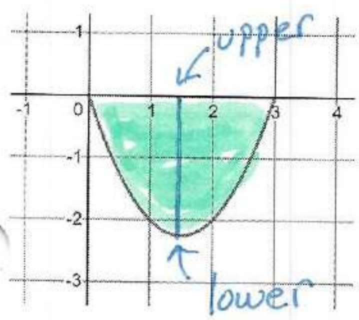
$0 = x^2+x-6$

$0 = (x+3)(x-2) \quad x=-3, 2$

$$\int_{-3}^{2} 6-x-x^2\,dx = 6x-\frac{x^2}{2}-\frac{x^3}{3}\Big|_{-3}^{2}$$

$$= 6\cdot 2-\frac{2^2}{2}-\frac{2^3}{3}-\left[6(-3)-\frac{(-3)^2}{2}-\frac{(-3)^3}{3}\right] \text{ un}^2$$

Find the area bounded by $y = x^2 - 3x$ and $y = 0$.




$0-(x^2-3x)$

$-x^2+3x$

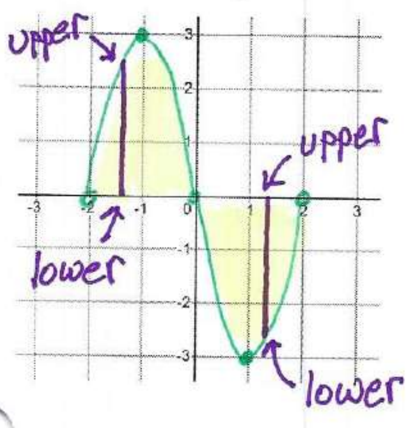
$$\int_0^3 -x^2+3x\,dx$$

$$-\frac{x^3}{3}+\frac{3x^2}{2}\Big|_0^3$$

$$-\frac{3^3}{3}+\frac{3\cdot 3^2}{2}-0$$

$$-9+\frac{27}{2} \qquad \frac{9}{2}$$

Find the area bounded by the graph of $y = x^3 - 4x$ and the $x$-axis.




y-int $(0,0)$

$0 = x^3-4x \qquad x^2-4=0$

$0 = x(x^2-4) \qquad x^2=4$

$x=0 \qquad x=\pm 2$

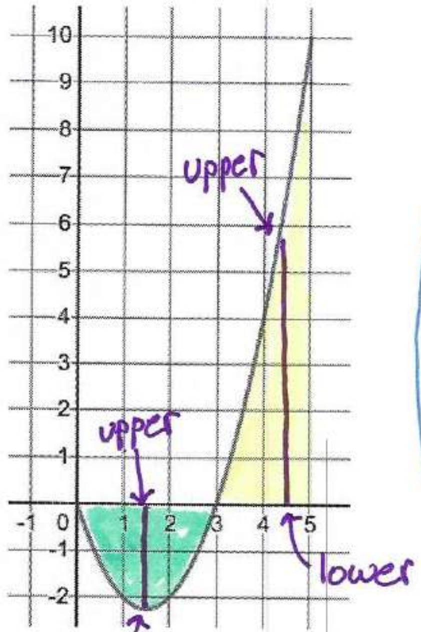
$$\int_{-2}^{0} x^3-4x\,dx + \int_0^2 -x^3+4x\,dx$$

$$\frac{x^4}{4}-2x^2\Big|_{-2}^{0} + -\frac{x^4}{4}+2x^2\Big|_0^2$$

$$0-\left[\frac{(-2)^4}{4}-2(-2)^2\right]+-\frac{2^4}{4}+2\cdot 2^2-0$$

**This will give the ACTUAL area, NOT positive above x-axis and negative below x-axis.**

Find the area bounded by the graph of $y = x^2 - 3x$, $x = 0$, $x = 5$, and $y = 0$.




lower

$0 - (x^2 - 3x)$

$-x^2 + 3x$

$$\int_0^3 -x^2+3x\,dx + \int_3^5 x^2-3x\,dx$$

$$-\frac{x^3}{3}+\frac{3x^2}{2}\Big|_0^3 + \frac{x^3}{3}-\frac{3x^2}{2}\Big|_3^5$$

$$-\frac{3^3}{3}+\frac{3\cdot 3^2}{2} - 0 + \frac{5^3}{3}-\frac{3\cdot 5^2}{2} - \left[\frac{3^3}{3}-\frac{3\cdot 3^2}{2}\right] \text{un}^2$$

Calc

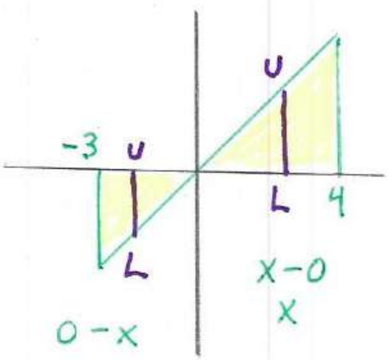
$$\int_0^5 |x^2-3x|\,dx = 13.167$$

**Assignment #1: Handout**

**Unit #9 Assignment # 1: Handout**

Find the area bounded by the functions given.

1. $y = x$ , $y = 0$, $x = -3$, $x = 4$




$x - 0$

$x$

$0 - x$

$-x$

$$\int_{-3}^{0} -x\,dx + \int_{0}^{4} x\,dx$$

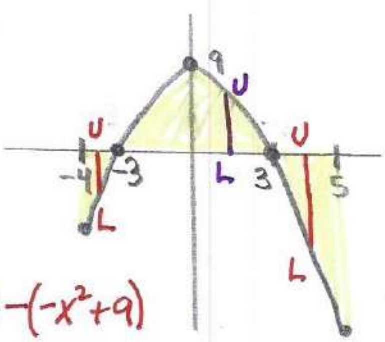
$$\frac{-x^2}{2}\Big|_{-3}^{0} + \frac{x^2}{2}\Big|_{0}^{4}$$

$$0 - \left[\frac{-(-3)^2}{2}\right] + \frac{4^2}{2} - 0 \quad un^2$$

2. $y = -x^2 + 9$ , $y = 0$, $x = -4$, $x = 5$

$0 = -x^2 + 9 \qquad 9 = x^2$

$0 = x^2 - 9 \qquad \pm 3 = x$




$0 - (-x^2 + 9)$

$x^2 - 9$

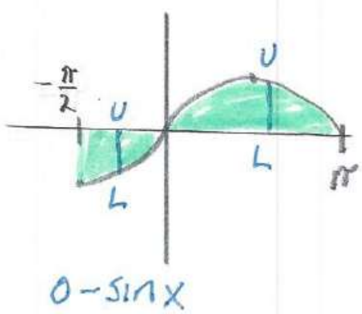
$$\int_{-4}^{-3} x^2 - 9\,dx + \int_{-3}^{3} -x^2 + 9\,dx + \int_{3}^{5} x^2 - 9\,dx$$

$$\frac{x^3}{3} - 9x\Big|_{-4}^{-3} + -\frac{x^3}{3} + 9x\Big|_{-3}^{3} + \frac{x^3}{3} - 9x\Big|_{3}^{5}$$

$$\frac{(-3)^3}{3} - 9(-3) - \left[\frac{(-4)^3}{3} - 9(-4)\right] + \frac{-3^3}{3} + 9\cdot 3 - \left[-\frac{(-3)^3}{3} + 9(-3)\right]$$

$$+ \frac{5^3}{3} - 9\cdot 5 - \left[\frac{3^3}{3} - 9\cdot 3\right] \quad un^2$$

3. $y = \sin x$ , $y = 0$, $x = -\pi/2$, $x = \pi$




$0 - \sin x$

$-\sin x$

$$\int_{-\frac{\pi}{2}}^{0} -\sin x\,dx + \int_{0}^{\pi} \sin x\,dx$$

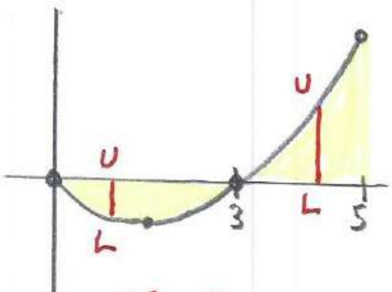
$$\cos x\Big|_{-\frac{\pi}{2}}^{0} + -\cos x\Big|_{0}^{\pi}$$

$$\cos 0 - \cos\left(-\frac{\pi}{2}\right) + -\cos \pi - {}^{-}\cos 0$$

$$1 - 0 + -{-1} + 1$$

$$3 \ un^2$$

4. Find the area bounded by $y = x^2 - 3x$, $x = 5$, and $y = 0$.




$0-(x^2-3x)$

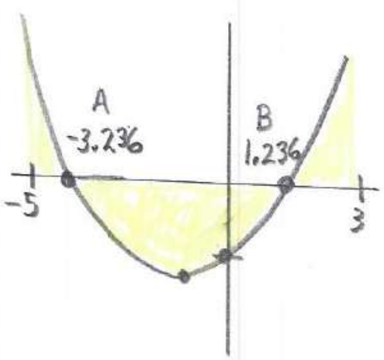
$-x^2+3x$

$$\int_0^3 -x^2+3x\,dx + \int_3^5 x^2-3x\,dx$$

$$\left.\frac{-x^3}{3}+\frac{3x^2}{2}\right|_0^3 + \left.\frac{x^3}{3}-\frac{3x^2}{2}\right|_3^5$$

$$-\frac{3^3}{3}+\frac{3\cdot 3^2}{2}-0+\frac{5^3}{3}-\frac{3\cdot 5^2}{2}-\left[\frac{3^3}{3}-\frac{3\cdot 3^2}{2}\right]$$

5. $y = x^2 + 2x - 4$, $y = 0$, $x = -5$, $x = 3$ (set up , then evaluate with a calculator)




$$\int_{-5}^{A} \underset{Y_1}{x^2+2x-4}\,dx + \int_{A}^{B} \underset{Y_1}{-x^2-2x+4}\,dx + \int_{B}^{3} \underset{Y_1}{x^2+2x-4}\,dx$$

$2.667\ un^2$

**Topic: Area Between Two Curves** **Lesson 2**

***Goal: Find the area bounded by two curves.***

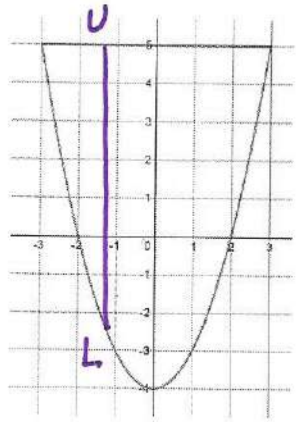
The area under a curve is defined to be the integral below:

$$\int_a^b (\text{upper} - \text{lower})\, dx$$

Note: Need an accurate drawing of the function and be able to identify the *bounded* region.

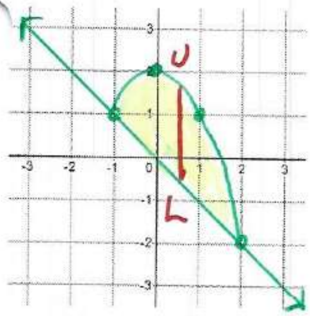
Find the points of intersection of the graphs.

Find the area bounded by the graphs of $y = 5$, $y = x^2 - 4$.



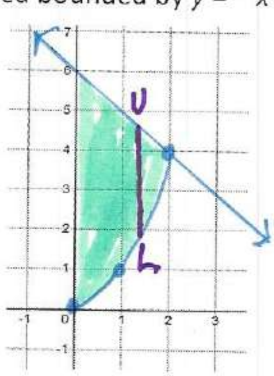

$$U - L = 5 - (x^2 - 4)$$
$$= 5 - x^2 + 4$$
$$= -x^2 + 9$$
$$\int_{-3}^{3} -x^2 + 9\, dx$$
$$-\frac{x^3}{3} + 9x \Big|_{-3}^{3}$$
$$-\frac{3^3}{3} + 9\cdot 3 - \left[-\frac{(-3)^3}{3} + 9(-3)\right]$$

Area enclosed by $y = 2 - x^2$, and $y = -x$.




$$U - L = 2 - x^2 - -x$$
$$= 2 + x - x^2$$
$$\int_{-1}^{2} 2 + x - x^2\, dx$$
$$2x + \frac{x^2}{2} - \frac{x^3}{3} \Big|_{-1}^{2}$$
$$2\cdot 2 + \frac{2^2}{2} - \frac{2^3}{3} - \left[2(-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3}\right] \text{ un}^2$$

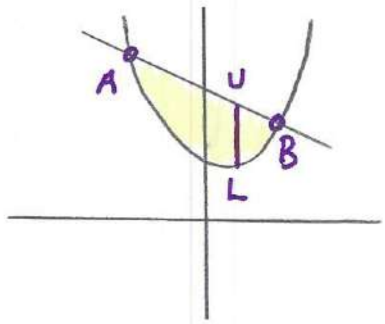
Area bounded by $y = -x + 6$, $y = x^2$, $x = 0$, and $x = 2$.




$$U - L = -x + 6 - x^2 = 6 - x - x^2$$
$$\int_0^2 6 - x - x^2\, dx$$
$$6x - \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^2$$
$$6\cdot 2 - \frac{2^2}{2} - \frac{2^3}{3} - 0$$

Find the area of the region bounded by $y = 2x^2 - 3x + 2$ and $y = \frac{-3}{2}x + 4$ using your graphing calculator.

$y_2$ $y_1$




$U - L = y_1 - y_2$

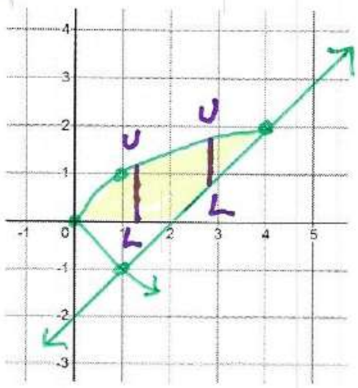
$$\int_A^B y_1 - y_2 \, dx = 3.249 \text{ un}^2$$

ALPHA TRACE $y_1$ − ALPHA TRACE $y_2$

$X = -0.693$
X enter
STO ALPHA A

Set up an integral to find area **bounded** on right by $y = x - 2$ on the left by $x = y^2$ and below by the $x$-axis.

$y = \sqrt{x}$

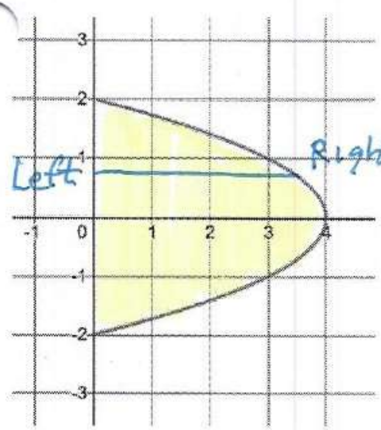



| U − L | U − L |
|---|---|
| $\sqrt{x} - 0$ | $\sqrt{x} - (x-2)$ |
| $\sqrt{x}$ | $\sqrt{x} - x + 2$ |
| $x^{1/2}$ | $x^{1/2} - x + 2$ |

$$\int_0^2 x^{1/2} \, dx + \int_2^4 x^{1/2} - x + 2 \, dx$$

Integral Changes at $x = 2$

Find the area bounded by $x = 4 - y^2$ and the $y$-axis.




height = right − left

$= 4 - y^2 - 0$

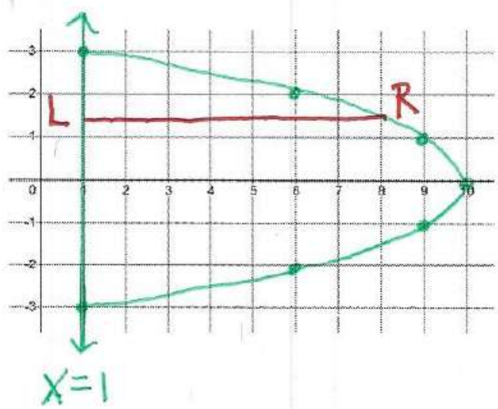
$\int_{y=-2}^{2} 4 - y^2 \, dy$

$4y - \frac{y^3}{3} \Big|_{-2}^{2}$

$4 \cdot 2 - \frac{2^3}{3} - \left[4(-2) - \frac{(-2)^3}{3}\right]$ un$^2$

Calculator $\int_{x=-2}^{2} 4 - x^2 \, dx$

Find the area bounded by $x = 10 - y^2$ and the line $x = 1$.




$R - L = 10 - y^2 - 1$

$= 9 - y^2$

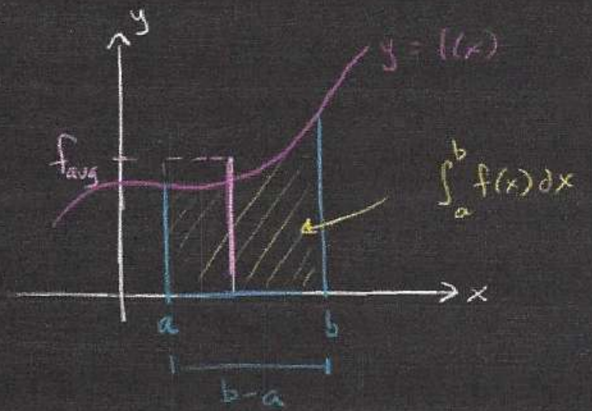
$\int_{y=-3}^{3} 9 - y^2 \, dy$

$9y - \frac{y^3}{3} \Big|_{-3}^{3}$

$9 \cdot 3 - \frac{3^3}{3} - \left[9(-3) - \frac{(-3)^3}{3}\right]$ un$^2$

**Assignment #2: Page 361: 1, 3, 4, 10, 13, 14, 22, 23, 27, 28, 29, 35, 36**

Average value of a function over $[a, b]$




$$f_{avg}(b-a) = \int_a^b f(x)\,dx$$

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

## Topic: Average Value

**Lesson 3**

***Goal: Find the average value over an interval.***

If *f* can be integrated on the closed interval [*a*, *b*], then the *average value* of *f* on the interval is

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Use your calculator to find the area between the curve $f(x) = 3x^2 - 2x$ and the x-axis on the interval [1, 4] by evaluating $\int_1^4 f(x)\,dx$.

$$\int_1^4 f(x)\,dx = 48$$

Find the average value of $f(x) = 3x^2 - 2x$ on the interval [1, 4].

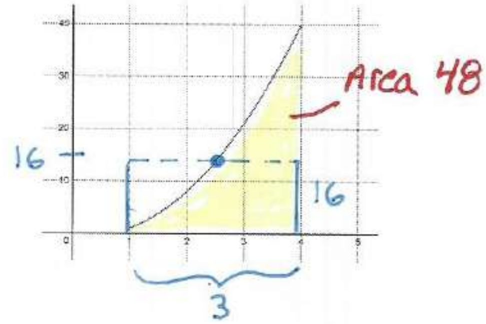
$$\frac{1}{4-1}\int_1^4 f(x)\,dx = 16$$

At what value(s) of x does the function assume the average value on the interval?

$$\underset{y_1}{3x^2 - 2x} = \underset{y_2}{16}$$

Find intersection on calculator

$$x = 2.667$$

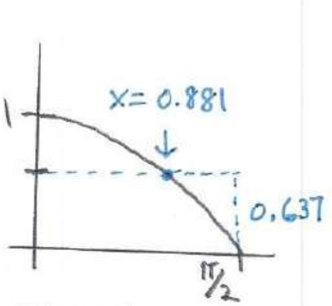



Find the average value of $f(x) = \cos x$ on the interval $[0, \pi/2]$.

$$\frac{1}{\frac{\pi}{2} - 0}\int_0^{\pi/2} \cos x\,dx = 0.6366197724$$

STO Alpha A

At what value(s) of x does the function assume the average value on the interval?




$$\underset{y_1}{\cos x} = \underset{y_2}{A}$$

$$x = 0.881$$

At $x = 0.881$ the height of the rectangle (0.637) times the width of the interval $[0, \frac{\pi}{2}]$ will equal the area under the curve

What is the average value of the function $f(x) = (9x^2 + \cos(x))^3$ on the interval $[0, 2\pi]$? Calculator OK.

$$\frac{1}{2\pi - 0}\int_0^{2\pi} (9x^2 + \cos x)^3\,dx = 6{,}440{,}521.741$$

Position $\rightarrow$ Velocity $\longrightarrow$ Acceleration

$x(t)$ or $s(t)$ $\quad v(t)=x'(t)$ $\quad a(t)=v'(t)$

If $v(t) = \frac{1}{1+t^3}$ then find the $\underbrace{\text{average velocity}}_{\text{Position Function } x(t)}$ of the function on the interval $[a, b]$.

$$\int_a^b v(t)\,dt = x(b) - x(a) \qquad \text{1st FTC}$$

$$\text{so} \quad \frac{1}{b-a}\int_a^b v(t)\,dt = \frac{x(b)-x(a)}{b-a} \qquad \text{Average of 1st FTC}$$

If $a = 2$ and $b = 4$, then what is the average velocity?

$$\frac{1}{4-2}\int_2^4 \frac{1}{1+t^3}\,dt = 0.044$$

**Assignment #3**: Page 375: 39–46 set it up then use your calculator to find the value.

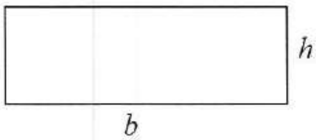
## Topic: Volumes by Cross Sections Lesson 4

*Goal: Find the volume of a 3D object by using cross sections.*

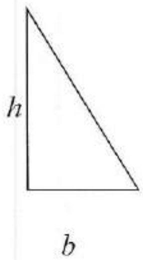
**Find the area of each shape below.**

**1. Square:** 

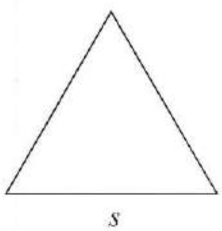

$A = s^2$

**2. Rectangle:** 

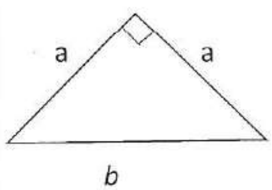

$A = b\,h$

**3. Right Triangle:** 


$A = \frac{1}{2}bh$

**4. Equilateral Triangle:** 


$A = \frac{\sqrt{3}}{4}s^2$

**Isosceles Right Triangle:** 

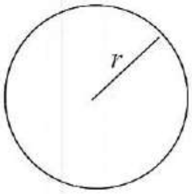

$A = \frac{1}{2}a^2$

$a^2 + a^2 = b^2$

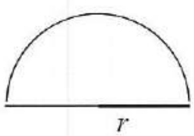
$2a^2 = b^2$

$a^2 = \frac{b^2}{2}$

$A = \frac{1}{2}\frac{b^2}{2} = \frac{b^2}{4}$

**Circle:** 


$A = \pi r^2$

**Semi Circle:** 


$A = \frac{1}{2}\pi r^2$

**Cross Sections: Slice the solid into an ∞ number of very thin cross sections. The area of the cross section, $A(x)$ depends upon what shape the cross sectional piece forms. Sum all of the cross sections together using integration. (Note: integration is an infinite sum.)**

$$V = \int A(x)dx$$ **where $A(x)$ is the area of the given shape**

height $= U - L = \sqrt{9-x^2} - -\sqrt{9-x^2}$

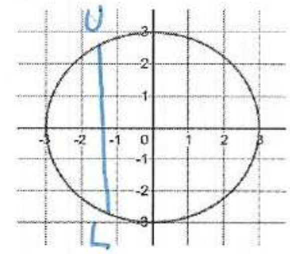
$= 2\sqrt{9-x^2}$

**All examples will use the same base that is given by the equation $x^2 + y^2 = 9$.**

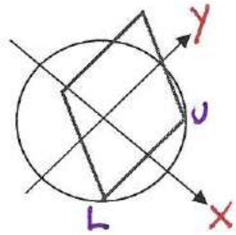
$-x^2 \quad -x^2$

$y^2 = 9 - x^2$

$y = \pm\sqrt{9-x^2}$




1) What is the volume of the solid with a base given by the equation $x^2 + y^2 = 9$ and with cross sections perpendicular to the $x$-axis that are squares?

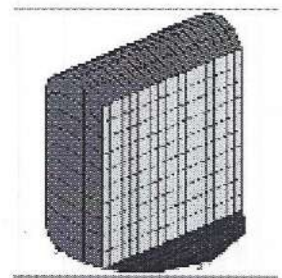



$A(x) = s^2$

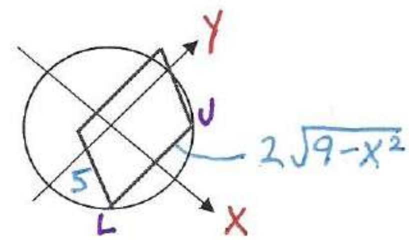
$s = 2\sqrt{9-x^2}$

$$\text{Vol} = \int_{-3}^{3} \left(2\sqrt{9-x^2}\right)^2 dx$$

$= 144$ calculator



2) What is the volume of the solid with a base given by the equation $x^2 + y^2 = 9$ and with cross sections perpendicular to the $x$-axis that are rectangles with a height of 5?

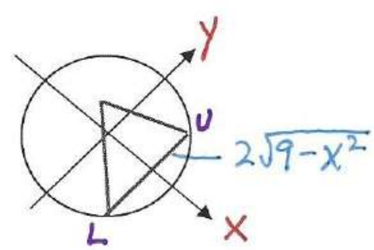


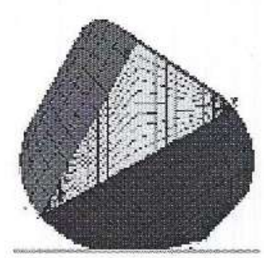

$A(x) = b\ h$

$b = 5 \qquad h = 2\sqrt{9-x^2}$

$$\int_{-3}^{3} 5 \cdot 2\sqrt{9-x^2}\,dx = 141.372$$

3) What is the volume of the solid with a base given by the equation $x^2 + y^2 = 9$ and with cross sections perpendicular to the $x$-axis that are equilateral triangles?



$A(x) = \frac{\sqrt{3}}{4} \cdot s^2$

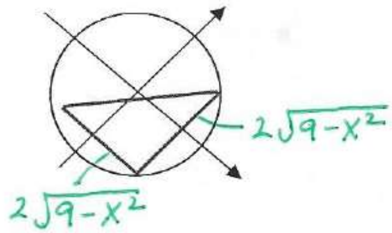
$s = 2\sqrt{9-x^2}$

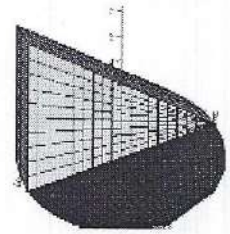
$$\int_{-3}^{3} \frac{\sqrt{3}}{4}\left(2\sqrt{9-x^2}\right)^2 dx = 62.354 \text{ un}^3$$

4) **What is the volume of the solid with a base given by the equation $x^2 + y^2 = 9$ and with cross sections perpendicular to the $x$-axis that are isosceles right triangles, with the hypotenuse in the base?**

$A(x) = \frac{1}{2}a^2$

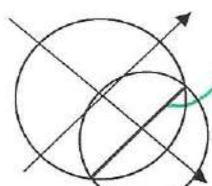
$= 2\sqrt{9-x^2}$



$$\int_{-3}^{3} \frac{1}{2}\left(2\sqrt{9-x^2}\right)^2 = 72 \text{ un}^3$$

5) **What is the volume of the solid with a base given by the equation $x^2 + y^2 = 9$ and with cross sections perpendicular to the $x$-axis that are circles?**




$$\int_{-3}^{3} \pi\left(\sqrt{9-x^2}\right)^2 dx = 113.097$$

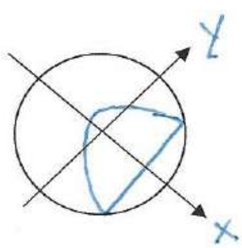
$A(x) = \pi r^2$

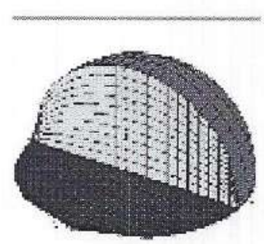
$r = \frac{1}{2} \cdot 2\sqrt{9-x^2}$

6) **What is the volume of the solid with a base given by the equation $x^2 + y^2 = 9$ and with cross sections perpendicular to the $x$-axis that are semicircles?**

$A(x) = \frac{1}{2}\pi r^2$

$r = \sqrt{9-x^2}$



$$\int_{-3}^{3} \frac{1}{2}\pi\left(\sqrt{9-x^2}\right)^2 dx = \frac{\pi}{2}\int_{-3}^{3} 9-x^2\, dx$$

$$= \frac{\pi}{2}\left[9x - \frac{x^3}{3}\Big|_{-3}^{3}\right] = \frac{\pi}{2}\left(9\cdot 3 - \frac{3^3}{3} - \left[9(-3) - \frac{(-3)^3}{3}\right]\right)$$

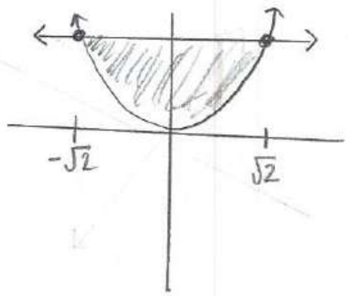
$$= \frac{\pi}{2}(18 + 18) = 18\pi \text{ un}^3$$

**Assignment #4: Handout**

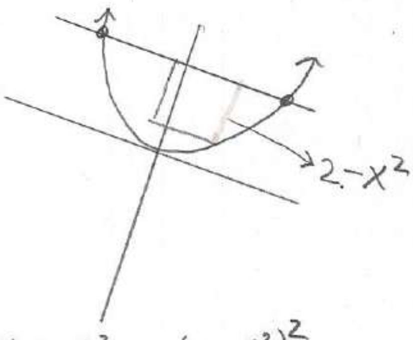
AP Calculus AB Y
**Unit #9 Assignment 4: Handout**



1. What is the volume of the solid with a base bounded by $y = x^2$ and the line $y = 2$ and with cross sections perpendicular to the $x$–axis that are squares? rectangles with a height of 6?




$U - L = 2 - x^2$




Squares $A(x) = s^2 = (2-x^2)^2$

$$\int_{-\sqrt{2}}^{\sqrt{2}} (2-x^2)^2\,dx = 6.034$$

Rectangles $A(x) = b \cdot h = (2-x^2)\,6$

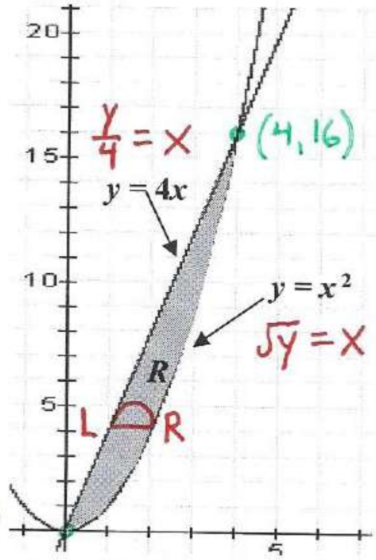
$$\int_{-\sqrt{2}}^{\sqrt{2}} (2-x^2)6\,dx = 22.627$$

## Topic: *Volumes by Cross Sections* Lesson 5

***Goal: Find the volume of a 3D object by using cross sections.***

**Further Examples**

1) What is the volume of the solid whose base is given by the region $R$ in the diagram and whose cross sections perpendicular to the $y$-axis are semi-circles?




$R - L$

$\sqrt{y} - \frac{y}{4}$

$A(x) = \frac{1}{2}\pi r^2$

$r = \frac{1}{2}\left(\sqrt{y} - \frac{y}{4}\right)$

$\sqrt{y} = x$

Intersection

$4x = x^2$

$0 = x^2 - 4x$

$0 = x(x-4)$

$x = 0, 4$

$y = 0, 16$

$$\int_{y=0}^{16} \frac{1}{2}\pi \left(\frac{1}{2}\left(\sqrt{y} - \frac{y}{4}\right)\right)^2 dy$$

or

$$\frac{\pi}{2}\int_{y=0}^{16} \left(\frac{\sqrt{y}}{2} - \frac{y}{8}\right)^2 dy$$

3.351

2) What is the volume of the solid whose base is given by the region $R$ in the diagram to the left and whose cross sections perpendicular to the $x$–axis are given by the formula $A(x) = \cos\left(\frac{\pi}{8}x\right)$? No calculator.

Area

$$\int_{x=0}^{4} \cos\left(\frac{\pi}{8}x\right) dx$$

$$\frac{8}{\pi} \sin \frac{\pi}{8}x \Big|_0^4$$

$$\frac{8}{\pi} \sin \frac{\pi}{8}\cdot 4 - \frac{8}{\pi} \sin 0$$

$$\frac{8}{\pi} \cdot 1 - 0$$

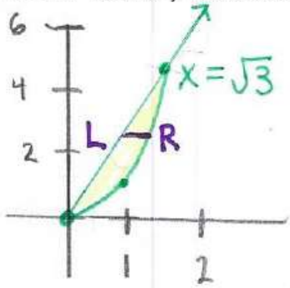
$$\frac{8}{\pi}$$

Do 1c w/class on HW

**Assignment #5: Page 373: 9, 10, 11, 13, 14, and Handout**

**Unit #9 Assignment #5: Cross Sections Handout**

1. Find the volume of the region bounded by $y = x^3$, $y = 3x$ in the first quadrant if the cross sections perpendicular to the $y$-axis are:




$x=\sqrt{3}$ $y=3\sqrt{3}$

Intersect $x^3 = 3x$ $\quad x^3-3x=0 \quad x(x^2-3)=0$

$x=0 \quad x^2-3=0 \quad x^2=3 \quad x=\pm\sqrt{3}$

$y=0 \quad y=3\sqrt{3}$

$y=x^3 \quad \sqrt[3]{y}=x$ R $\qquad y=3x \quad \frac{y}{3}=x$ L

a) Squares *(1.188)*

$A(x) = s^2$

$S = R-L = \sqrt[3]{y}-\frac{y}{3}$

$$\int_{y=0}^{3\sqrt{3}} \left(\sqrt[3]{y}-\frac{y}{3}\right)^2 dy = 1.188$$

b) equilateral triangles *(0.514)*

$A(x) = \frac{\sqrt{3}}{4}s^2$

$s = \sqrt[3]{y}-\frac{y}{3}$

$$\int_{y=0}^{3\sqrt{3}} \frac{\sqrt{3}}{4}\left(\sqrt[3]{y}-\frac{y}{3}\right)^2 dy = 0.514$$

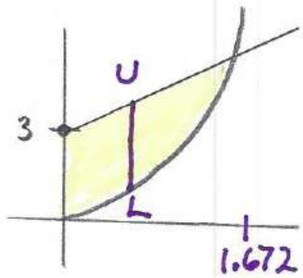
c) Isosceles Right Triangles with the hypotenuse in the base *(0.297)*

$A(x) = \frac{b^2}{4}$

$b = R-L = \sqrt[3]{y}-\frac{y}{3}$

$$\int_{y=0}^{3\sqrt{3}} \frac{1}{4}\left(\sqrt[3]{y}-\frac{y}{3}\right)^2 dy = 0.297$$

2. Find the volume of the region bounded by $y = x^3$, $y = x + 3$ in the first quadrant if the cross sections perpendicular to the $x$-axis are rectangles and the height of the rectangles are given by the function $h(x) = x + 1$. ***(7.598)***




Intersection – use calc

$x^3 = x + 3$

$y^1 \quad y^2$

$x = 1.6716999$

Sto Alpha A

$U - L$

$x + 3 - x^3$

$A(x) = bh$

$b = x + 1$

$h = x + 3 - x^3$

$$\int_{x=0}^{A} (x+3-x^3)(x+1)\,dx = 7.598$$

3. Find the volume of the region bounded by $y = x^3$, $y = x + 3$, and $0 \leq x \leq 1$ in the first quadrant with cross sections perpendicular to the $x$-axis, if the area of the cross sections is given by $A(x) = \sin^4(x)\cos(x)$. ***(0.084)***

Lesson 6

**Topic:** ***Volumes of Solids of Revolution by Disks or Washers***

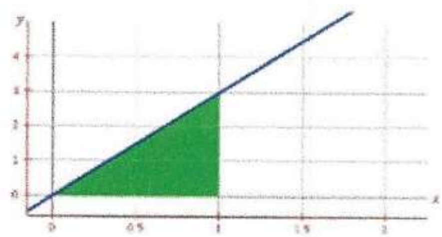
***Goal:*** *Find the volume of a 3D object by using disks or washers.*

**Definition: If a region is revolved about a line, the resulting solid is called a *solid of revolution* and the line is called the *axis of revolution*.**

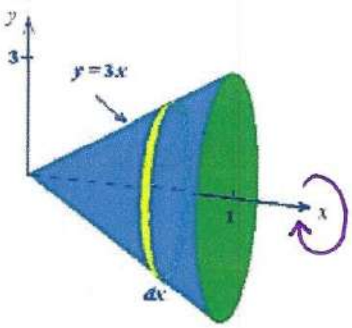
**Disk Method: Slice the solid into an infinite number of very thin disks. A disk is just a circle whose area is $\pi r^2$. Sum all of the disks together using integration. (Note: integration is an infinite sum.)**

If you slice these solids through the axis they are being rotated around, you will end up with a solid disk.

Consider the area bounded by the straight line y = 3x, the x-axis, and x = 1:



When the shaded area is rotated 360° about the x-axis, we observe that a volume is generated. The resulting solid is a cone and the radius of the cone is the function *f(x)*.




To find the volume of the cone,
Integrate pi times the radius squared.

Area ⊙ times change in X

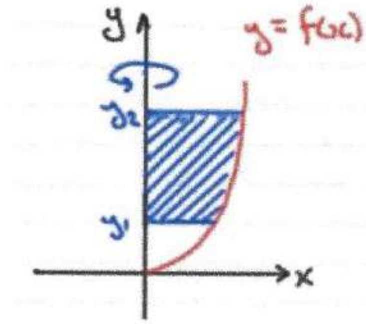
$$\text{Volume} = \int_0^b \pi\,(3x)^2\,dx$$

$\pi r^2$

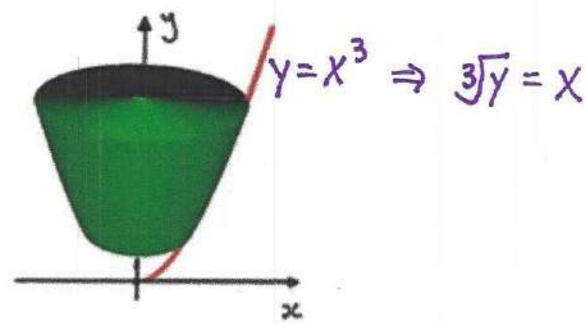
Bring the pi outside and you get

$$\text{Volume} = \pi\int_0^b (3x)^2\,dx$$

Consider the area bounded by $y = x^3$, the y-axis, and the lines y = 1 and y = 8:




When the shaded area is rotated about the y-axis, we observe that a volume is generated.
The resulting solid a cone with the tip sliced off.




To find the volume of the cone, integrate with respect to y.

$\pi r^2$

$$\text{Volume} = \int_{y=1}^{8} \pi\,(\sqrt[3]{y})^2\,dy \qquad \text{or}$$

$$\text{Volume} = \pi\int_{y=1}^{8} (\sqrt[3]{y})^2\,dy$$

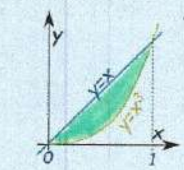
## Washer Method

Washers: Disks with Holes

What if we want the volume **between two functions**?

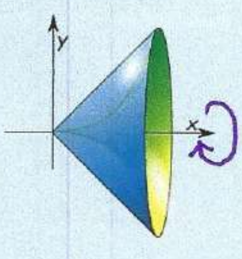
**Example: Volume between the functions $y=x$ and $y=x^3$ from $x=0$ to 1**

These are the functions:

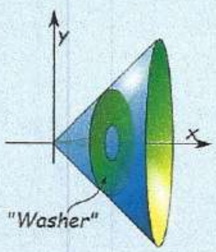



Similar to Lesson 5 #1

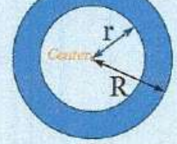
Rotated around the x-axis:




The disks are now "washers":




And they have the area of an annulus:




In our case $\mathbf{R} = \mathbf{x}$ and $r = x^3$

In effect this is the **same as the disk method,** except we subtract one disk from another.

And so our integration looks like:

$$\text{Volume} = \int_0^1 \pi (x)^2 - \pi (x^3)^2 \, dx$$

Have pi outside (on both functions):

$$\text{Volume} = \pi \int_0^1 (x)^2 - (x^3)^2 \, dx$$

Simplify:

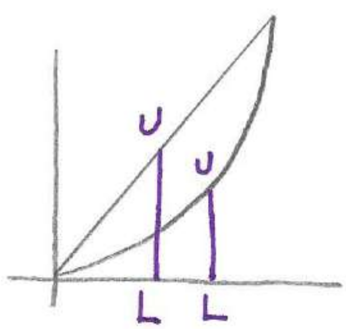
$$\text{Volume} = \pi \int_0^1 x^2 - x^6 \, dx$$

The integral of $x^2$ is $x^3/3$ and the integral of $x^6$ is $x^7/7$

And so, going between 0 and 1 we get:

$$\text{Volume} = \pi \left[ (1^3/3 - 1^7/7) - (0-0) \right]$$

$$\approx 0.598...$$




R r

$$\pi \int_A^B R^2 - r^2 \, dx$$

or

$$\int_A^B \pi R^2 - \pi r^2 \, dx$$

## *Disk and Washer Method*

**Disk Method: Slice the solid into an $\infty$ number of very thin disks. A disk is just a circle whose area is $\pi r^2$. Sum of the disks together using integration. (Note: integration is an infinite sum.)**

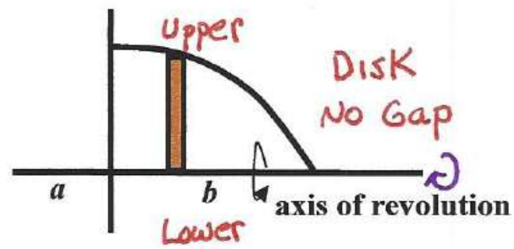
**DISK** $V = \int \pi r^2\, dx$ or **WASHER** $V = \int \pi (R^2 - r^2)\, dx$

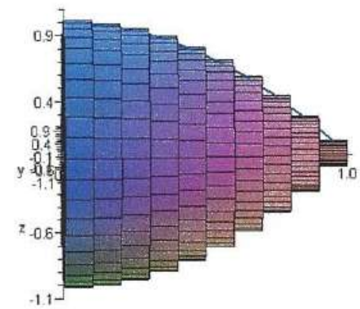
or $\int \pi R^2 - \pi r^2 dx$

**Hint: Disks are always $\perp$ to the axis of revolution**

$$V = \int_{x=a}^{b} \pi r^2\, dx$$

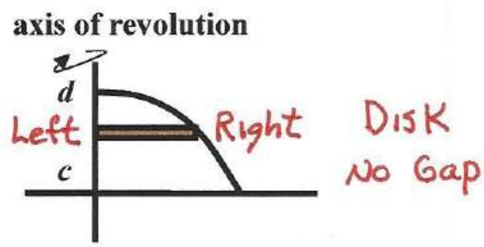
**The axis of revolution is the** x-axis.

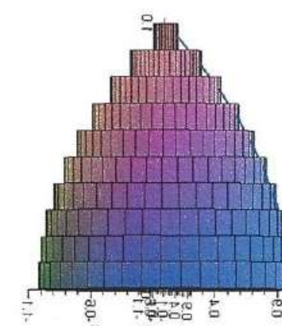


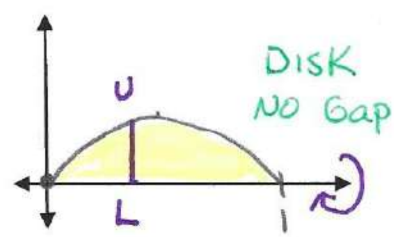
$$V = \int_{y=c}^{d} \pi r^2\, dy$$

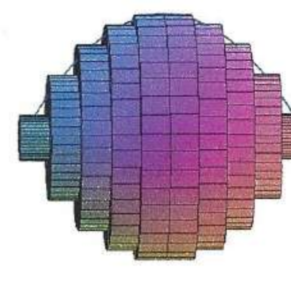
**The axis of revolution is the** y-axis.



**Example 1**: Find the volume of the solid formed by revolving the region bounded by $y = -x^2 + x$ and $y = 0$ about the x–axis.



Identify the region by graphing.

Cross sections are $\perp$ to x-axis

Find the radius.

$r = U - L = -x^2 + x - 0$
$= -x^2 + x$

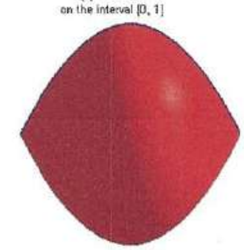
Find the limits of integration

$$\int_0^1 \pi(-x^2+x)^2\, dx$$

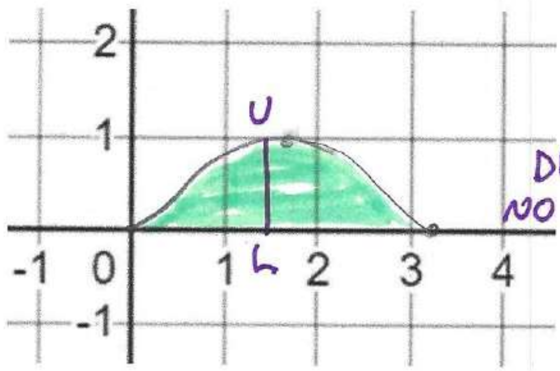
0.105

Set up the integral.

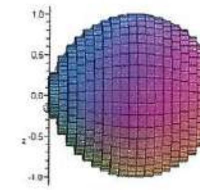
The Volume of Revolution Around the Horizontal Axis of
f(x) = -x^2+x
on the interval [0, 1]

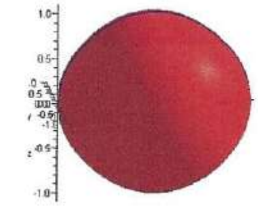


**Example 2**: Find the volume of the solid formed by revolving the region bounded by $y = \sin x$ and $y = 0$ about the $x$–axis, $0 \le x \le \pi$.




DISK
NO GAP





Identify the region by graphing.

Cross sections are $\perp$ to

X-axis

Find the radius.

$r = U - L = \sin x - 0 = \sin x$

Find the limits of integration

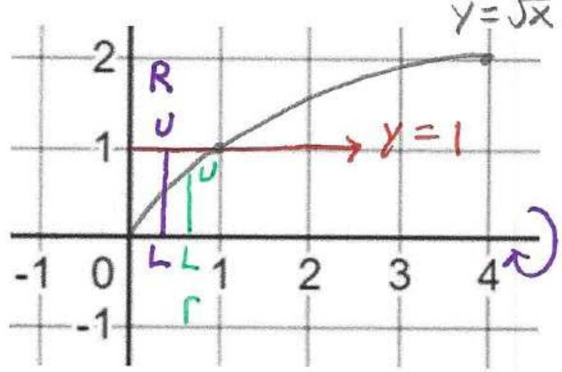
$0 - \pi$

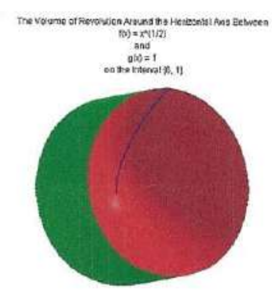
Set up the integral.

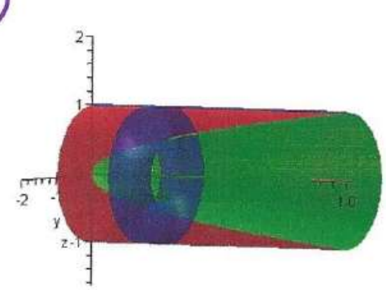
$$\int_0^{\pi} \pi(\sin x)^2\, dx = 4.935$$

**Example 3**: Find the volume of the solid formed by revolving the region bounded by $y = \sqrt{x}$, $y = 1$, and $x = 0$ about the $x$–axis. (hole--washer)

Washer



Identify the region by graphing.

Cross sections are $\perp$ to

X-axis

Find both radii.

$R = 1 - 0 = 1$ $\quad r = \sqrt{x} - 0 = \sqrt{x}$

Find the limits of integration

$0 - 1$

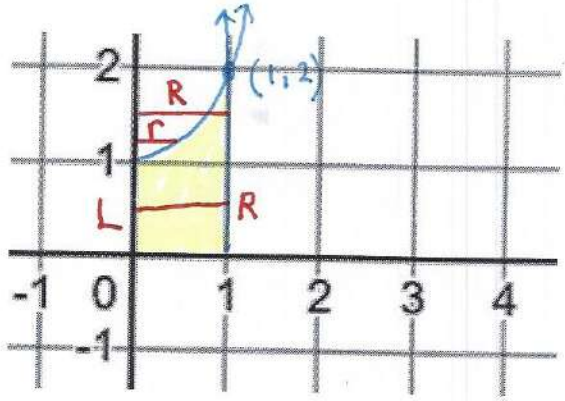
Set up the integral.

$$\int_0^1 \pi \cdot 1^2 - \pi \sqrt{x}^2\, dx = \int_0^1 \pi - \pi x\, dx$$

$$= \pi x - \pi \frac{x^2}{2} \Big|_0^1$$

$$= \pi - \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

**Example 4**: Find the volume of the solid formed by revolving the region bounded by $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the $y$–axis.




Washer
$y = 1$ To 2

Disk
$y = 0$ To 1

$y = x^2 + 1$
$y - 1 = x^2$
$\sqrt{y-1} = x$

Identify the region by graphing.

Cross sections are ⊥ to

y-axis

Find the radii.

Find the limits of integration

$y = 0$ To 1
$R - L = 1$

$y = 1$ To 2
$R = 1 \quad r = \sqrt{y-1}$

Set up the integral.

$$\int_{y=0}^{1} \pi \cdot 1^2 \, dy + \int_{y=1}^{2} \pi \cdot 1^2 - \pi \sqrt{y-1}^{\,2} \, dy$$

$\pi(y-1)$

$$\pi \int_0^1 1 \, dy + \pi \int_1^2 1 - (y-1) \, dy$$

$2 - y$

$$\pi y \Big|_0^1 + \pi\left(2y - \frac{y^2}{2}\right)\Big|_1^2$$

$$\pi - 0 + \pi\left[2 \cdot 2 - \frac{2^2}{2} - \left(2 \cdot 1 - \frac{1^2}{2}\right)\right]$$

$4 - 2 - \left(2 - \frac{1}{2}\right)$

$$\pi + \pi\left(\frac{1}{2}\right)$$

$$\frac{3\pi}{2}$$

**Disk and Washer Method--Summary**

**Solid:** $\int \pi r^2 dx$ or $\int \pi r^2 dy$

**Hollow:** $\int \pi(R^2 - r^2) dx$ or $\int \pi(R^2 - r^2) dy$

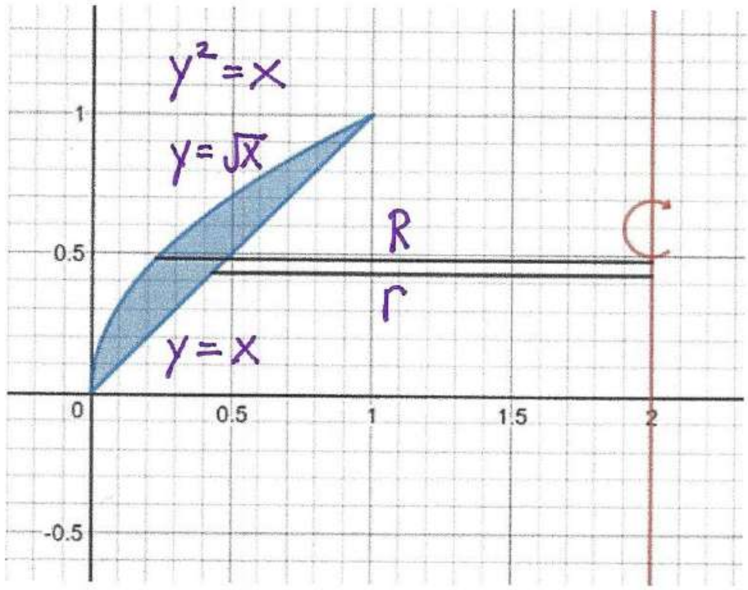
**Assignment #6: Volume of Revolution Worksheet—Disk and Washer Method: 1-4, 5a, 5b, 6a, 6b, 7a, 7b, 8a, 8b, 9a. 10a (see handout after lesson 7)** 6+7

## Lesson 7

**Topic:** ***Volumes of Solids of Revolution by Disks -- Non-Adjacent Axis of Revolution***

***Goal: Find the volume of a 3D object by using disks or washers for an axis of revolution not the x-axis or y-axis.***

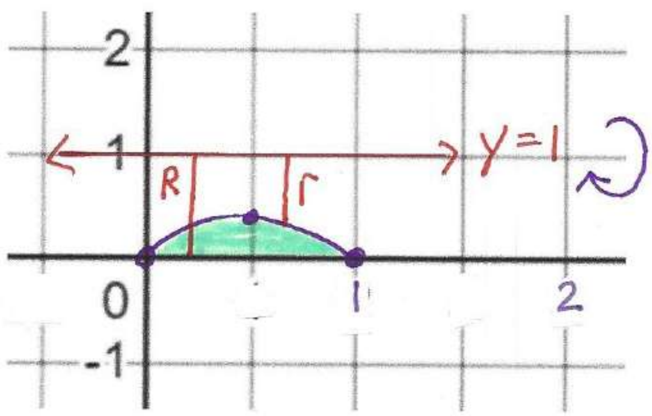
**Non-adjacent axis of revolution means the region that is being rotated is not necessarily the *x*-axis or *y*-axis.**




$R = 2-y^2 \quad r = 2-y$

$\pi \int_{y=0}^{1} (2-y^2)^2 - (2-y)^2 \, dy$

**Example 5**: Find the volume of the solid formed by revolving the region bounded by $y = -x^2 + x$ and $y = 0$ about $y = 1$.




Identify the region by graphing.

Cross sections are ⊥ to

x-axis

Find the radii.

$R = 1-0 = 1 \quad r = 1-(-x^2+x) = x^2-x+1$

Find the limits of integration

$x=0-1$

Set up the integral.

$\int_{x=0}^{1} \pi \cdot 1^2 - \pi(x^2-x+1)^2 \, dx$

or

$\pi \int_0^1 1^2-(x^2-x+1)^2 \, dx = 0.943$

**Example 6**: Find the volume of the solid formed by revolving the region bounded by $y = -x^2 + x$ and $y = 0$ about $y = -1$.

Identify the region by graphing.

Identify the axis of revolution.

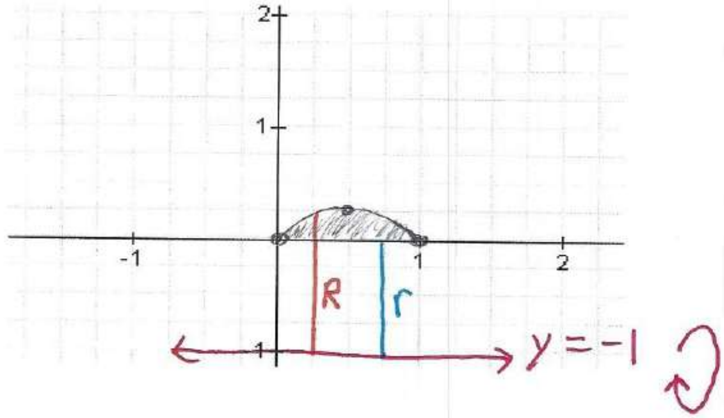
Cross sections are ⊥ to

x-axis

Find the radii.

Find the limits of integration

Set up the integral.




$R = -x^2 + x - (-1)$ $\quad r = 0 - (-1)$

$= -x^2 + x + 1$ $\quad r = 1$

$$\pi \int_{x=0}^{1} \left[(-x^2+x+1)^2 - 1^2\right] dx = 1.152$$

**Example 7**: Find the volume of the solid formed by revolving the region bounded by $y = x$, $0 \le x \le 1$, and y = 0, about $x = 2$.

Identify the region by graphing.

Identify the axis of revolution.

Cross sections are ⊥ to

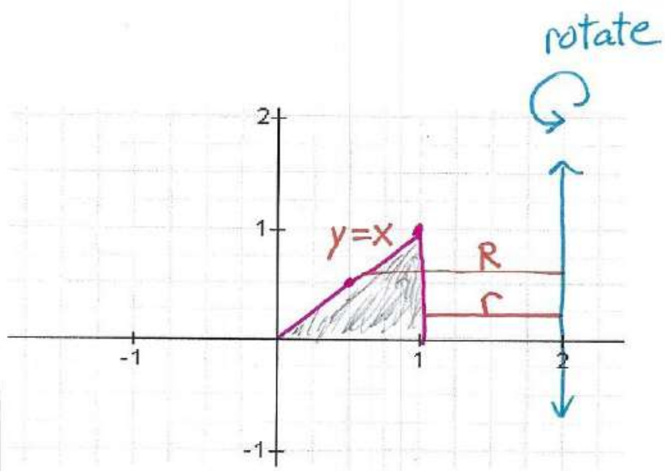
y-axis

Find the radii.

Find the limits of integration

$y = 0$ To $y = 1$

Set up the integral.




$R = 2 - y$ $\quad r = 2 - 1 = 1$

$$\pi \int_{y=0}^{1} \left[(2-y)^2 - 1^2\right] dy = 4.189$$

**Example 8**: Find the volume of the solid formed by revolving the region bounded by $y = x$, $0 \le x \le 1$, and $y = 0$, about $x = 4$.

Identify the region by graphing.

Identify the axis of revolution.

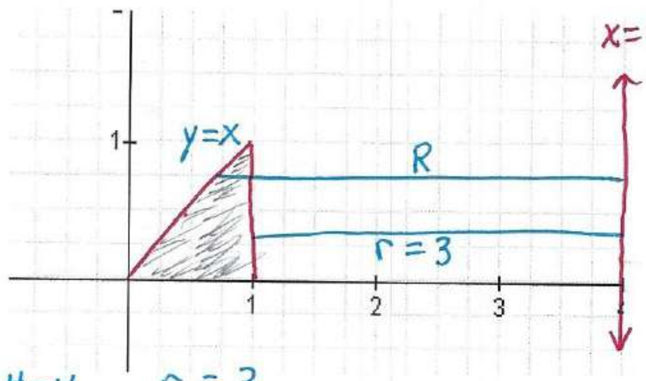
Cross sections are $\perp$ to

y-axis

Find the radii.

Find the limits of integration

Set up the integral.




$R = 4 - y \quad r = 3$

$$\pi \int_0^1 \left[(4-y)^2 - 3^2\right] dy = 10.472$$

**Example 9**: Find the volume of the solid formed by revolving the region bounded by $y = x$, $0 \le x \le 1$, and $y = 0$, about $x = 1$.

Identify the region by graphing.

Identify the axis of revolution.

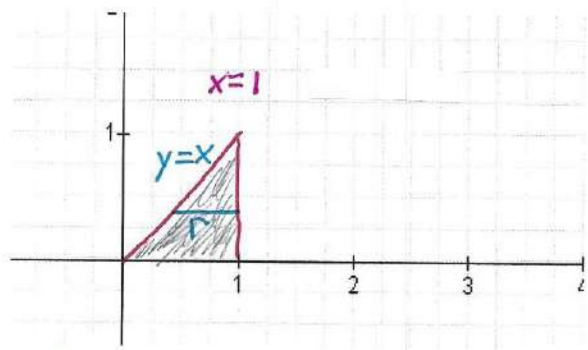
Cross sections are $\perp$ to

y-axis

Find the radii.

Find the limits of integration

Set up the integral.




$r = 1 - y$

$$\pi \int_0^1 (1-y)^2 \, dy = 1.047$$

**Assignment** 7

**Volume of Revolution Worksheet—Disk and Washer Method: 5c, 6c, 7c, 7d, 8c, 9b, 10b, 11**

6 + 7