| $h$ (feet) | 0 | 2 | 5 | 10 |
|---|---|---|---|---|
| $A(h)$ (square feet) | 50.3 | 14.4 | 6.5 | 2.9 |

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height $h$ feet is given by the function $A$, where $A(h)$ is measured in square feet. The function $A$ is continuous and decreases as $h$ increases. Selected values for $A(h)$ are given in the table above.

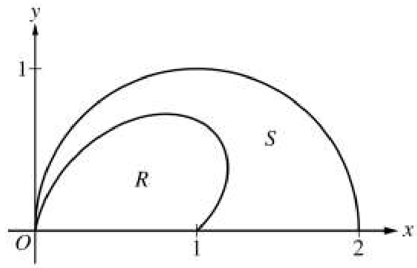
(a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.

(b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.

(c) The area, in square feet, of the horizontal cross section at height $h$ feet is modeled by the function $f$ given by $f(h) = \dfrac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.

(d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

| | |
|---|---|
| | 1 : units in parts (a), (c), and (d) |
| (a) Volume $= \int_0^{10} A(h)\,dh$ <br> $\approx (2-0)\cdot A(0) + (5-2)\cdot A(2) + (10-5)\cdot A(5)$ <br> $= 2\cdot 50.3 + 3\cdot 14.4 + 5\cdot 6.5$ <br> $= 176.3$ cubic feet | 2 : { 1 : left Riemann sum; 1 : approximation } |
| (b) The approximation in part (a) is an overestimate because a left Riemann sum is used and $A$ is decreasing. | 1 : overestimate with reason |
| (c) $\int_0^{10} f(h)\,dh = 101.325338$ <br> The volume is 101.325 cubic feet. | 2 : { 1 : integral; 1 : answer } |
| (d) Using the model, $V(h) = \int_0^h f(x)\,dx$. <br> $\left.\frac{dV}{dt}\right\|_{h=5} = \left[\frac{dV}{dh}\cdot\frac{dh}{dt}\right]_{h=5}$ <br> $= \left[f(h)\cdot\frac{dh}{dt}\right]_{h=5}$ <br> $= f(5)\cdot 0.26 = 1.694419$ <br> When $h = 5$, the volume of water is changing at a rate of 1.694 cubic feet per minute. | 3 : { 2 : $\frac{dV}{dt}$; 1 : answer } |




2. The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Let $R$ be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the $x$-axis. Let $S$ be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the $x$-axis.

(a) Find the area of $R$.

(b) The ray $\theta = k$, where $0 < k < \frac{\pi}{2}$, divides $S$ into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of $k$.

(c) For each $\theta$, $0 \le \theta \le \frac{\pi}{2}$, let $w(\theta)$ be the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Write an expression for $w(\theta)$. Find $w_A$, the average value of $w(\theta)$ over the interval $0 \le \theta \le \frac{\pi}{2}$.

(d) Using the information from part (c), find the value of $\theta$ for which $w(\theta) = w_A$. Is the function $w(\theta)$ increasing or decreasing at that value of $\theta$ ? Give a reason for your answer.

(a) $\frac{1}{2}\int_0^{\pi/2} (f(\theta))^2 \, d\theta = 0.648414$

The area of $R$ is 0.648.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\int_0^k \left((g(\theta))^2 - (f(\theta))^2\right) d\theta = \frac{1}{2}\int_0^{\pi/2} \left((g(\theta))^2 - (f(\theta))^2\right) d\theta$

— OR —

$\int_0^k \left((g(\theta))^2 - (f(\theta))^2\right) d\theta = \int_k^{\pi/2} \left((g(\theta))^2 - (f(\theta))^2\right) d\theta$

$2 : \begin{cases} 1 : \text{integral expression} \\ \quad\ \text{for one region} \\ 1 : \text{equation} \end{cases}$

(c) $w(\theta) = g(\theta) - f(\theta)$

$$w_A = \frac{\int_0^{\pi/2} w(\theta) \, d\theta}{\frac{\pi}{2} - 0} = 0.485446$$

The average value of $w(\theta)$ on the interval $\left[0, \frac{\pi}{2}\right]$ is 0.485.

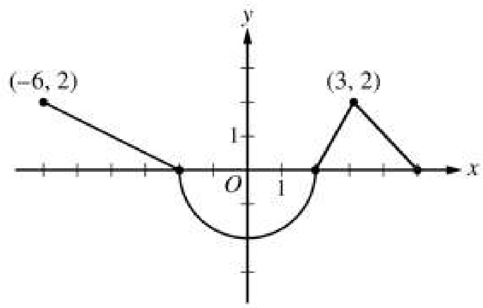
$3 : \begin{cases} 1 : w(\theta) \\ 1 : \text{integral} \\ 1 : \text{average value} \end{cases}$

(d) $w(\theta) = w_A$ for $0 \le \theta \le \frac{\pi}{2} \Rightarrow \theta = 0.517688$

$w(\theta) = w_A$ at $\theta = 0.518$ (or 0.517).

$w'(0.518) < 0 \Rightarrow w(\theta)$ is decreasing at $\theta = 0.518$.

$2 : \begin{cases} 1 : \text{solves } w(\theta) = w_A \\ 1 : \text{answer with reason} \end{cases}$




Graph of $f'$

3. The function $f$ is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of $f'$, the derivative of $f$, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of $f(-6)$ and $f(5)$.

(b) On what intervals is $f$ increasing? Justify your answer.

(c) Find the absolute minimum value of $f$ on the closed interval $[-6, 5]$. Justify your answer.

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

(a) $f(-6) = f(-2) + \int_{-2}^{-6} f'(x)\,dx = 7 - \int_{-6}^{-2} f'(x)\,dx = 7 - 4 = 3$

$f(5) = f(-2) + \int_{-2}^{5} f'(x)\,dx = 7 - 2\pi + 3 = 10 - 2\pi$

$3: \begin{cases} 1: \text{uses initial condition} \\ 1: f(-6) \\ 1: f(5) \end{cases}$

(b) $f'(x) > 0$ on the intervals $[-6, -2)$ and $(2, 5)$.
Therefore, $f$ is increasing on the intervals $[-6, -2]$ and $[2, 5]$.

2 : answer with justification

(c) The absolute minimum will occur at a critical point where $f'(x) = 0$ or at an endpoint.

$f'(x) = 0 \Rightarrow x = -2, x = 2$

| $x$ | $f(x)$ |
|---|---|
| $-6$ | $3$ |
| $-2$ | $7$ |
| $2$ | $7 - 2\pi$ |
| $5$ | $10 - 2\pi$ |

The absolute minimum value is $f(2) = 7 - 2\pi$.

$2: \begin{cases} 1: \text{considers } x = 2 \\ 1: \text{answer with justification} \end{cases}$

(d) $f''(-5) = \dfrac{2 - 0}{-6 - (-2)} = -\dfrac{1}{2}$

$\lim_{x \to 3^-} \dfrac{f'(x) - f'(3)}{x - 3} = 2$ and $\lim_{x \to 3^+} \dfrac{f'(x) - f'(3)}{x - 3} = -1$

$f''(3)$ does not exist because
$\lim_{x \to 3^-} \dfrac{f'(x) - f'(3)}{x - 3} \neq \lim_{x \to 3^+} \dfrac{f'(x) - f'(3)}{x - 3}$.

$2: \begin{cases} 1: f''(-5) \\ 1: f''(3) \text{ does not exist, with explanation} \end{cases}$

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time $t$ minutes can be modeled by the function $H$ that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of $H$ at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time $t$ minutes is the function $G$ that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$ ?

| Solution | Scoring |
| --- | --- |
| (a) $H'(0) = -\frac{1}{4}(91 - 27) = -16$<br>$H(0) = 91$<br><br>An equation for the tangent line is $y = 91 - 16t$.<br><br>The internal temperature of the potato at time $t = 3$ minutes is approximately $91 - 16 \cdot 3 = 43$ degrees Celsius. | 3 : { 1 : slope<br>1 : tangent line<br>1 : approximation |
| (b) $\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H - 27) = \frac{1}{16}(H - 27)$<br><br>$H > 27$ for $t > 0 \Rightarrow \frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0$ for $t > 0$<br>Therefore, the graph of $H$ is concave up for $t > 0$. Thus, the answer in part (a) is an underestimate. | 1 : underestimate with reason |
| (c) $\frac{dG}{(G - 27)^{2/3}} = -dt$<br>$\int \frac{dG}{(G - 27)^{2/3}} = \int (-1)\, dt$<br>$3(G - 27)^{1/3} = -t + C$<br>$3(91 - 27)^{1/3} = 0 + C \Rightarrow C = 12$<br>$3(G - 27)^{1/3} = 12 - t$<br>$G(t) = 27 + \left(\frac{12 - t}{3}\right)^3$ for $0 \le t < 10$<br><br>The internal temperature of the potato at time $t = 3$ minutes is $27 + \left(\frac{12 - 3}{3}\right)^3 = 54$ degrees Celsius. | 5 : { 1 : separation of variables<br>1 : antiderivatives<br>1 : constant of integration and uses initial condition<br>1 : equation involving $G$ and $t$<br>1 : $G(t)$ and $G(3)$<br><br>Note: max 2/5 [1-1-0-0-0] if no constant of integration<br><br>Note: 0/5 if no separation of variables |

5. Let $f$ be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of $f$ at $x = 3$.

(b) Find the $x$-coordinate of each critical point of $f$ in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

(c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\int_5^{\infty} f(x)\,dx$ or show that the integral diverges.

(d) Determine whether the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

| Solution | Points |
|---|---|
| (a) $f'(x) = \dfrac{-3(4x-7)}{\left(2x^2 - 7x + 5\right)^2}$ <br> $f'(3) = \dfrac{(-3)(5)}{(18 - 21 + 5)^2} = -\dfrac{15}{4}$ | 2 : $f'(3)$ |
| (b) $f'(x) = \dfrac{-3(4x-7)}{\left(2x^2 - 7x + 5\right)^2} = 0 \Rightarrow x = \dfrac{7}{4}$ <br> The only critical point in the interval $1 < x < 2.5$ has $x$-coordinate $\dfrac{7}{4}$. <br> $f'$ changes sign from positive to negative at $x = \dfrac{7}{4}$. <br> Therefore, $f$ has a relative maximum at $x = \dfrac{7}{4}$. | 2 : { 1 : $x$-coordinate; 1 : relative maximum with justification |
| (c) $\int_5^{\infty} f(x)\,dx = \lim\limits_{b\to\infty} \int_5^b \dfrac{3}{2x^2 - 7x + 5}\,dx = \lim\limits_{b\to\infty} \int_5^b \left(\dfrac{2}{2x-5} - \dfrac{1}{x-1}\right) dx$ <br> $= \lim\limits_{b\to\infty} \left[\ln(2x-5) - \ln(x-1)\right]_5^b = \lim\limits_{b\to\infty} \left[\ln\left(\dfrac{2x-5}{x-1}\right)\right]_5^b$ <br> $= \lim\limits_{b\to\infty} \left[\ln\left(\dfrac{2b-5}{b-1}\right) - \ln\left(\dfrac{5}{4}\right)\right] = \ln 2 - \ln\left(\dfrac{5}{4}\right) = \ln\left(\dfrac{8}{5}\right)$ | 3 : { 1 : antiderivative; 1 : limit expression; 1 : answer |
| (d) $f$ is continuous, positive, and decreasing on $[5, \infty)$. <br> The series converges by the integral test since $\int_5^{\infty} \dfrac{3}{2x^2 - 7x + 5}\,dx$ converges. <br> — OR — <br> $\dfrac{3}{2n^2 - 7n + 5} > 0$ and $\dfrac{1}{n^2} > 0$ for $n \ge 5$. <br> Since $\lim\limits_{n\to\infty} \dfrac{\frac{3}{2n^2 - 7n + 5}}{\frac{1}{n^2}} = \dfrac{3}{2}$ and the series $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n^2}$ converges, <br> the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges by the limit comparison test. | 2 : answer with conditions |

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function $f$ has derivatives of all orders for $-1 < x < 1$. The derivatives of $f$ satisfy the conditions above. The Maclaurin series for $f$ converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for $f$ are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for $f$.

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the $n$th-degree Taylor polynomial for $g$ about $x = 0$ evaluated at $x = \frac{1}{2}$, where $g$ is the function defined in part (c). Use the alternating series error bound to show that

$$\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| < \frac{1}{500}.$$

(a) $f(0) = 0$
$f'(0) = 1$
$f''(0) = -1(1) = -1$
$f'''(0) = -2(-1) = 2$
$f^{(4)}(0) = -3(2) = -6$

The first four nonzero terms are
$0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$

The general term is $\frac{(-1)^{n+1}x^n}{n}$.

$3: \begin{cases} 1: f''(0),\ f'''(0), \text{ and } f^{(4)}(0) \\ 1: \text{verify terms} \\ 1: \text{general term} \end{cases}$

(b) For $x = 1$, the Maclaurin series becomes $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

The series does not converge absolutely because the harmonic series diverges.

The series alternates with terms that decrease in magnitude to 0, and therefore the series converges conditionally.

2 : converges conditionally with reason

(c) $$\int_0^x f(t)\,dt = \int_0^x \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + \frac{(-1)^{n+1}t^n}{n} + \cdots\right) dt$$
$$= \left[\frac{t^2}{2} - \frac{t^3}{3 \cdot 2} + \frac{t^4}{4 \cdot 3} - \frac{t^5}{5 \cdot 4} + \cdots + \frac{(-1)^{n+1}t^{n+1}}{(n+1)n} + \cdots\right]_{t=0}^{t=x}$$
$$= \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^{n+1}x^{n+1}}{(n+1)n} + \cdots$$

$3: \begin{cases} 1: \text{two terms} \\ 1: \text{remaining terms} \\ 1: \text{general term} \end{cases}$

(d) The terms alternate in sign and decrease in magnitude to 0. By the alternating series error bound, the error $\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right|$ is bounded by the magnitude of the first unused term, $\left|-\frac{(1/2)^5}{20}\right|$.

Thus, $\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| \leq \left|-\frac{(1/2)^5}{20}\right| = \frac{1}{32 \cdot 20} < \frac{1}{500}.$

1 : error bound