More Quarter test review
Section 4.1
Composite Functions

Composite Functions

Given two functions *f* and *g*, the **composite function**, denoted by $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$

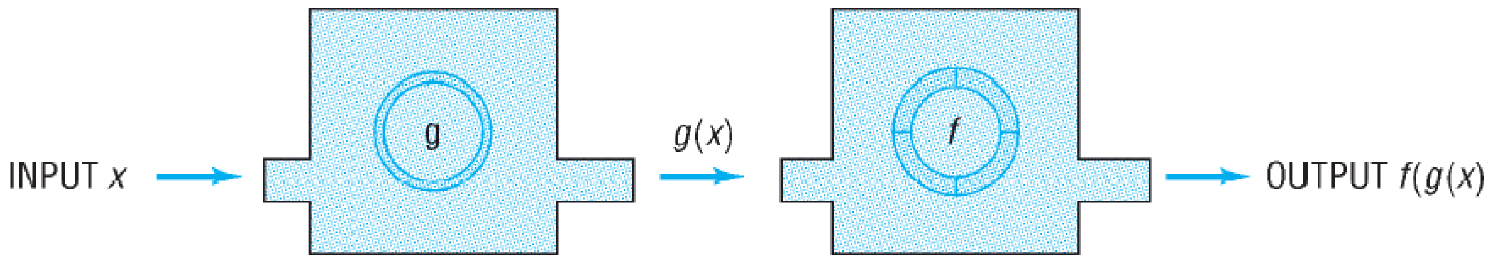
"*f* composed with g"

"*f* composition g"

"*f* of g"

It's kind of like if f(x) = 2x - 1, then
f(2x) = 2(2x) – 1 = 4x - 1

The domain of $f \circ g$ is the set of all numbers *x* in the domain of *g* such that *g*(*x*) is in the domain of *f*… that is, if *g*(*x*) is not in the domain of *f*, then *f*(*g*(*x*)) is not defined. Because of this, the domain of $f \circ g$ is a subset of the domain of *g*; the range of $f \circ g$ is a subset of *f*.



(Notice that the "inside" function *g* in *f(g(x))* is done first.)

Evaluating a Composition

To evaluate $(f \circ g)(x)$, find *g(x)* first, then plug that answer into *f(x)*.

Example: Suppose that *f(x)* = $2x^2 - 3$ and *g(x)* = 4*x*. Find:

a) $(f \circ g)(1)$

First solve for *g*(1). *Substitute 1 in the place of x for g(x)=4x*

g(1) = 4(1) = 4

Use this new value in place of x in the equation f(x) = $2x^2 - 3$:

$f(4) = 2(4)^2 - 3$

$= 2(16) - 3$

$= 32 - 3 = 29$

b) $(g \circ f)(1)$ First, find f(1): $f(1) = 2(1)^2 - 3 = -1$

Use this value in g(x):

$g(-1) = 4(-1) = -4$

c) $(f \circ f)(-2)$ First, find f(-2): $f(-2) = 2(-2)^2 - 3 = 5$

Use this value in f(x):

$f(5) = 2(5)^2 - 3 = 47$

d) $(g \circ g)(-1)$ First, find g(-1): $g(-1) = 4(-1) = -4$

Use this value to find g(x):

$g(-4) = 4(-4) = -16$

Finding a Composite Function and its Domain

$f \circ g$ can be written as a simplified function whose domain is based first on g, then on $f \circ g$.

Find the domain of the "inside" function first, then the domain of the two together…

Example: Suppose that $f(x) = x^2 + 3x - 1$ and $g(x) = 2x + 3$. Find $f \circ g$ and $g \circ f$ and their domains.

To find $f \circ g$, substitute g(x) into f(x):

$f(g) = (2x + 3)^2 + 3(2x + 3) - 1$ *FOIL and distribute*

$= (4x^2 + 12x + 9) + 6x + 9 - 1$ *Simplify*

$= 4x^2 + 18x + 17$

$$f \circ g = 4x^2 + 18x + 17$$

D: all real numbers

To find $g \circ f$, substitute f(x) into g(x):

$g(f) = 2(x^2 + 3x - 1) + 3$ *Distribute*

$= 2x^2 + 6x - 2 + 3$ *Simplify*

$= 2x^2 + 6x + 1$

$$g \circ f = 2x^2 + 6x + 1$$

D: all real numbers

Example:

Suppose that $f(x)=\dfrac{1}{x+2}$ and $g(x)=\dfrac{4}{x-1}$. Find $f\circ g$ and $f\circ f$ and their domains.

To find $f\circ g$ *substitute g(x) into f(x):*

$$f\left(\frac{4}{x-1}\right)=\frac{1}{\dfrac{4}{x-1}+2}$$

Simplify by multiplying numerator and denominator by x - 1

This will get rid of the fraction in the denominator

$$=\frac{1}{\dfrac{4}{\cancel{x-1}}+2}\bullet\frac{x-1}{\cancel{x-1}}$$

Simplify

$$=\frac{x-1}{4+2(x-1)}\quad=\frac{x-1}{4+2x-2}$$

Simplify by factoring the denominator

$$=\frac{x-1}{2x+2}=\frac{x-1}{2(x+1)}$$

$$f \circ g = \frac{x-1}{2(x+1)}$$

To find the domain, first find the domain of g(x):

$$g(x) = \frac{4}{x-1} \qquad D: \; x \neq 1$$

Now find the domain of $f \circ g$ *:*

$$f \circ g = \frac{x-1}{2(x+1)} \qquad D: \; x \neq -1$$

Put the two together:

$$Domain: \; x \neq \pm 1$$

Suppose that $f(x)=\dfrac{1}{x+2}$ and $g(x)=\dfrac{4}{x-1}$. Find $f\circ g$ and $f\circ f$ and their domains.

To find $f\circ f$ substitute f(x) into f(x):

Simplify by multiplying numerator and denominator by x + 2

$$f\left(\frac{1}{x+2}\right)=\frac{1}{\dfrac{1}{x+2}+2}$$

This will get rid of the fraction in the denominator

$$=\frac{1}{\dfrac{1}{\cancel{x+2}}+2}\bullet\frac{x+2}{\cancel{x+2}}$$

Simplify

$$=\frac{x+2}{1+2(x+2)}$$

$$=\frac{x+2}{1+2x+4}\quad=\frac{x+2}{2x+5}$$

$$f \circ f = \frac{x+2}{2x+5}$$

To find the domain, first find the domain of f(x):

$$f(x) = \frac{1}{x+2} \qquad D:\ x \neq -2$$

Now find the domain of $f \circ f$ *:*

$$f \circ f = \frac{x+2}{2x+5} \qquad D:\ x \neq -5/2$$

Put the two together:

$$Domain:\ x \neq -2,\ -5/2$$

Showing that Two Composite Functions are Equal

…that is, prove that $(f \circ g)(x) = (g \circ f)(x) = x$

Example: If $f(x) = 3x - 4$ and $g(x) = \frac{1}{3}(x+4)$, show that $(f \circ g)(x) = (g \circ f)(x) = x$, for every x in the domain of $f \circ g$ and $g \circ f$.

Find $f \circ g$ by substituting g(x) into f(x):

$f \circ g$

$$f(g) = 3\left(\tfrac{1}{3}(x+4)\right) - 4$$

Simplify

$$= (x \cancel{+ 4)} \cancel{- 4}$$

$$= x$$

$$f \circ g = x$$

Now find $g \circ f$ by substituting f(x) into g(x):

Simplify inside the parentheses

$$g(f) = \tfrac{1}{3}(3x \cancel{- 4 + 4})$$

Multiply

$$= \tfrac{1}{3}(3x)$$

$$= x$$

$$g \circ f = x$$

$$\therefore f \circ g = g \circ f$$

"therefore"

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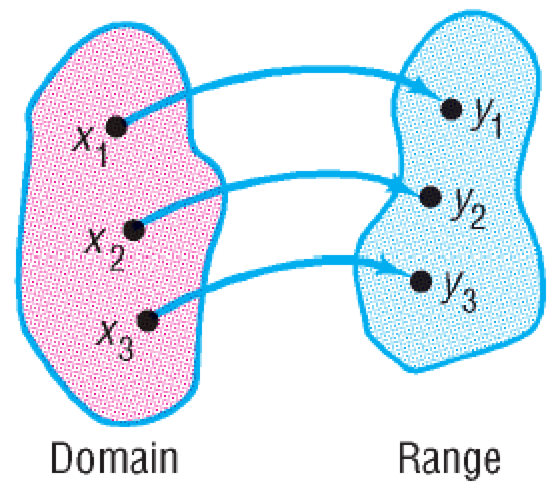
One-to-One and Inverse Functions

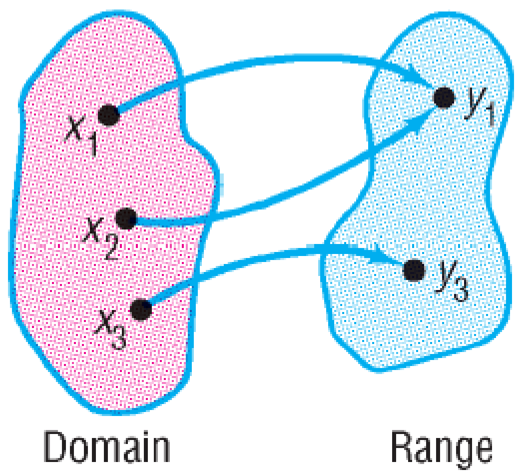
One-to-One Functions

A function is **one-to-one** if no y in the range is in the image of more than one x in the domain…that is, if any two different inputs in the domain correspond to two different outputs in the range.

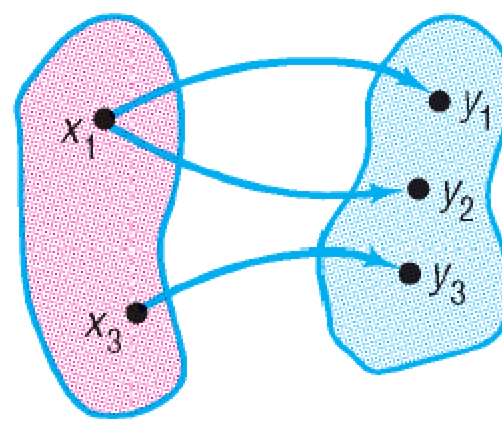
One-to-One



Not one-to-one

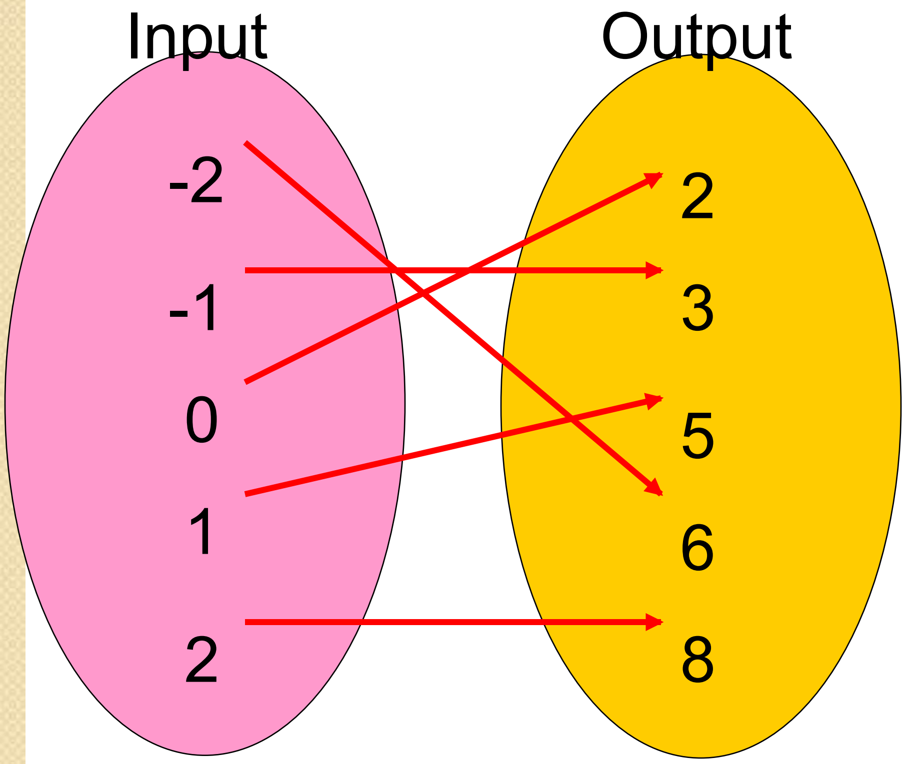


Not a function



Example: Determine whether the set is a one-to-one function.

$\{(-2, 6), (-1, 3), (0, 2), (1, 5), (2, 8)\}$



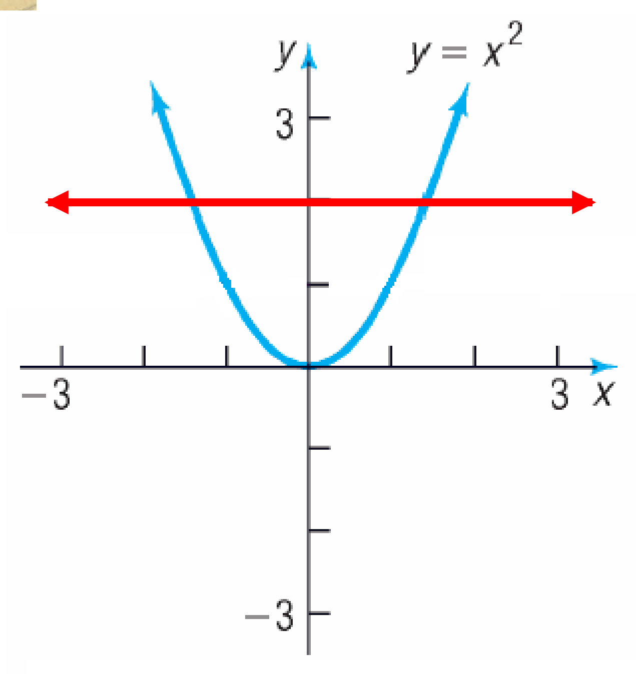
Yes, the function is one-to-one.

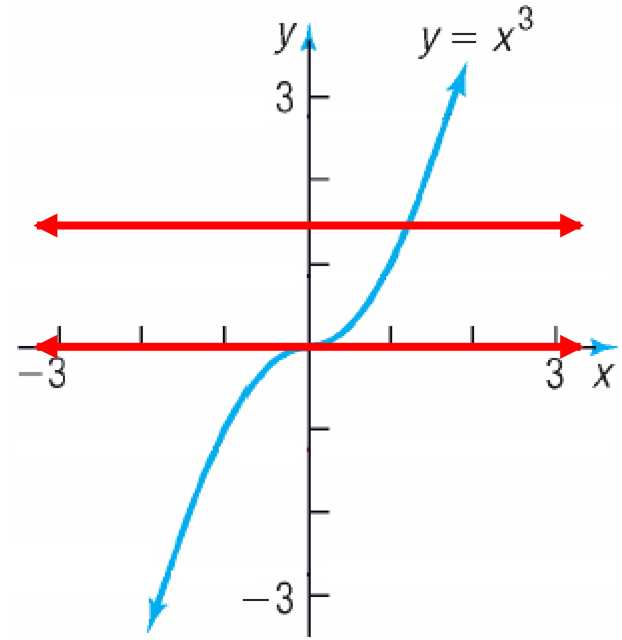
To determine if a graph represents a one-to-one function:

Horizontal-line Test – If every horizontal line intersects the graph of a function f in at most one point, then f is one-to-one.

- A function that is increasing on an interval I is a one-to-one function on I.
- A function that is decreasing on an interval I is a one-to-one function on I.

Example: Use a horizontal line test to determine whether the graph represents a one-to-one function.





No, the function is not one-to-one.

Yes, the function is one-to-one.

Inverse Functions

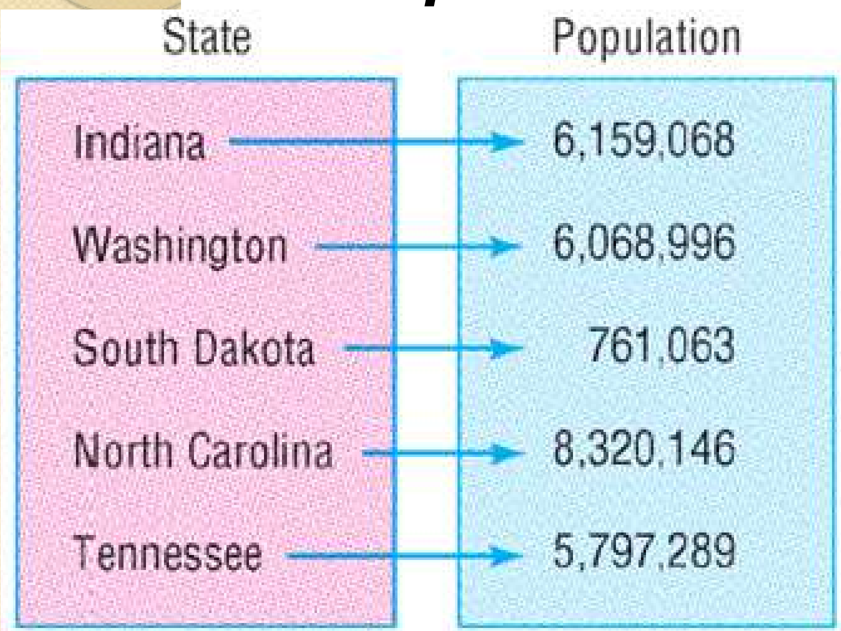
- If (x, y) is a point in f, then (y, x) is point in its inverse, f^{-1}.
- Domain of f = Range of f^{-1} and

* Range of f = Domain of f^{-1}.
- For the inverse of a function f to itself be a function, f must be one-to-one.
- $f \circ f^{-1} = f^{-1} \circ f = x$

To verify they are inverses…
- The graph of f and the graph of f^{-1} are symmetric with respect to $y = x$.

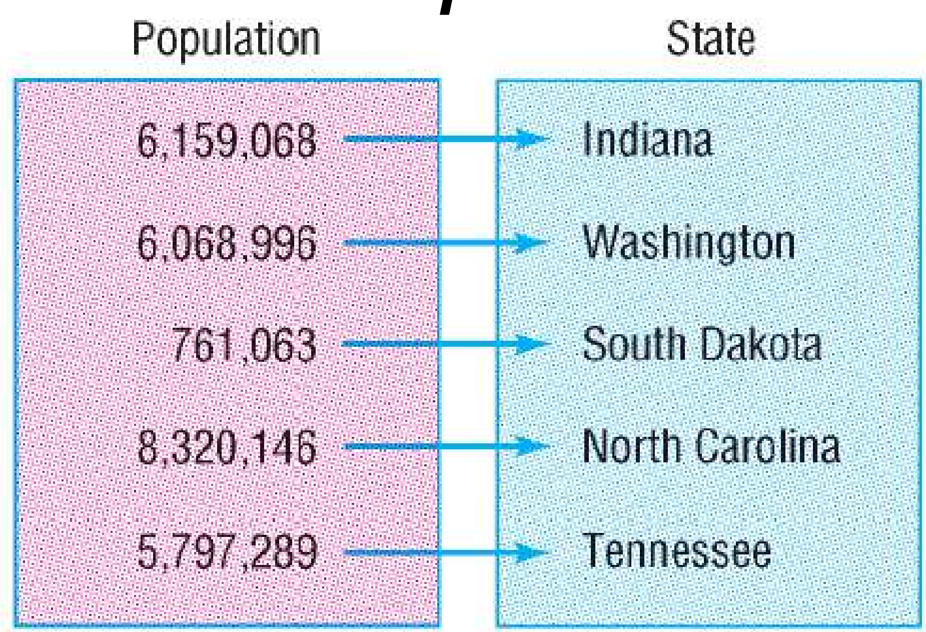
Both are functions, both are one-to-one…

f



Domain Range

f^{-1}

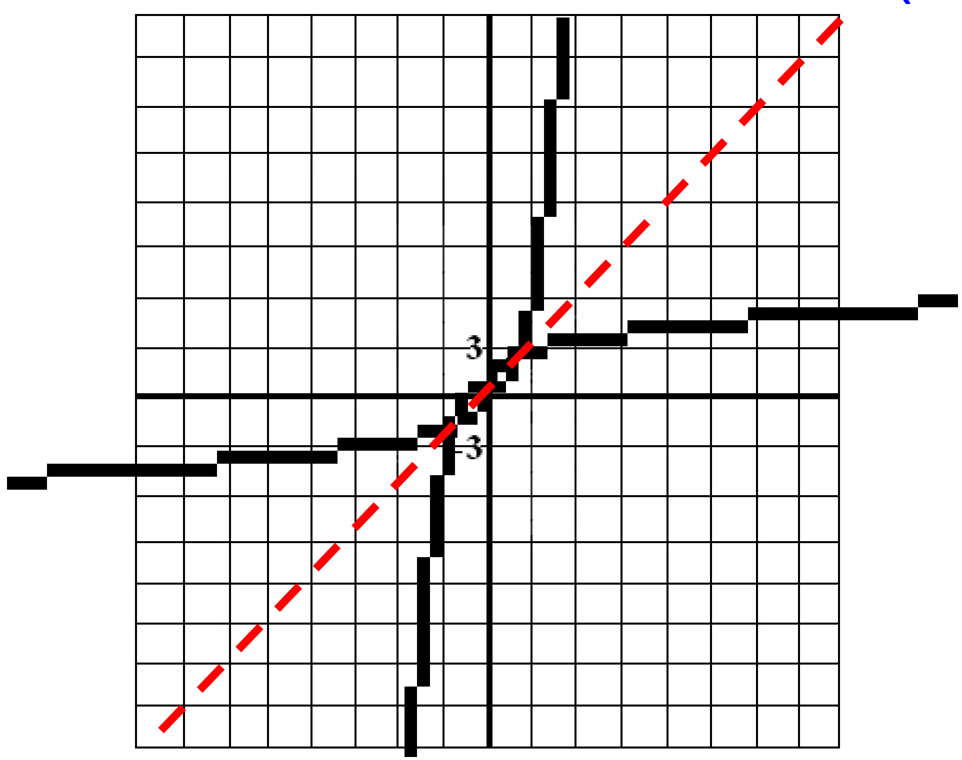


Domain Range

Example: Find the inverse of the following one-to-one function:

{(-3, -27), (-2, -8), (-1, -1), (0, 0), (1, 1), (2, 8), (3, 27)}

{(-27, -3), (-8, -2), (-1, -1), (0, 0), (1, 1), (8, 2), (27, 3)}



Example: Find the inverse of $f(x) = 2x + 3$. Identify the domain and range of f and f^{-1}, then graph both on the same coordinate axes.

1. Switch x and y:

$$y = 2x + 3$$

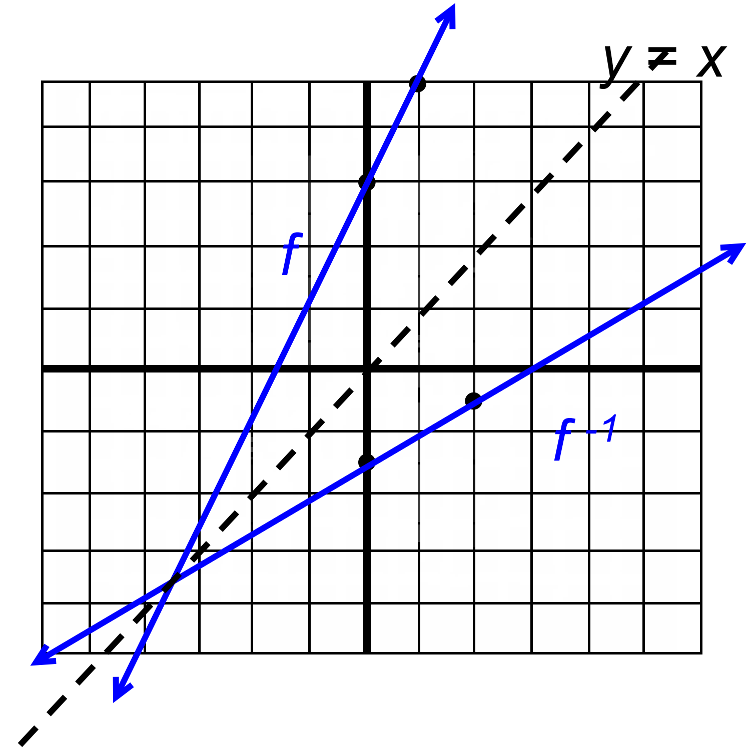
$$x = 2y + \cancel{3}$$

2. Solve for y:

$$\begin{array}{r} x = 2y + \cancel{3} \\ -3 \qquad\quad -\cancel{3} \\ \hline x - 3 = 2y \end{array}$$

$$\frac{\cancel{2}y}{\cancel{2}} = \frac{x}{2} - \frac{3}{2}$$

$$\boxed{f^{-1} = \frac{1}{2}x - \frac{3}{2}}$$



Domain: $\Re$

Range: $\Re$

To verify inverse functions, prove that

$$f \circ f^{-1} = f^{-1} \circ f = x$$

Prove that the functions in the above example are inverses:

$$f(x) = 2x + 3$$

$$f^{-1}(x) = \tfrac{1}{2}x - \tfrac{3}{2}$$

$$f \circ f^{-1} = f(f^{-1}) = f\left(\tfrac{1}{2}x - \tfrac{3}{2}\right) = 2\left(\tfrac{1}{2}x - \tfrac{3}{2}\right) + 3 = (x \cancel{- 3) + 3} = x$$

$$f^{-1} \circ f = f^{-1}(f) = f^{-1}(2x+3) = \tfrac{1}{2}(2x+3) - \tfrac{3}{2} = \left(x \cancel{+ \tfrac{3}{2}\right) - \tfrac{3}{2}} = x$$

$\therefore$ *f* and f^{-1} are inverses

Example: Verify that the inverse of $g(x) = \dfrac{1}{x-1}$ is $g^{-1}(x) = \dfrac{1}{x} + 1$

First show that $g \circ g^{-1} = x$

$$g \circ g^{-1} = g(g^{-1}) = g\left(1/x + 1\right) = \frac{1}{\left(\frac{1}{x} \cancel{+1}\right) \cancel{-1}} = \frac{1}{1/\cancel{x}} \bullet \frac{x}{\cancel{x}} = \frac{x}{1} = x$$

Now show that $g^{-1} \circ g = x$

$$g^{-1} \circ g = g^{-1}(g) = g^{-1}\left(\frac{1}{x-1}\right) = \frac{1}{\left(\frac{1}{x-1}\right)} + 1 = \frac{1}{\cancel{\left(\frac{1}{x-1}\right)}} * \frac{x-1}{\cancel{x-1}} + 1$$

$$= \frac{x-1}{1} + 1 = x \cancel{-1+1} = x$$

$\therefore$ *g* and *g* $^{-1}$ are inverses

Example: Find the inverse of $f(x) = \frac{2x+1}{x-1}$, $x \neq 1$. Verify your result and find the domain and range of both *f* and *f* -1.

First find f^{-1}

$$y = \frac{2x+1}{x-1}$$

switch *x* and *y*

$$(y-1) * x = \frac{2y+1}{\cancel{y-1} * (y-1)}$$

multiply both sides by *y* – 1

$$x(y-1) = 2y+1$$

distribute *x*

$$\begin{array}{rcl} xy - \cancel{x} & = & \cancel{2}y + 1 \\ -2y + \cancel{x} & & -2y \quad + x \\ \hline \end{array}$$

gather *y*'s on same side

$$xy - 2y = 1 + x$$

factor out *y*

$$\frac{y(\cancel{x-2})}{\cancel{x-2}} = \frac{1+x}{x-2}$$

divide by *x* – 2

$$\boxed{f^{-1}(x) = \frac{x+1}{x-2}}$$

Example: Find the inverse of $f(x) = \dfrac{2x+1}{x-1}$, $x \neq 1$

Verify your result and find the domain and range of both *f* and *f*$^{-1}$.

$$\boxed{f^{-1}(x) = \frac{x+1}{x-2}}$$

Can verify by graphing & check for symmetry

Now find the domain & range:

$$f(x) = \frac{2x+1}{x-1}$$

Domain: $x \neq 1$

Range: $y \neq 2$

If you know the domain of one, you know the range of the other…

$$f^{-1}(x) = \frac{x+1}{x-2}$$

Domain: $x \neq 2$

Range: $y \neq 1$

Summary

1. If a function f is one-to-one, then it has an inverse function f^{-1}.
2. Domain of f = Range of f^{-1}; Range of f = Domain of f^{-1}.
3. To verify that f^{-1} is the inverse of f, show that $f^{-1}(f(x)) = x$ for every x in the domain of f and $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}.
4. The graphs of f and f^{-1} are symmetric with respect to the line $y = x$.
5. To find the range of a one-to-one function f, find the domain of the inverse function f^{-1}.