Section 6.6

Work

All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Introduction

- In this section we will use our integration skills to study some basic principles of "work", which is one of the fundamental concepts in physics and engineering.
- A simple example:
  - When you push a car that has run out of gas for a certain distance you are performing work and the effect of your work is to make the car move.
  - The energy of motion caused by the work is called the kinetic energy of the car.
  - There is a principle of physics called the work-energy relationship. We will barely touch on this principle.
  - Our main goal (on later slides) is to see how integration is related.

6.6.1 DEFINITION If a constant force of magnitude F is applied in the direction of motion of an object, and if that object moves a distance d, then we define the *work* W performed by the force on the object to be

$$W = F \cdot d \qquad (1)$$

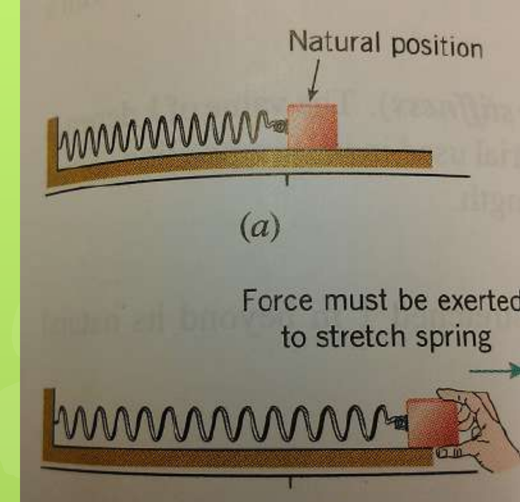
Examples of W = F * d

- An object moves five feet along a line while subjected to a constant force of 100 pounds in its direction of motion.
  - The work done is W = F * d = 100 * 5 = 500 ft-lbs

- An object moves 25 meters along a line while subjected to a constant force of four Newtons in its direction of motion.
  - The work done is W = F * d = 4*25 = 100 N-m

    =100 Joules

Work Done by a Variable Force Applied in the Direction of Motion



- If we wanted to pull the block attached to the spring horizontally, then we would have to apply more and more force to the block to overcome the increasing force of the stretching spring.

6.6.2 PROBLEM Suppose that an object moves in the positive direction along a coordinate line while subjected to a variable force $F(x)$ that is applied in the direction of motion. Define what is meant by the *work* W performed by the force on the object as the object moves from $x = a$ to $x = b$, and find a formula for computing the work.

- We will break the interval of the stretch [a,b] into subintervals and approximate the work on each subinterval.
- By adding the approximations to the work we will obtain a Riemann sum.
- The limit of the Riemann sum as n increases will give us an integral for work W.

Integral for Work, W (with a Variable Force)

6.6.3 DEFINITION Suppose that an object moves in the positive direction along a coordinate line over the interval $[a, b]$ while subjected to a variable force $F(x)$ that is applied in the direction of motion. Then we define the ***work*** W performed by the force on the object to be

$$W = \int_a^b F(x)\,dx \tag{2}$$

Example

▸ **Example 4** An astronaut's *weight* (or more precisely, *Earth weight*) is the force exerted on the astronaut by the Earth's gravity. As the astronaut moves upward into space, the gravitational pull of the Earth decreases, and hence so does his or her weight. If the Earth is assumed to be a sphere of radius 4000 mi, then it follows from Newton's Law of Universal Gravitation that an astronaut who weighs 150 lb on Earth will have a weight of

$$w(x) = \frac{2,400,000,000}{x^2} \text{ lb}, \quad x \geq 4000$$

at a distance of x miles from the Earth's center (Exercise 25). Use this formula to estimate the work in foot-pounds required to lift the astronaut 220 miles upward to the International Space Station.

Solution. Since the Earth has a radius of 4000 mi, the astronaut is lifted from a point that is 4000 mi from the Earth's center to a point that is 4220 mi from the Earth's center. Thus, from (2), the work W required to lift the astronaut is

$$\begin{aligned} W &= \int_{4000}^{4220} \frac{2,400,000,000}{x^2}\, dx \\ &= -\frac{2,400,000,000}{x}\bigg]_{4000}^{4220} \\ &\approx -568,720 + 600,000 \\ &= 31,280 \text{ mile-pounds} \\ &= (31,280 \text{ mi·lb}) \times (5280 \text{ ft/mi}) \\ &\approx 1.65 \times 10^8 \text{ ft·lb} \end{aligned}$$

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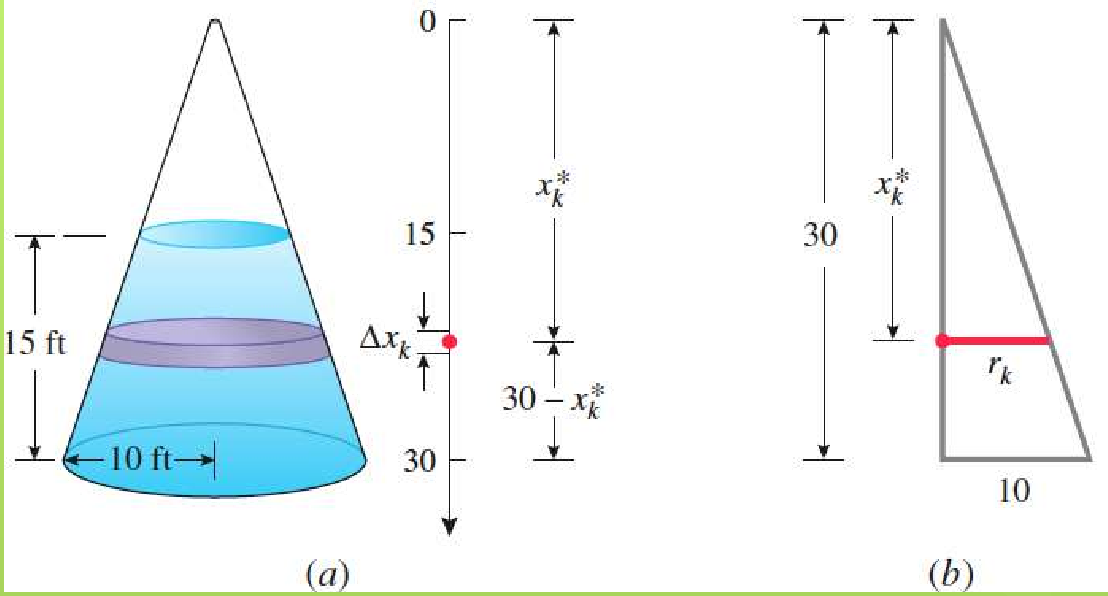
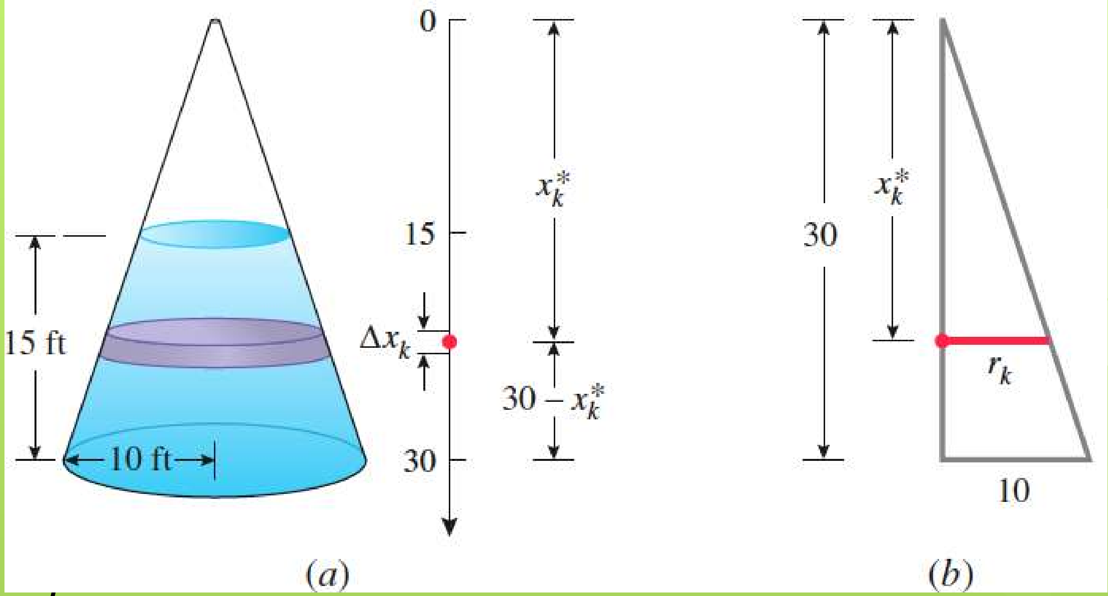
Calculating Work from Basic Principles Example



- As you can see in the figure above right, we have a cone shaped (conical) container of radius 10 ft and height 30 ft. Suppose that this container is filled with water to a depth of 15 ft. How much work is required to pump all of the water out through the hole in the top of the container?
  - We will divide the water into thin layers, approximate the work required to move each layer to the top of the container and add them to obtain a Riemann sum.
  - The limit of the Riemann sum produces an integral for the total work.

Calculating Work from Basic Principles Example con't



- The volume of a cone is $\frac{1}{3}\ {}^{2}\mathrm{h}$
- By similar triangles the ratio of radius to height $\frac{}{h} = \frac{10}{30} = \frac{1}{3}$
- When these are combined with the weight density of water which is 62.4 pounds per cubic foot, we get the following Riemann sum and integral:

$$W = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \frac{62.4\pi}{9}(x_k^*)^3 \Delta x_k = \int_{15}^{30} \frac{62.4\pi}{9} x^3\, dx$$

$$= \frac{62.4\pi}{9}\left(\frac{x^4}{4}\right)\Bigg]_{15}^{30} = 1{,}316{,}250\pi \approx 4{,}135{,}000 \text{ ft·lb}$$

6.6.4 NEWTON'S SECOND LAW OF MOTION If an object with mass m is subjected to a force F, then the object undergoes an acceleration a that satisfies the equation

$$F = ma \tag{5}$$

Chalk Art on the Street