Chapter 18

Confidence Intervals for Proportions

Chapter 18

Homework

p488 1, 3, 5, 7, 11, 13, 15, 18, 20, 27, 29, 31, 33, 35

Chapter 18

Objective

Students determine confidence intervals for a proportion from a simple random sample.

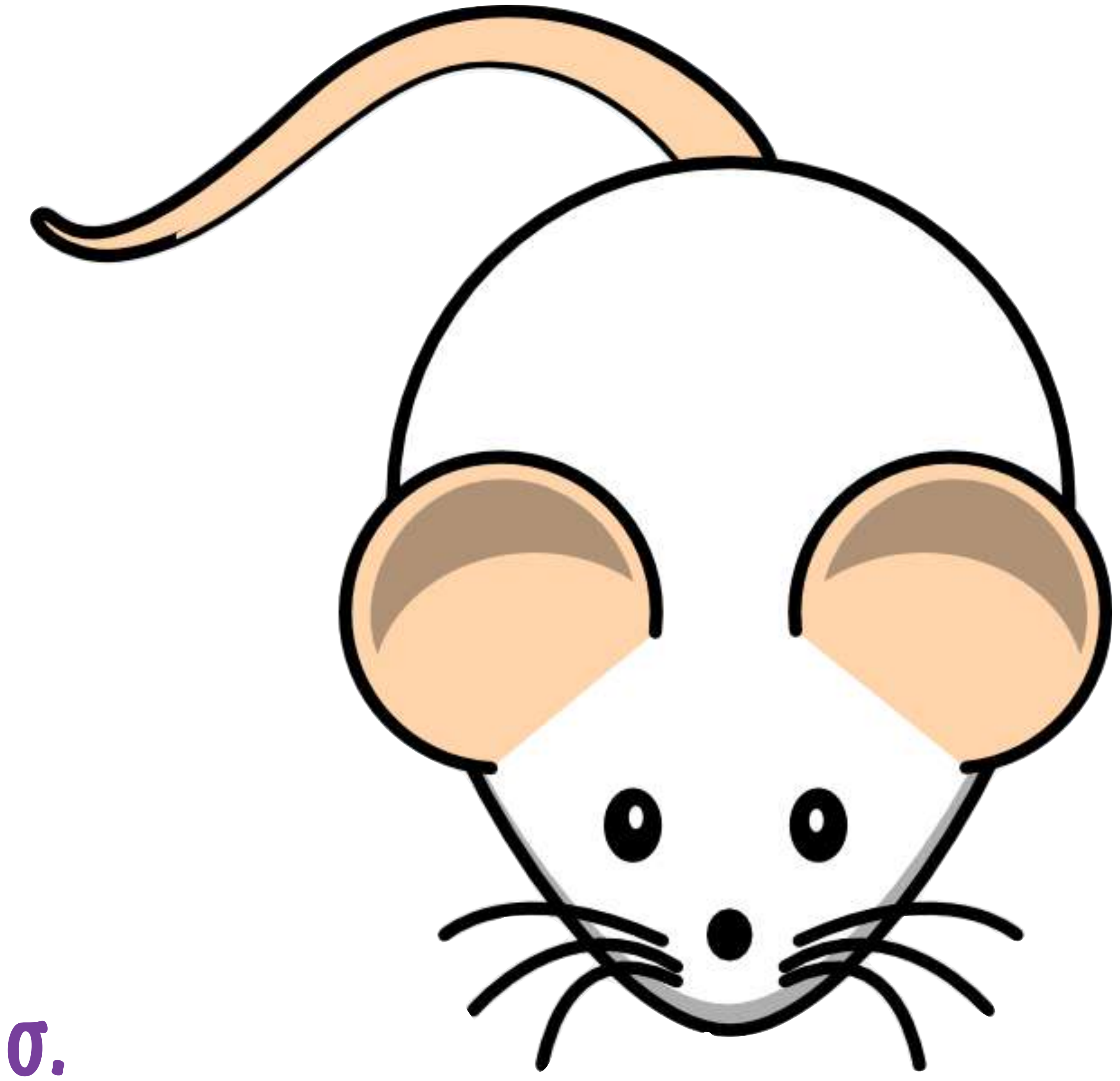
Standard Error

Remember

- Both of the **sampling distributions** (mean and proportion) we have examined have been Normal with standard deviations (standard error) of:

- For proportions $$SD(\hat{P}) = \sqrt{\frac{pq}{n}}$$

- For means $$SD(\bar{Y}) = \frac{\sigma}{\sqrt{n}}$$



- These formulae use the **parameters** p, q, and σ.

Standard Error

- So, when we don't know p or σ, we're stuck, right?

  - Nosiree bob.

- We will use sample statistics to estimate these population parameters.

  - Whenever we estimate the standard deviation of a sampling distribution, we call it a **standard error**. Your book, and AP use a slightly different definition. The standard error involves using the sample statistic (s or $\hat{p}$) to calculate standard error.

Standard Error

Chpt 19 Objective: Students determine confidence intervals for a proportion from a simple random sample.

- Whenever we **estimate** the standard deviation of a sampling distribution, we call it a **standard error**.

- For a sample **proportion**, the standard error is

$$SE(\hat{p}) = \sqrt{\frac{\hat{p}\hat{q}}{n}}$$

- For the sample **mean**, the standard error is

$$SE(\bar{y}) = \frac{s}{\sqrt{n}}$$

- Note the use of the sample statistics, p̂, q̂, or s in place of the parameters p, q, or σ.

A Confidence Interval

- Recall that the **sampling distribution** model of $\hat{p}$ is centered at p, with standard deviation $\sqrt{\frac{pq}{n}}$.
  - We use an estimate of that value **to create an interval**, within which we **expect** to find the population parameter.

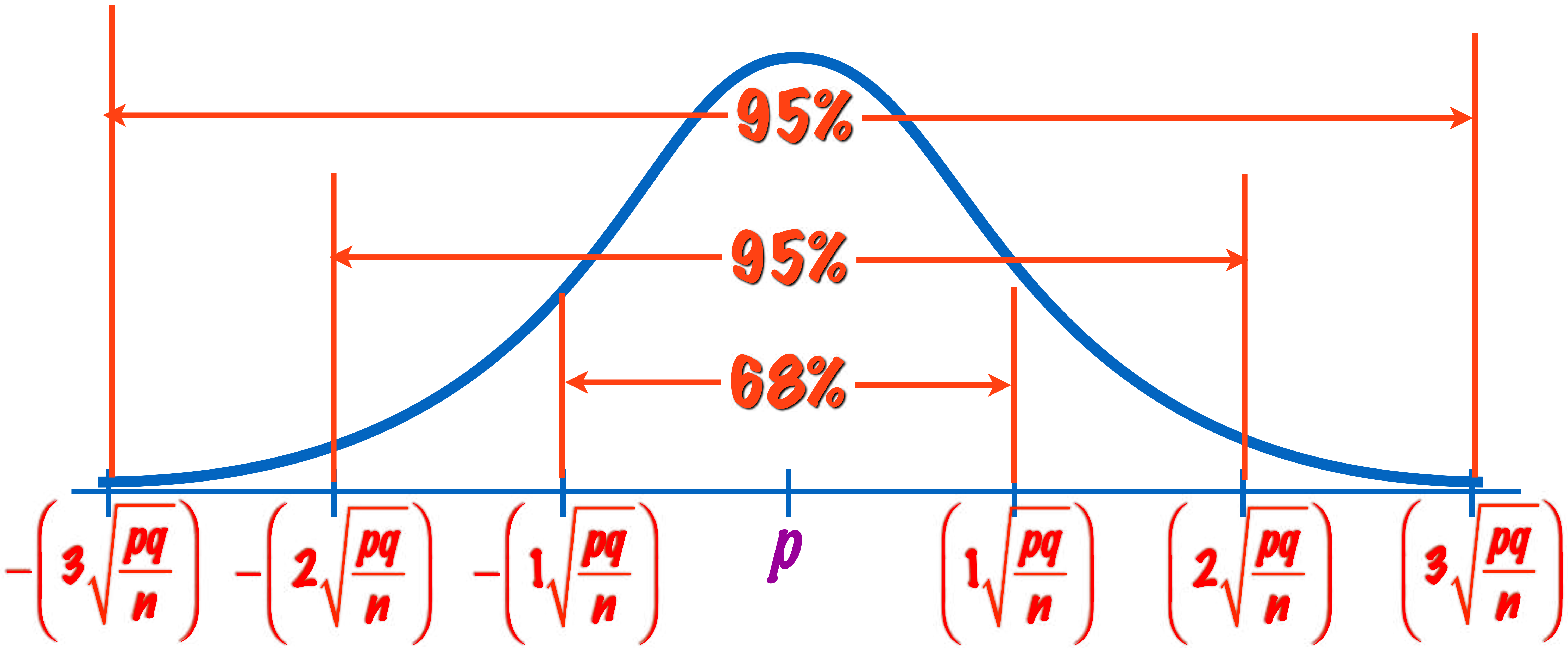
$$SE(\hat{p}) = \sqrt{\frac{\hat{p}\hat{q}}{n}}$$

A Confidence Interval

- From the 68-95-99.7% Rule, we know

$$N\left(p, \sqrt{\frac{pq}{n}}\right)$$

- about 68% of all samples will have $\hat{p}$'s within 1 SE of p
- about 95% of all samples will have $\hat{p}$'s within 2 SEs of p
- about 99.7% of all samples will have $\hat{p}$'s within 3 SEs of p



- This is the graph of the distribution of sample proportions (sampling distribution).

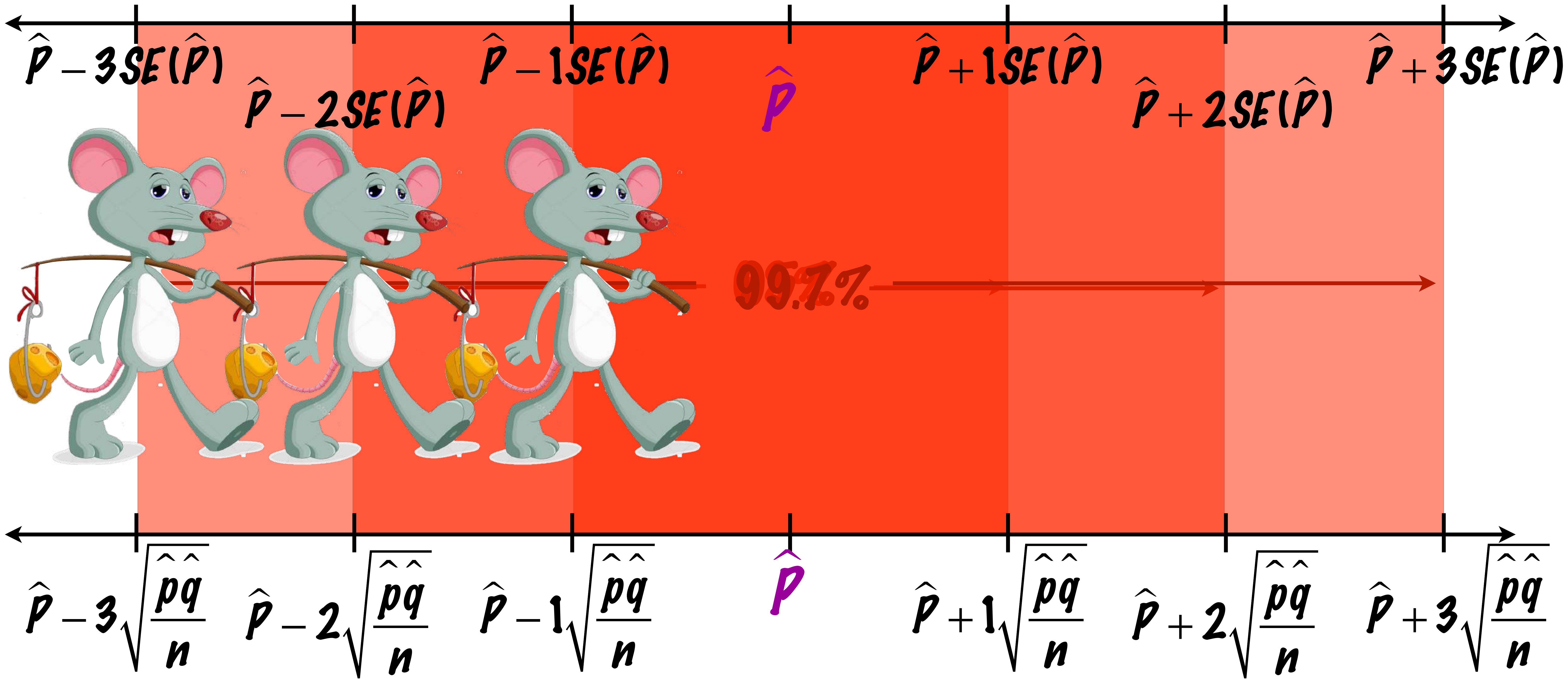
- Now let us turn things around and look at the situation from $\hat{p}$'s point of view…
  - In other words, we know $\hat{p}$, so we stand at $\hat{p}$ and search for p
  - Consider the 95% level:
  - We look at that interval from where $\hat{p}$ is located.
  - There's a 95% chance that p is no more than about 2 SEs away from $\hat{p}$.
  - So, if we reach out 2 SEs, we are 95% sure that p will be in that interval. In other words, if we reach out 2 SEs in either direction of $\hat{p}$, we can be 95% confident that this interval contains the population proportion, p.
  - The interval that is approximately two standard errors wide, centered at $\hat{p}$ is called a…

95% confidence interval

A Confidence Interval

Chpt 19 Objective: Students determine confidence intervals for a proportion from a simple random sample.

- 68 - 95 - 99.7 Confidence intervals



- Remember a sample **statistic** is simply an point **estimate** for the true population **parameter**.

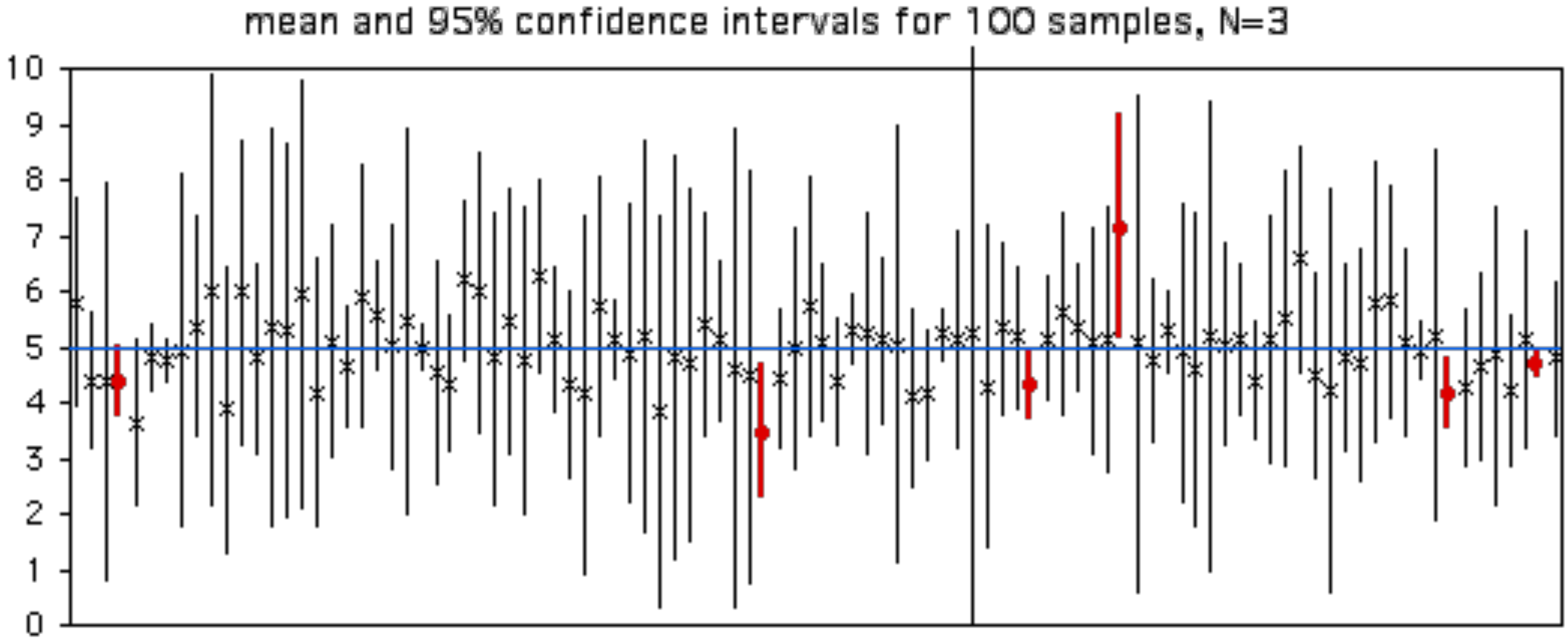
- Each confidence interval uses a sample **statistic** to **estimate** an interval within which we hope to find the true population **parameter**.

- Remember that samples vary. Therefore, the statistics we find from those samples will vary. That means the confidence intervals we construct from those sample statistics will also vary.

- Each sample will determine a different confidence interval. The confidence interval created is from **that specific sample** and will likely to be very different than an interval developed from another sample.

Different sample = Different Interval

- The figure below shows that some confidence intervals (from 100 random samples) capture the true proportion (the blue horizontal line), while others (red) do not capture the true proportion:
  - Each vertical line is a confidence interval built around the proportion from that sample (the x).



Interpreting a Confidence Interval

Chpt 19 Objective: Students determine confidence intervals for a proportion from a simple random sample.

- The confidence value refers to the **process** of **constructing** the interval, not in any one interval itself.
  - When we say we are 95% confident, we are saying that the **methodology** we used will, in fact, capture the true population parameter 95% of the time.
  - The confidence interval itself, the one created from our sample, either captures the population parameter or it does not.
  - What we are claiming is that our **methodology** will capture the true value 95% of the time.
  - **What we are NOT saying** is that "there is a 95% chance our interval contains the true population value.
  - We are saying that 95% of all the 95% confidence intervals we could create would contain the true parameter that the statistics are estimating.

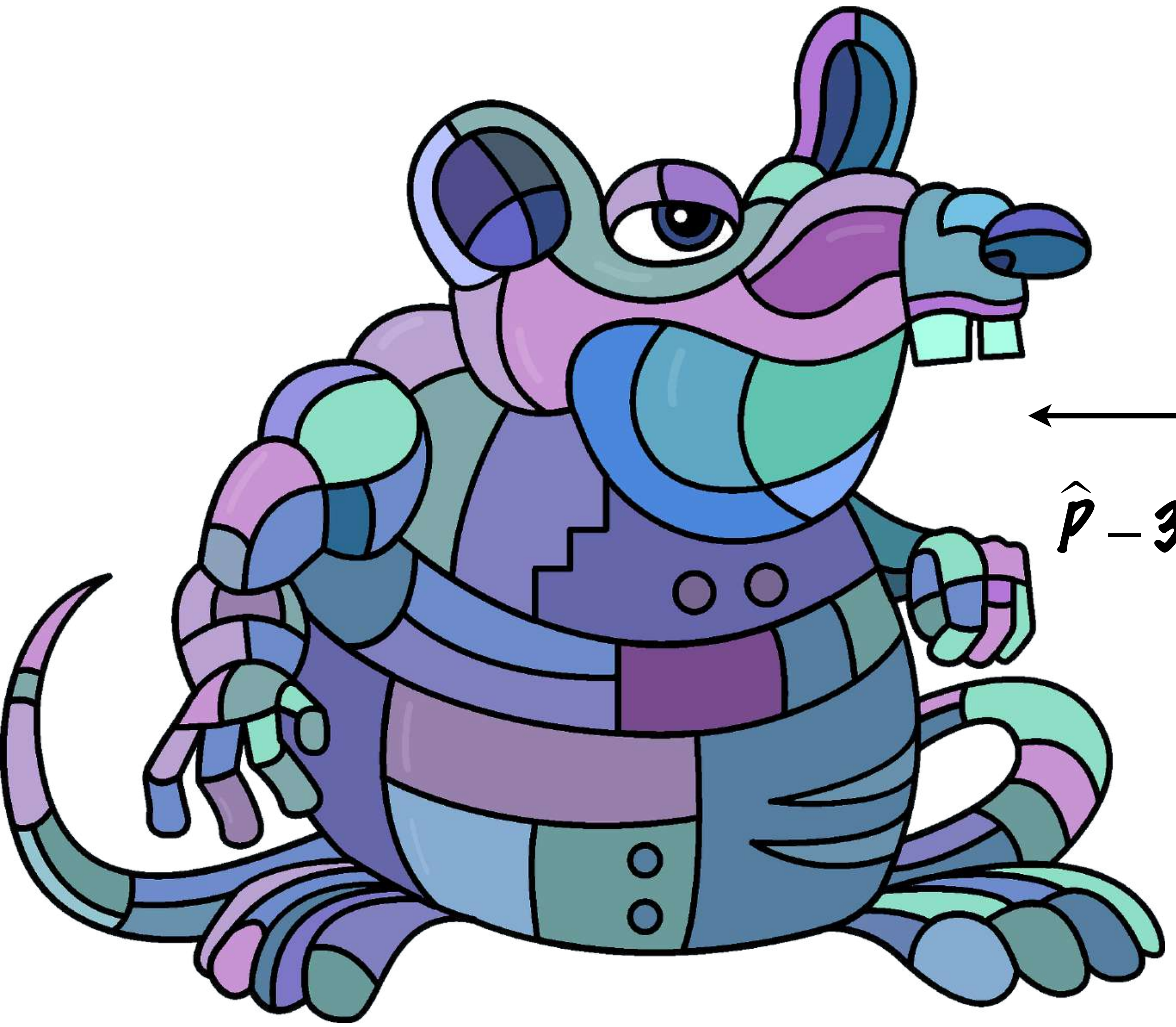
ME: Certainty vs. Precision

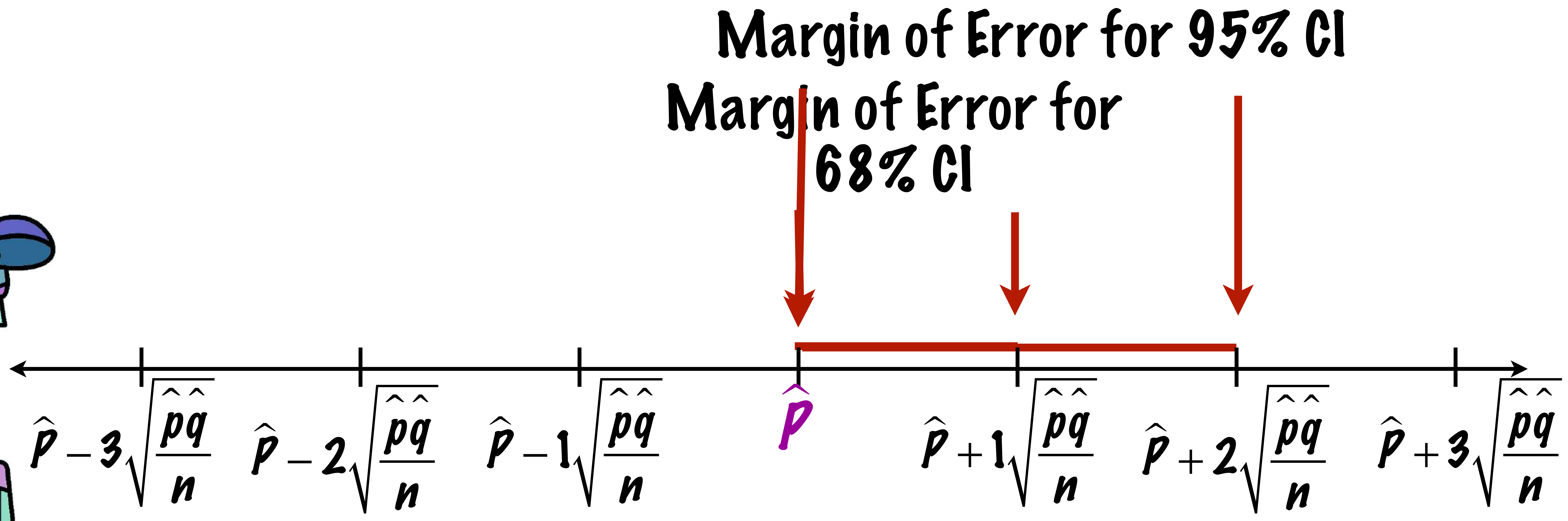
- We can claim, with 95% confidence, that the interval $\hat{p} \pm 2SE(\hat{p})$ contains the true population proportion

- The extent of the interval on either side of $\hat{p}$ is called the **margin of error (ME).**
  - Confidence intervals typically have the form **statistic ± ME.**
  - The more confident we want to be, the larger our **margin of error (ME).** needs to be, making the interval wider.
  - The wider the interval (greater margin of error) the greater the confidence, the narrower the interval (greater precision) the lower our confidence.

Certainty vs. Precision

Chpt 19 Objective: Students determine confidence intervals for a proportion from a simple random sample.





ME: Certainty vs. Precision

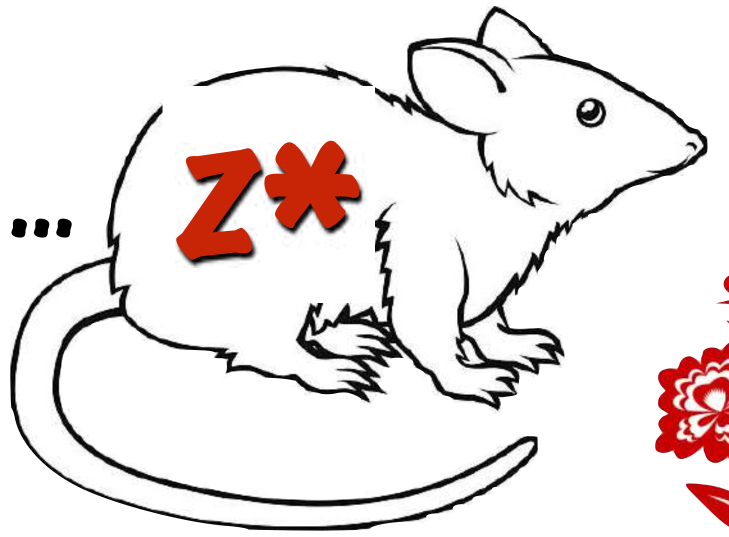
Chpt 19 Objective: Students determine confidence intervals for a proportion from a simple random sample.

- To be more confident, we wind up being less precise.
- Therefore you must decide the balance between confidence and accuracy.
- Fortunately, in most cases we can manage to be both **sufficiently** certain and **sufficiently** precise to make useful statements.
- The most commonly chosen confidence levels are **90%**, **95%**, and **99%** (but any percentage can be used).

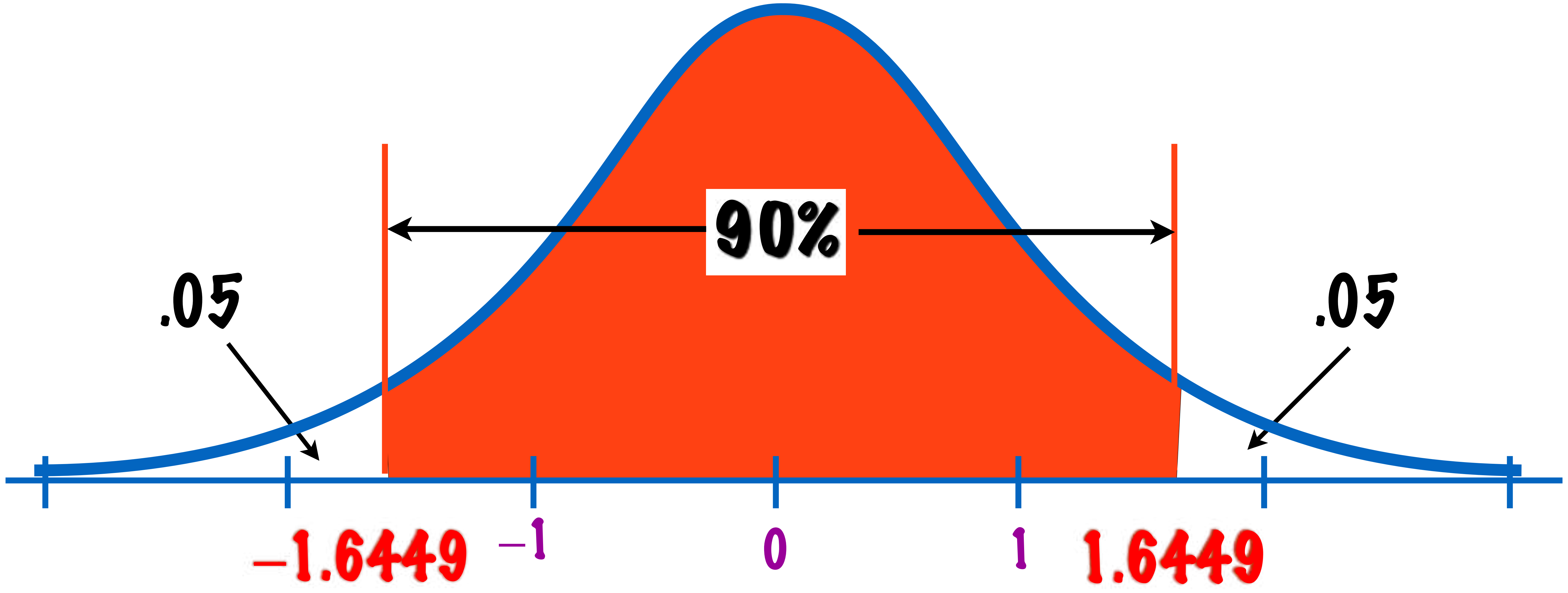
Critical Values

Chpt 19 Objective: Students determine confidence intervals for a proportion from a simple random sample.

- Up to this point in constructing confidence intervals we have been cheating a little bit.
- The '2' in $\hat{p} \pm 2SE(\hat{p})$ in our 95% confidence interval) came from the 68-95-99.7% Rule.
  - But $\pm 2SE$ is not exactly 95% of the population, it is closer to 95.45% of the distribution
  - Using the TI-84, find a more exact value for 95% confidence interval.
    - (InvNorm(.975, 0, 1) = 1.959963986
- The value 1.96 is called the **critical value of the test statistic** and is denoted ... z^*
  - You will become intimately familiar with the **critical value 1.96**

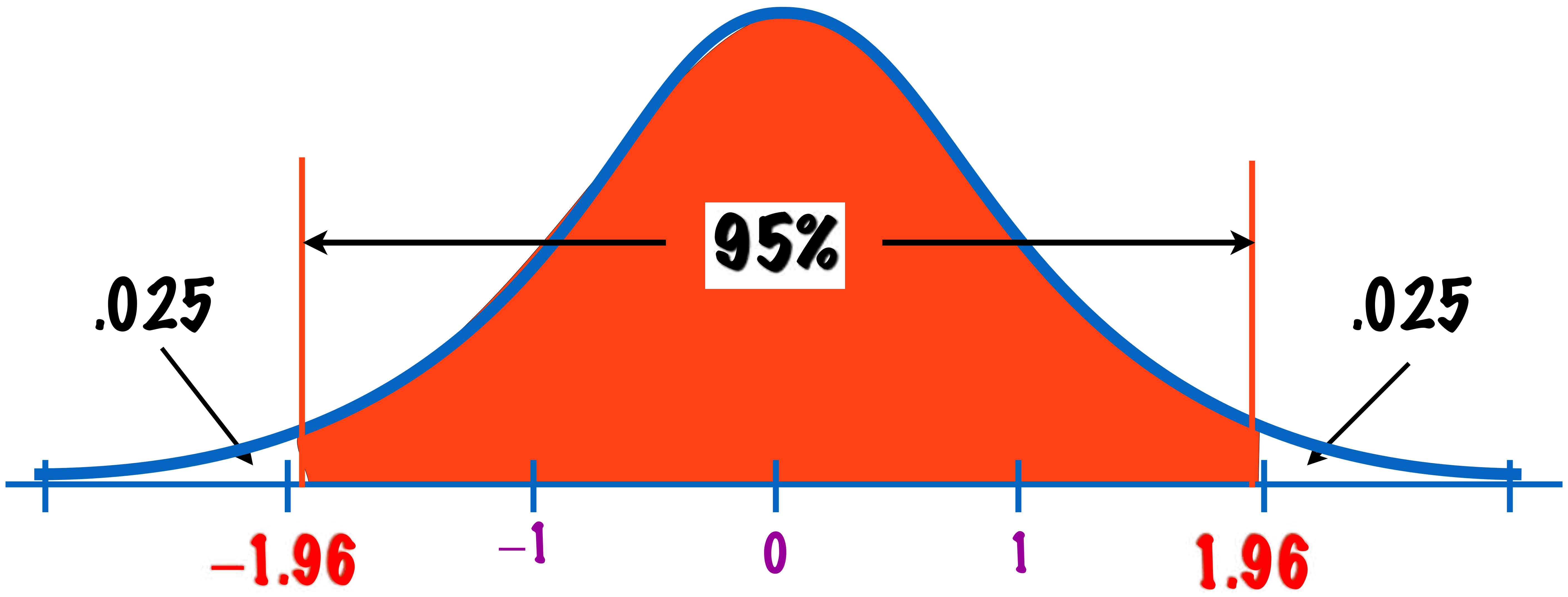


- Example: For a **90%** confidence interval, the **critical value** is **1.6449**:
  - (InvNorm(.95, 0, 1) =1.644853626



Critical Value of Test Statistic

- For 95% confidence interval, the **critical value** is **1.96**:
  - (InvNorm(.975, 0, 1) =1.959963986



Assumptions and Conditions

Chpt 19 Objective: Students determine confidence intervals for a proportion from a simple random sample.

- As before, our statistical models are based upon **assumptions and conditions.**

- Different models require different assumptions.
- If those assumptions are not true, the model is most likely inappropriate and our conclusions based on that model are most likely garbage.

- You can never be sure that an assumption is true, but as alway, we decide whether an assumption is plausible by checking a related **condition**.

Assumptions and Conditions

Chpt 19 Objective: Students determine confidence intervals for a proportion from a simple random sample.

- Here are the assumptions and the corresponding conditions **you must acknowledge** before creating a confidence interval for a proportion:

  - **Independence Assumption:** Do not just blithely assume independence. You first need to ask yourself if independence is reasonable. The data will **not** tell you anything about independence. So we check two conditions to decide whether independence is reasonable.

  - **Randomization Condition:** Were the data sampled at random? A representative sample collected randomly makes independence likely.

  - **10% Condition:** Is the sample size no more than 10% of the population?

Assumptions and Conditions

Chpt 19 Objective: Students determine confidence intervals for a proportion from a simple random sample.

- **Sample Size Assumption:** The sample must be large enough for us to use the Central Limit Theorem. A large sample obviates the need for the original distribution to be unimodal and symmetric.

  obviate: to anticipate and prevent (as a situation) or make unnecessary (as an action) - Merriam Webster Dictionary

  - **Success/Failure Condition:** To ensure the sample is sufficiently large for estimating a **proportion**, we must expect at least 10 "successes" and at least 10 "failures." (**np & $nq \geq 10$**)

- When the conditions are met, we are ready to find the confidence interval for the population proportion, **p**.

  - The confidence interval is

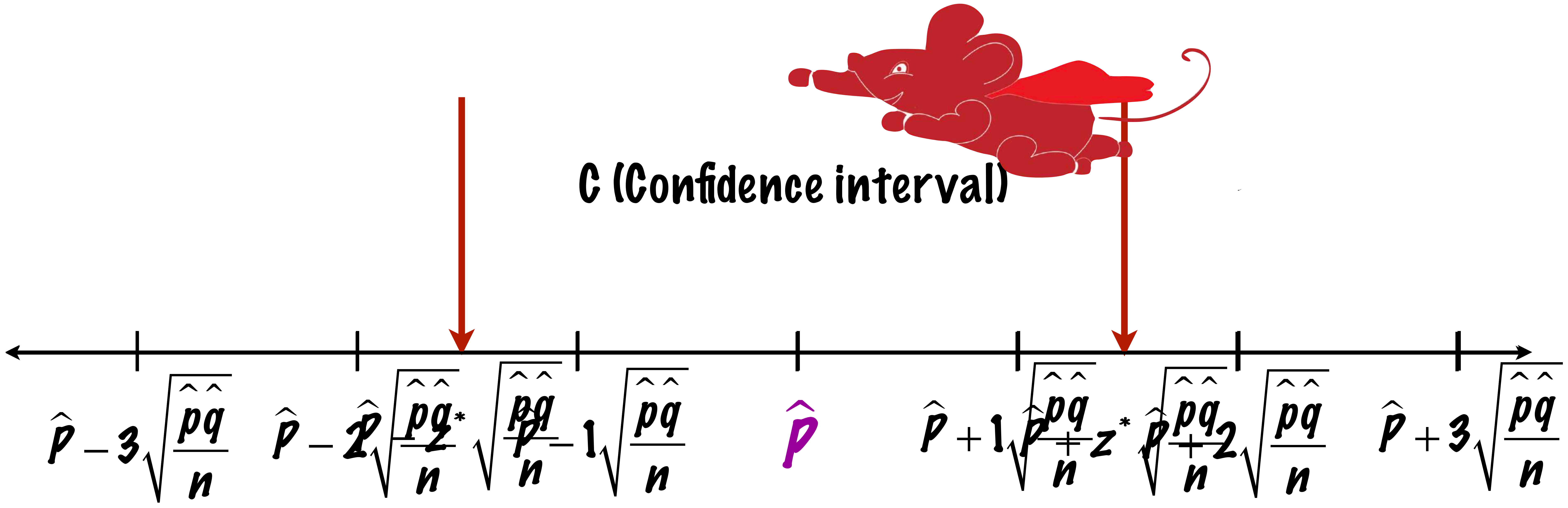
$$\hat{p} \pm z^{*}\left(SE(\hat{p})\right)$$

$$SE(\hat{p}) = \sqrt{\frac{\hat{p}\hat{q}}{n}}$$

$$ME = z^{*}\sqrt{\frac{\hat{p}\hat{q}}{n}}$$

One-Proportion z-Interval

- The critical value, z^*, is determined by the particular confidence level, C, that you specify. Rather than 1 or 2 or 3 standard deviations, we first decide **Confidence** (say 95%), then find the number of standard deviations that will result in that interval.



First Example

- A sample of 200 people indicate that 42% wash their hair every day. Find a 99% confidence interval for the population proportion.
  - For the moment we will neglect the assumptions.
    - z^* = InvNorm (.995, 0, 1) = 2.575829303

99% CI $\quad \hat{p} \pm z_{\alpha/2} \dfrac{\sqrt{\hat{p}\hat{q}}}{\sqrt{n}} = .42 \pm 2.5758 \dfrac{\sqrt{.42 \bullet .58}}{\sqrt{200}} \approx .42 \pm .0899 \quad .3301 < p < 0.5099$

Based on a sample of 200 people, we are 99% confident the proportion of the population who wash their hair every day is between 33% and 51%.

- Start with the calculator

Note: x must be a whole number

STAT ➢ TESTS A:1-PropZInterval

x: 84
n: 200
C-Level: .99
Calculate

(.3301, .5099)
p^=.42
n: 200

Choosing Your Sample Size

Chpt 19 Objective: Students determine confidence intervals for a proportion from a simple random sample.

- The determination of how large a sample we need to take is an important, and mostly misunderstood, step in planning any study.
  - To determine an appropriate sample size we must first choose how accurate we want to be (Margin of Error (ME)) and how accurate we want to be (Confidence Interval Level (C)).
  - We can simply use the formula $ME = z^* \sqrt{\frac{\hat{p}\hat{q}}{n}}$, fill in the values we know and solve for n.

  - Or, if you must, solve the formula for n. $n = \frac{\left(z^*\right)^2 \left(\hat{p}\hat{q}\right)}{\left(ME^2\right)}$

- The formula requires $\hat{p}$ which we do not yet have since we have not taken the sample. A good estimate for $\hat{p}$, which will yield the largest value for $\hat{p}\hat{q}$ (and therefore for n) is 0.50.

Choosing Your Sample Size

- In general, the sample size needed to produce a confidence interval with a given margin of error at a given confidence level is:

$$ME = z^* \sqrt{\frac{\hat{p}\hat{q}}{n}} \qquad n = \frac{(z^*)^2(\hat{p}\hat{q})}{(ME^2)} \qquad n = \frac{(z^*)^2(.25)}{(ME^2)}$$

  - where z* is the critical value for your confidence level.
    - z* = InvNorm (Area, 0, 1)

- Remember samples consist of a discrete number of data values (observations) and cannot be decimal values, **round up the sample size you obtain.**

PANIC: The Method for Confidence Intervals

- I ain't big on mnemonic devices but the process for finding confidence intervals is to "PANIC".

- P: Parameter of interest - define it
- A: Assumptions/conditions
- N: Name the interval
- I: Interval (confidence)
- C: Conclude in context

P: Parameters

- Parameter: the statistical values of a population (represented by a Greek letter)
- Define in the first step of confidence interval
  - p = the proportion of people voting yes for proposition 4A.
  - μ= the true mean summer weight for NFL players

A: Assumptions/Conditions

- Random Sampling Condition:
- Independence Condition:
- 10% Condition:
- Success/Failure Condition:

N: Name Interval

- Currently we are finding z-interval for a proportion.
- Soon we will find other intervals for means and proportions

I: Interval

- Calculate the interval $\hat{p} \pm z^* \left(SE(\hat{p}) \right)$

C: Conclude in Context

- Based on the data from our sample of size n , we are C % confidence the true value of the population proportion is between __$\hat{p} - z^* \left(SE(\hat{p}) \right)$__ and __$\hat{p} + z^* \left(SE(\hat{p}) \right)$__.

Sentence Frame

- When stating the conclusion for a confidence interval, use the following sentence structure.

Sample size n

Confidence Level

Based on the data from our sample of size ___ , we are ____ % confident the true value of the population _______ is between _____ and _____.

Parameter & Variable

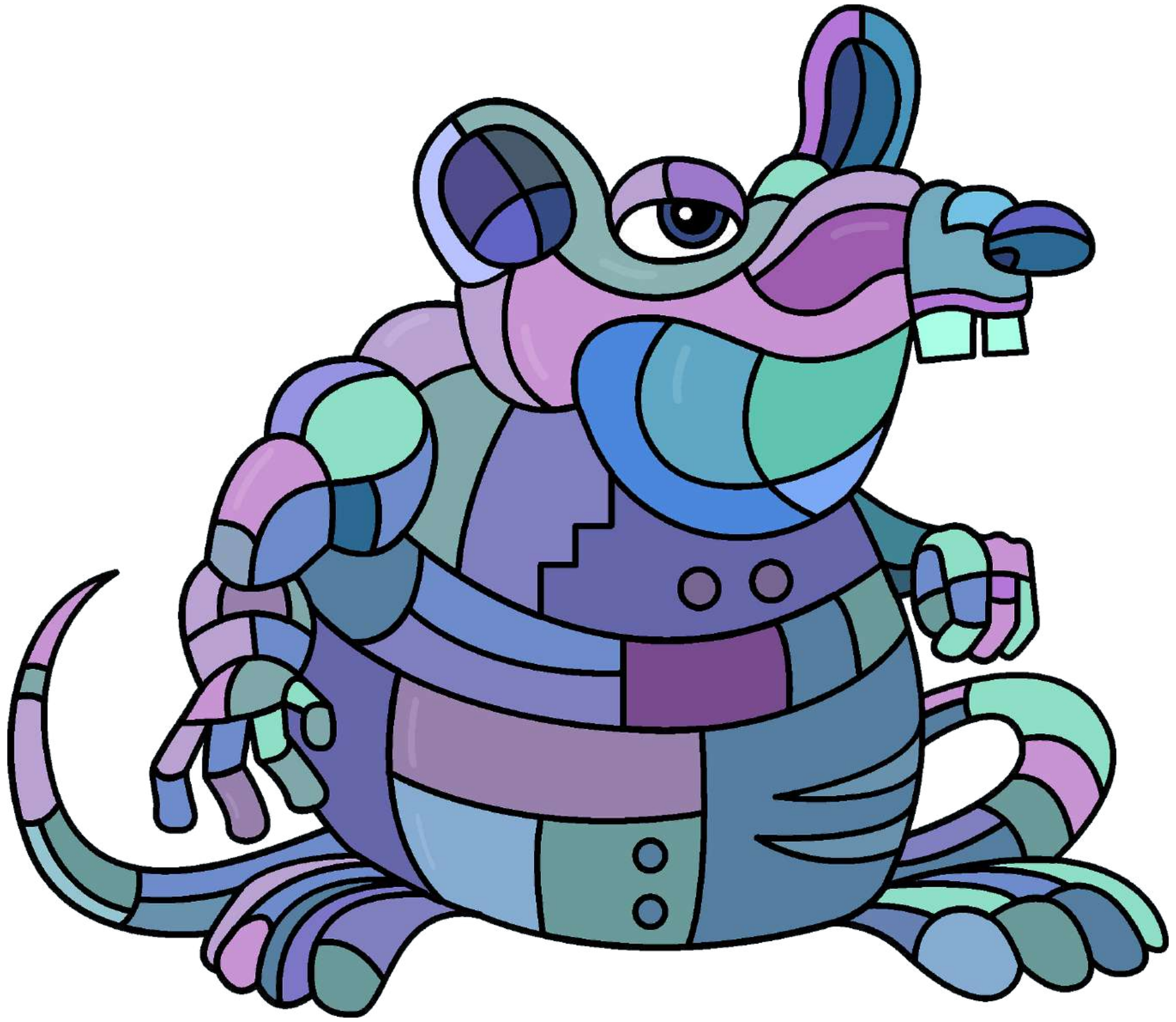
Lower Boundary

Upper Boundary

What Can Go Wrong?

Chpt 19 Objective: Students determine confidence intervals for a proportion from a simple random sample.

- **Know what we are actually doing!**
  - The parameter does not vary. The parameter is a fixed value we are estimating.
  - One sample is, ONE SAMPLE. Other samples are equally valid and will return different intervals.
  - We rarely truly know the parameter, do not state any certainty.
  - Remember: It's about the parameter (not the statistic).



- Stick to your finding, (acknowledge the uncertainty), but remember that you chose the confidence.
- Treat the whole interval equally. Any values that fall within the interval are equally valid.

What Can Go Wrong?

Chpt 19 Objective: Students determine confidence intervals for a proportion from a simple random sample.

- **Margin of Error Too Large to Be Useful:**
  - We cannot be exact, but how precise do we need to be?
  - If you make the margin of error smaller without changing sample size the result is to reduce your level of confidence. (That may not be a useful solution.)
  - You need to think about your margin of error when you plan your study.

  To get a narrower interval without giving up confidence, you need to have less variability.

- What do you suppose might increase accuracy (narrower interval) without sacrificing confidence?
  - Increase sample size.

Example

- Your local newspaper polls a random sample of 330 voters, finding 144 who say they will vote "yes" on the upcoming school budget. Create a 95% confidence interval for actual sentiment of all voters. Concentrate on the conditions and the interpretation.

330 voters, finding 144, 95% CI

P: Parameters

- We want a 95% confidence interval for the proportion of all voters in the population voting "yes" on the upcoming school budget.

- Stat - Tests - Scroll up to A:1-PropZInt (x, n, C)

STAT ➢ TESTS A:1-PropZInterval

x: 144
n: 330
C-Level: .95
Calculate

- Enter x - number of successes. This must be a whole number. If you have been given $\hat{p}$, calculate x = $n\hat{p}$, then round to the next integer.
- Enter n - number of trials.
- Enter C-Level (confidence level) - C level must be in decimal form (95% = .95)
- The calculator will report back with the interval in interval notation, $\hat{p}$, and n.

Example

A: Assumptions/Conditions

330 voters, finding 144, 95% CI

- **Independence Assumption**: It is reasonable to think that the responses were mutually independent.
- **Random Sampling Condition**: The problem stated voters were sampled randomly.
- **10% Condition**: Assuming there are more than 3300 eligible voters in our population, 330 is less than 10% of the population of voters.
- **Success/Failure Condition**: $n\hat{p} = 144$ and $n\hat{q} = 186$, which are both at least 10, so the sample is large enough.

Example

N: Name Interval

330 voters, finding 144, 95% CI

The conditions are satisfied, so I can use the Normal model N(4364, .0273) to find a one proportion z-interval with 95% confidence.

I: Find the Interval

- n = 330 $\hat{p}$ = 144/330 ≈ 04364
- $SE(\hat{p}) = \sqrt{\frac{\hat{p}\hat{q}}{n}} = \sqrt{\frac{(.4364)(.5636)}{330}} = .0273$

N(4364, .0273)

- z* = InvNorm (.975, 0, 1) = 1.96
- The margin of error (ME) = (z*) SE(p^) ≈ 1.96(0.0273) ≈ 0.0535
- The confidence interval is 04364 ± 0.0535 or (0.3829, 04899)

Example

I: Find the Interval TI-84

330 voters, finding 144, 95% CI

- Stat - Tests - A:1-PropZInt (144, 330, .95)
- (.38286, .48987) → $\hat{p}$ = 0.4363636364

C: Conclude in Context

- Based on a sample of 330 voters, we are **95% confident** that the proportion of voters voting yes on the budget is between 38.3% and 49.0%.

Another Example

- An experiment finds that 27% of 53 subjects report improvement after using a new medicine. Create a 95% confidence interval for the actual rate of improvement.

27% of 53 = 14.31, 95% CI

- We will find a 95% confidence interval for the proportion of all people who will improve after using a new medication.
  - **Independence Condition**: One patient response to the medication should not have an influence on other patients responses to the medication.
  - **Random Sampling Condition**: The patients were part of an experiment, which hopefully included **random assignment** of volunteers to treatment groups..
  - **10% Condition**: In this instance 10% is not relevant. We are testing the medication and we certainly can expect independence of treatment results.
  - **Success/Failure Condition:** $n\hat{p}$ = 14.31 and $n\hat{q}$ = 38.69 , which are both at least 10, so the sample size is sufficient.

Another Example

- An experiment finds that 27% of 53 subjects report improvement after using a new medicine. Create a 95% confidence interval for the actual rate of improvement.

27% of 53 = 14.31, 95% CI

N(.27, .0610)

- $SE(\hat{P}) = \sqrt{\frac{\hat{p}\hat{q}}{n}} = \sqrt{\frac{(.27)(.73)}{53}} = .0610$
- Invnorm(.975, 0, 1) = 1.96
- The margin of error (ME) = (z*) SE(p̂) ≈ 1.96(.061) ≈ 0.1195
- The confidence interval is **0.27 ± 0.1195** or **(0.1505, 0.3895)**
- Keep in mind the TI needs whole numbers for the number of successes.
  - Stat - Tests - A:1-PropZInt (**15**, 53, .95) = **(0.16174, 040429)**
  - Stat - Tests - A:1-PropZInt (**14**, 53, .95) = **(0.14546, 0.38285)**
- I would use 14 as it is closer to 14.3 and is the more conservative interval. I will accept either one as an acceptable response.

- **Based on a sample of 53 subjects, we are 95% confident that the proportion of patients improving is between 15.0% and 39.0%.**

Another Example

- 15% - 39% is a pretty wide interval. If we want to be 95% confident of capturing the true (population) value we need to use a wide net. A sample of 53 is too small to create a narrow confidence interval. Perhaps a 90% interval will be better.

27% of 53 = 14.31, 95% CI

N(.27, .0610)

- To find the 90% confidence interval, the only thing changing is the critical value, z^*.
  - Invnorm(.95, 0, 1) = 1.6449
  - The margin of error (ME) = $(z^*)\,SE(\hat{p}) \approx 1.6449(.061) \approx 0.1003$
  - The confidence interval is 0.27 ± 0.1003 or (0.17, 0.37)
  - We are 90% confident that between 17.0% and 37.0% of participants will improve after using the medication.
  - That did not help much. 17% - 37% is still a pretty wide interval. If we want to be 90% confident of capturing the true (population) value we can use not so wide a net. But a sample of 53 is still too small to allow small intervals. So .....
  - What shall we do? Hmmmmmm?

Sample Size

- 15% - 39% is too wide an interval to be truly useful but it does indicate the treatment is potentially useful.
  - What sample size would you suggest in a follow-up study if we want a margin of error of 5% with 98% confidence?

$$ME = z^* \sqrt{\frac{\hat{p}\hat{q}}{n}}$$

- z^* = InvNorm (.99, 0, 1) = 2.326347877

$$.05 = 2.3263\sqrt{\frac{(.27)(.73)}{n}} \qquad \sqrt{n} = \frac{2.3263}{.05}\sqrt{(.27)(.73)} \qquad n = \left(\frac{2.3263}{.05}\right)^2 \left((.27)(.73)\right) = 426.6562$$

- **To be 98% confident with a margin of error of 5% we would need 427 subjects.**

Another Example

- If we did not want to run a pilot study and thus did not have the proportion .27 from our sample but we wished to determining the sample size necessary for a margin of error of .05; we could use .5 for both $\hat{p}$ and $\hat{q}$.

$$ME = z^* \sqrt{\frac{\hat{p}\hat{q}}{n}}$$

- z^* = InvNorm (.99, 0, 1) = 2.326347877

$$.05 = 2.3263\sqrt{\frac{(.5)(.5)}{n}} \qquad n = \frac{2.3263^2(.25)}{.05^2} \qquad n = 23.263^2 = 541.16$$

- **To be 98% confident with a margin of error of 5% we would need 542 subjects.**

- Note that using .5 results in a more conservative (wider) interval.

Another Example

- A study of 1792 adult men found that 1548 were free of cardiovascular disease (CVD) and 244 presented with cardiovascular disease. Estimate the prevalence of CVD among adult men.
- We will find a 95% confidence interval for the **proportion** of adult men with cardovascular disease.
  - **Independence Condition:** Is is certainly reasonable to believe CVD among men is independent.
  - **Random Sampling Condition:** it does not state that the men were **randomly chosen** but we will assume random selection.
  - **10% Condition:** 1792 men is certainly less than 10% of the men in the U.S.
  - **Success/Failure Condition:** $n\hat{p} = 244$ and $n\hat{q} = 1548$, which are both at least 10, so the sample size is sufficient.

    - $\hat{p} = 244/1792 = .1362$ and $\hat{q} = 1548/1792 = .8638$

- A study of 1792 adult men found that 1548 were free of cardiovascular disease (CVD) and 244 presented with cardiovascular disease. Estimate the prevalence of CVD among adult men.
- We will construct a **95% confidence z interval for the proportion of the population** that present CVD.
  - $\hat{p} = 244/1792 = .1362$ and $\hat{q} = 1548/1792 = .8638$
  - $SE(\hat{P}) = \sqrt{\frac{\hat{p}\hat{q}}{n}} = \sqrt{\frac{(.1362)(.8638)}{1792}} = .0081$
    - Invnorm(.975, 0, 1) = 1.96
    - The margin of error (ME) = $(z^*)\, SE(\hat{p}) \approx 1.96(.0081) \approx 0.0159$
    - The confidence interval is **0.1362 ± 0.0159** or **(0.1203, 0.1521)**

N(.1362, .0081)

- Keep in mind the TI needs whole numbers (not the proportion) for the **number** of successes.
  - Stat - Tests - A:1-PropZInt (**244**, 1792, .95) = **(0.12028, 0.15204)**
- **Based on a sample of 1792 men, we are 95% confident that the proportion of men presenting with CVD is between 12% and 15.2%.**