7.8

Applications of Matrices and Determinants

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What You Should Learn

- Use determinants to find areas of triangles.
- Use determinants to decide whether points are collinear.
- Use Cramer’s Rule to solve systems of linear equations.
- Use matrices to encode and decode messages.

Area of a Triangle

Area of a Triangle

In this section, you will study some additional applications of matrices and determinants.

The first involves a formula for finding the area of a triangle whose vertices are given by three points on a rectangular coordinate system.

Area of a Triangle

The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

where the symbol $(\pm)$ indicates that the appropriate sign should be chosen to yield a positive area.

Example 1 – *Finding the Area of a Triangle*

Find the area of the triangle whose vertices are (1, 0), (2, 2), and (4, 3) as shown in Figure 7.28.

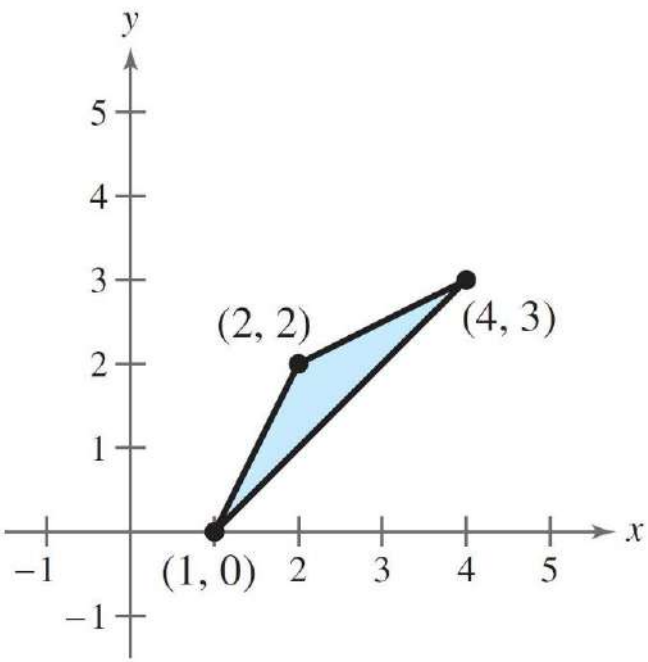

Figure 7.28

Example 1 – *Solution*

Begin by letting $(x_1, y_1) = (1, 0)$,

$(x_2, y_2) = (2, 2)$, and

$(x_3, y_3) = (4, 3)$.

Then, to find the area of the triangle, evaluate the determinant.

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= 1(-1)^2 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix})^3 \quad + 1 \ (\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} \qquad \begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix}$$

Example 1 – *Solution*

cont'd

= 1(–1) + 0 + 1(–2)

= –3

Using this value, you can conclude that the area of the triangle is

Area $$= -\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

(–3) $$= -\frac{1}{2}$$

square units. $$= \frac{3}{2}$$

Collinear Points

Collinear Points

What if the three points in Example 1 had been on the same line?

What would have happened had the area formula been applied to three such points? The answer is that the determinant would have been zero.

Consider, for instance, the three collinear points (0, 1), (2, 2) and (4, 3) as shown in Figure 7.29.

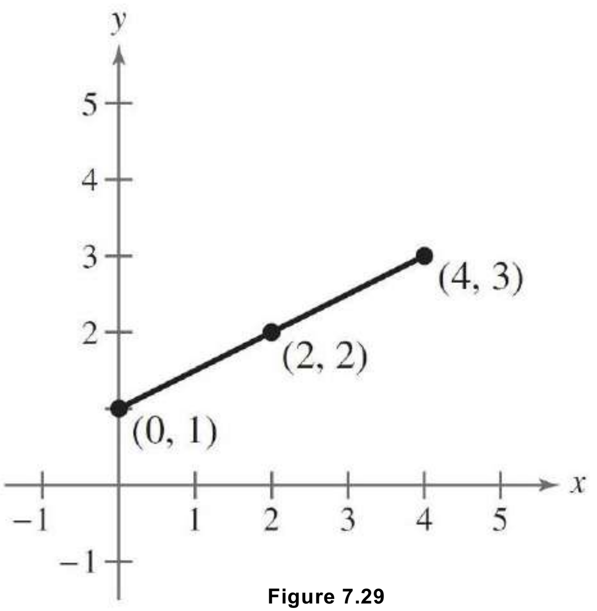

Figure 7.29

Example 2 – *Testing for Collinear Points*

Determine whether the points
(–2, –2), (1, 1), and (7, 5)
are collinear. (See Figure 7.30.)

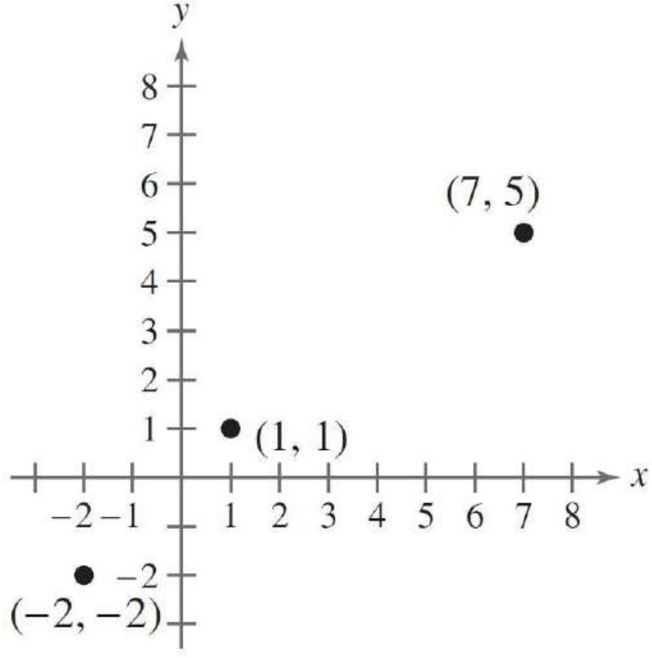

Figure 7.30

Example 2 – *Solution*

Begin by letting $(x_1, y_1) = (–2, –2)$,

$(x_2, y_2) = (1, 1)$, and

$(x_3, y_3) = (7, 5)$.

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} -2 & -2 & 1 \\ 1 & 1 & 1 \\ 7 & 5 & 1 \end{vmatrix}$$

$$= -2(-1)^2\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} + (-2)(-1)^3\begin{vmatrix} 1 & 1 \\ 7 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix}$$

Example 2 – *Solution*

cont'd

$= -2(-4) + 2(-6) + 1(-2)$

$= -6.$

Because the value of this determinant is *not* zero, you can conclude that the three points are not collinear.

Cramer's Rule

Cramer's Rule

So far, you have studied three methods for solving a system of linear equations: substitution, elimination with equations, and elimination with matrices.

You will now study one more method, **Cramer's Rule,** named after Gabriel Cramer (1704–1752).

This rule uses determinants to write the solution of a system of linear equations.

Read slides #15&16 so that slide #17 makes sense, but do not copy them down.

Cramer's Rule

Cramer's rule states that the system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

has a solution

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$$

and

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

provided that

$a_1b_2 - a_2b_1 = 0$.

Cramer's Rule

Each numerator and denominator in this solution can be expressed as a determinant, as follows.

$$x = \frac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$y = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Cramer's Rule

Relative to the original system, the denominators of and and are simply the determinant of the *coefficient* matrix of the system.

This determinant is denoted by *D.*

The numerators of and are denoted by D_x and D_y, respectively.

Cramer's Rule

They are formed by using the column of constants as replacements for the coefficients of *x* and *y* as follows.

Coefficient Matrix	*D*	D_x	D_y
$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$	$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$	$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

Cramer's Rule

For example, given the system

$$\begin{cases} 2x - 5y = 3 \\ -4x + 3y = 8 \end{cases}$$

the coefficient matrix, D, D_x, and D_y are as follows.

Coefficient Matrix	D	D_x	D_y
$\begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix}$	$\begin{vmatrix} 2 & -5 \\ -4 & 3 \end{vmatrix}$	$\begin{vmatrix} 3 & -5 \\ 8 & 3 \end{vmatrix}$	$\begin{vmatrix} 2 & 3 \\ -4 & 8 \end{vmatrix}$

Cramer's Rule

Cramer's Rule

If a system of n linear equations in n variables has a coefficient matrix A with a *nonzero* determinant $|A|$, then the solution of the system is

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}, \quad \ldots, \quad x_n = \frac{|A_n|}{|A|}$$

where the ith column of A_i is the column of constants in the system of equations. If the determinant of the coefficient matrix is zero, then the system has either no solution or infinitely many solutions.

Example 3 – *Using Cramer's Rule for a 2 × 2 System*

Use Cramer's Rule to solve the system

$$\begin{cases} 4x - 2y = 10 \\ 3x - 5y = 11 \end{cases}$$

Solution:

To begin, find the determinant of the coefficient matrix.

$$D = \begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix} = -20 - (-6)$$

$$= -14$$

Example 3 – *Solution*

cont'd

Because this determinant is not zero, apply Cramer's Rule.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 10 & -2 \\ 11 & -5 \end{vmatrix}}{-14}$$

$$= \frac{(-50) - (-22)}{-14}$$

$$= \frac{-28}{-14}$$

$$= 2$$

Example 3 – *Solution*

cont'd

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 4 & 10 \\ 3 & 11 \end{vmatrix}}{-14}$$

$$= \frac{44 - 30}{-14}$$

$$= \frac{14}{-14}$$

$= –1$

So, the solution is $x = 2$ and $y = –1$.

Cryptography

Cryptography

A **cryptogram** is a message written according to a secret code. (The Greek word *kryptos* means “hidden.”)

Matrix multiplication can be used to encode and decode messages.

Cryptography

To begin, you need to assign a number to each letter in the alphabet (with 0 assigned to a blank space), as follows.

0 = _	9 = I	18 = R
1 = A	10 = J	19 = S
2 = B	11 = K	20 = T
3 = C	12 = L	21 = U
4 = D	13 = M	22 = V
5 = E	14 = N	23 = W
6 = F	15 = O	24 = X
7 = G	16 = P	25 = Y
8 = H	17 = Q	26 = Z

Cryptography

Then the message is converted to numbers and partitioned into **uncoded row matrices,** each having *n* entries, as demonstrated in Example 6.

Example 6 – *Forming Uncoded Row Matrices*

Write the uncoded row matrices of dimension 1 × 3 for the message

MEET ME MONDAY.

Solution:

Partitioning the message (including blank spaces, but ignoring punctuation) into groups of three produces the following uncoded row matrices.

$[13\quad 5\quad 5]$	$[20\quad 0\quad 13]$	$[5\quad 0\quad 13]$	$[15\quad 14\quad 4]$	$[1\quad 25\quad 0]$
M E E	T M	E M	O N D	A Y

Note that a blank space is used to fill out the last uncoded row matrix.

Cryptography

To encode a message, choose an *n* × *n* invertible matrix such as

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

and multiply the uncoded row matrices by *A* (on the right) to obtain **coded row matrices.** Here is an example.

Uncoded Matrix	*Encoding Matrix A*	*Coded Matrix*

$$\begin{bmatrix} 13 & 5 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -26 & 21 \end{bmatrix}$$