Section 6.1

Area Between Two Curves

All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Introduction

- In the last chapter, the definite integral was introduced as the limit of Riemann sums and we used them to find area:

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*) \Delta x$$

- However, Riemann sums and definite integrals have applications that extend far beyond the area problem.
- In this chapter, we will use Riemann sums and definite integrals to find volume and surface area of a solid, length of a plane curve, work done by a force, etc.
- While these problems sound different, the calculations we will use will all follow a nearly identical procedure.

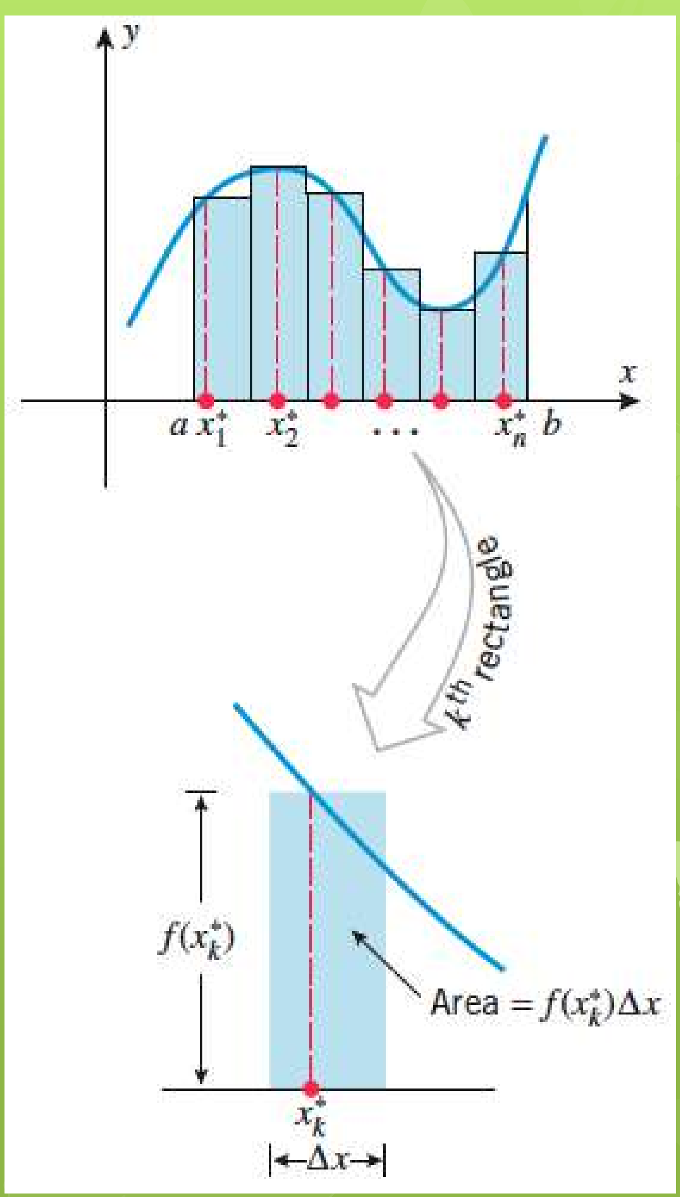
A Review of Riemann Sums

- The distance between a and b is b-a.
- Since we divided that distance into n subintervals, each is :

$$\Delta x = \frac{b-a}{n}$$

- In each subinterval, draw a rectangle whose height is the value of the function f(x) at an arbitrarily selected point in the subinterval (a.k.a. x_k*) which gives $f(x_k^*)$.
- Since the area of each rectangle is base * height, we get the formula you see on the right for each rectangle:
- Area=

$b*h = \Delta x * f(x_k^*) = f(x_k^*)\Delta x$.



Review of Riemann Sums - continued

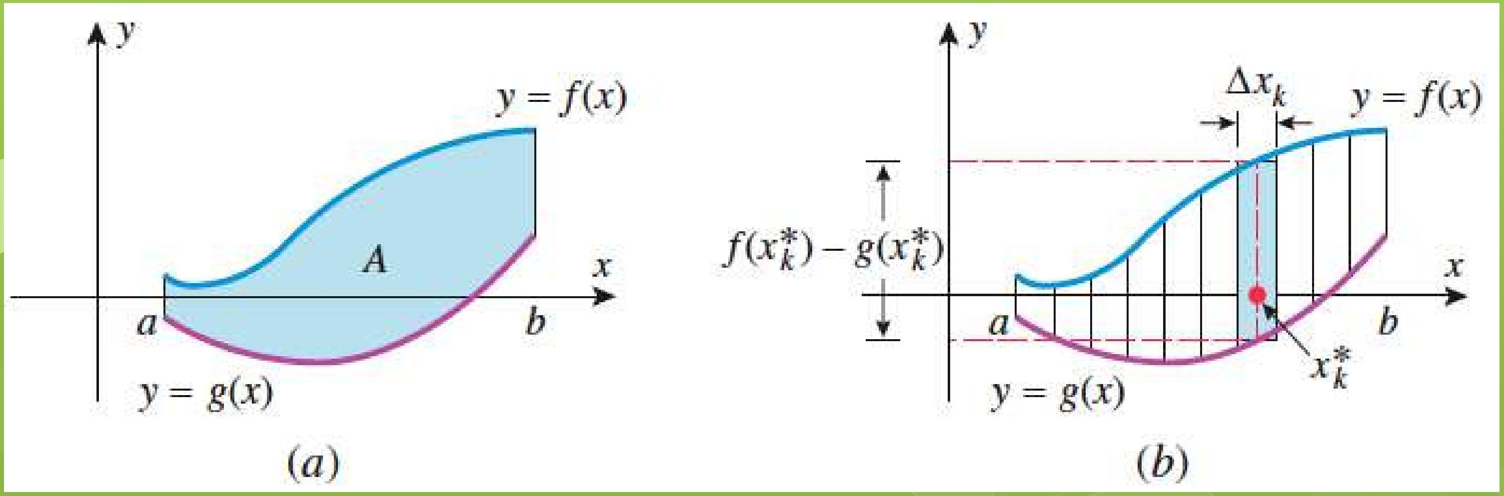
- Remember, that was the area for each rectangle. We need to find the sum of the areas of all of the rectangles between a and b which is why we use sigma notation.
- As we discussed in a previous section, the area estimate is more accurate with the more number of rectangles used. Therefore, we will let n approach infinity.

5.4.3 DEFINITION (*Area Under a Curve*) If the function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$, then the ***area*** A under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \qquad (2)$$

Area Between y = f(x) and y = g(x)

- To find the area between two curves, we will divide the interval [a,b] into n subintervals (like we did in section 5.4) which subdivides the area region into n strips (see diagram below).



Area Between y = f(x) and y = g(x) continued

- To find the height of each rectangle, subtract the function output values $f(x_k^*) - g(x_k^*)$. The base is Δx_k .
- Therefore, the area of each strip is base * height = Δx_k * $[f(x_k^*) - g(x_k^*)]$.
- We do not want the area of one strip, we want the sum of the areas of all of the strips. That is why we need the sigma.
- Also, we want the limit as the number of rectangles "n" increases to approach infinity, in order to get an accurate area.

- NOTE: insert slide here discussing Riemann sums and how to relate to the integral

Assuming One Curve is Always Above the Other

6.1.1 FIRST AREA PROBLEM Suppose that f and g are continuous functions on an interval $[a, b]$ and

$$f(x) \geq g(x) \quad \text{for} \quad a \leq x \leq b$$

[This means that the curve $y = f(x)$ lies above the curve $y = g(x)$ and that the two can touch but not cross.] Find the area A of the region bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.1.3a).

6.1.2 AREA FORMULA If f and g are continuous functions on the interval $[a, b]$, and if $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\, dx \qquad (1)$$

Summary of Steps Involved

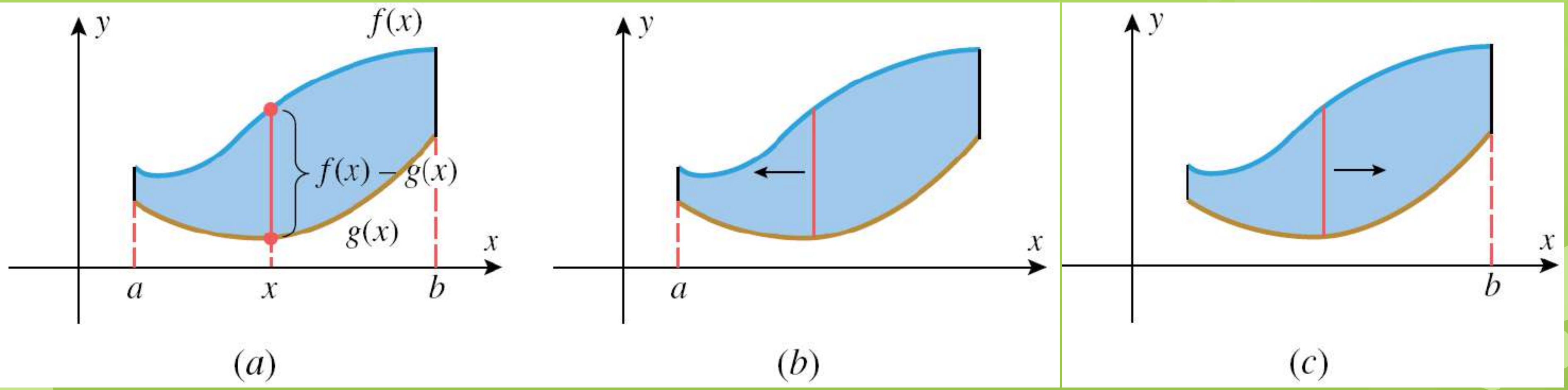
Finding the Limits of Integration for the Area Between Two Curves

Step 1. Sketch the region and then draw a vertical line segment through the region at an arbitrary point x on the x-axis, connecting the top and bottom boundaries (Figure 6.1.9a).

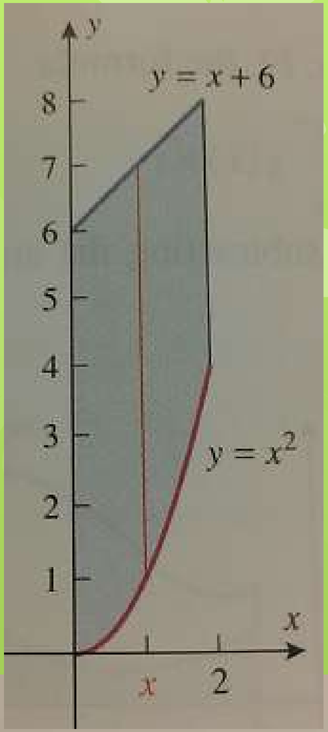
Step 2. The y-coordinate of the top endpoint of the line segment sketched in Step 1 will be $f(x)$, the bottom one $g(x)$, and the length of the line segment will be $f(x) - g(x)$. This is the integrand in (1).

Step 3. To determine the limits of integration, imagine moving the line segment left and then right. The leftmost position at which the line segment intersects the region is $x = a$ and the rightmost is $x = b$ (Figures 6.1.9b and 6.1.9c).

Picture of Steps Two and Three From Previous Slide:



Straightforward Example

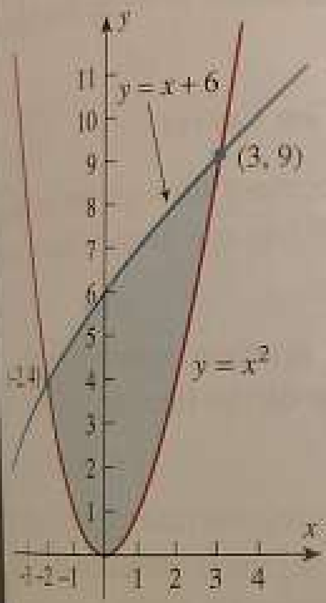


▶ **Example 1** Find the area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$, and bounded on the sides by the lines $x = 0$ and $x = 2$.

Solution. The region and a cross section are shown in Figure 6.1.4. The cross section extends from $g(x) = x^2$ on the bottom to $f(x) = x + 6$ on the top. If the cross section is moved through the region, then its leftmost position will be $x = 0$ and its rightmost position will be $x = 2$. Thus, from (1)

$$A = \int_0^2 [(x+6) - x^2]\,dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^2 = \frac{34}{3} - 0 = \frac{34}{3}$$ ◀

Sometimes, you will have to find the limits of integration by solving for the points of intersection first:



…re 6.1.6

Example 2 Find the area of the region that is enclosed between the curves $y = x^2$ and $y = x + 6$.

Solution. A sketch of the region (Figure 6.1.6) shows that the lower boundary is $y = x^2$ and the upper boundary is $y = x + 6$. At the endpoints of the region, the upper and lower boundaries have the same y-coordinates; thus, to find the endpoints we equate

$$y = x^2 \quad \text{and} \quad y = x + 6 \tag{2}$$

This yields

$$x^2 = x + 6 \quad \text{or} \quad x^2 - x - 6 = 0 \quad \text{or} \quad (x + 2)(x - 3) = 0$$

from which we obtain

$$x = -2 \quad \text{and} \quad x = 3$$

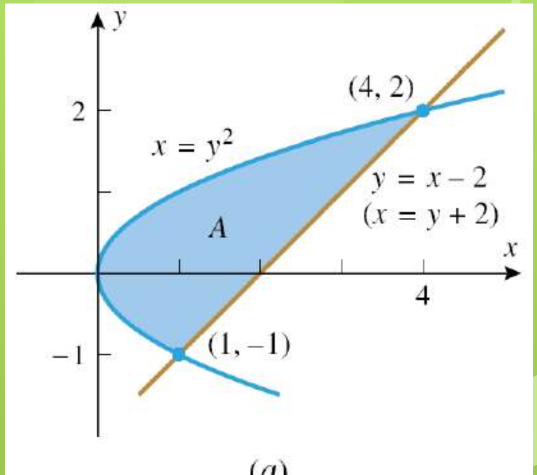
Although the y-coordinates of the endpoints are not essential to our solution, they may be obtained from (2) by substituting $x = -2$ and $x = 3$ in either equation. This yields $y = 4$ and $y = 9$, so the upper and lower boundaries intersect at $(-2, 4)$ and $(3, 9)$.

- Then solve for the area as we did in the previous example:

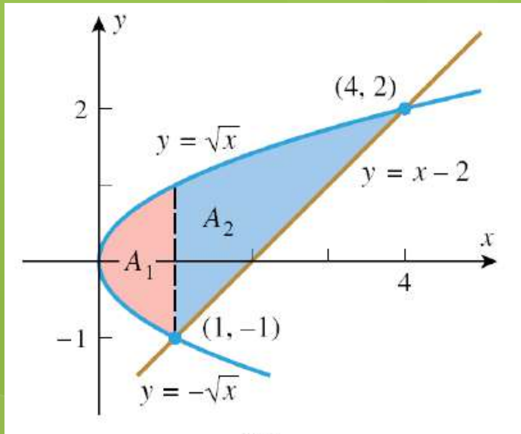
$$A = \int_{-2}^{3} [(x + 6) - x^2]\, dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3} = \frac{27}{2} - \left(-\frac{22}{3}\right) = \frac{125}{6}$$

Inconsistent Boundaries

- If you look at the area in figure (a), the upper and lower boundaries are not the same for the left portion of the graph as they are for the right portion.
- On the left, the $x = y^2$ curve is the upper and lower boundary.
- On the right, the $x = y^2$ curve is the upper boundary, but the line $y = x - 2$ is the lower boundary.
- Therefore, in order to calculate the area using x as our variable, we must divide the region into two pieces, find the area of each, then add those areas to find the total area (see figure (b)).
- See work on page 417 if interested.



(a)



(b)

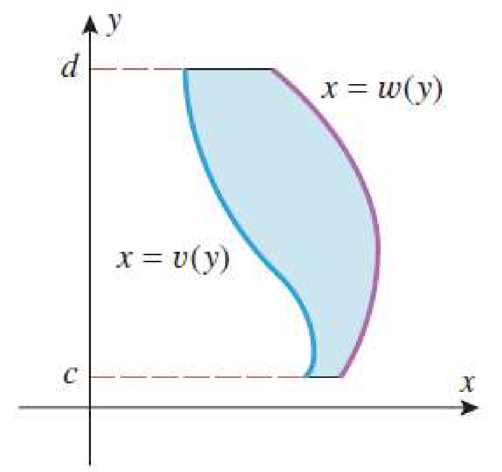
Reversing the Roles of x and y

- Instead, we could reverse the roles of x and y to make it easier to find the area.
- Solve for x in terms of y, find the lower and upper limits of integration in terms of y, and integrate with respect to y.

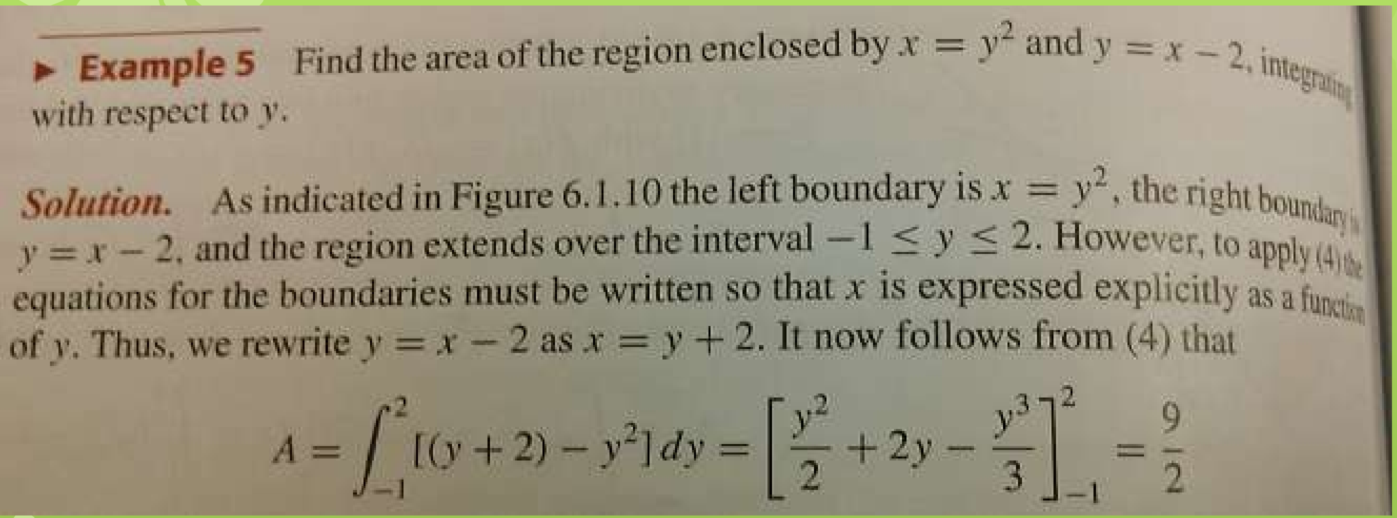
6.1.3 SECOND AREA PROBLEM Suppose that w and v are continuous functions of y on an interval $[c, d]$ and that

$$w(y) \geq v(y) \quad \text{for} \quad c \leq y \leq d$$

[This means that the curve $x = w(y)$ lies to the right of the curve $x = v(y)$ and that the two can touch but not cross.] Find the area A of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, and above and below by the lines $y = d$ and $y = c$ (Figure 6.1.11).



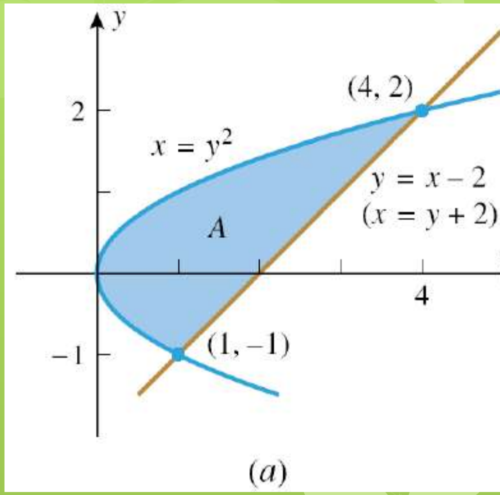
Reverse x and y to find the area on slide #13 instead of breaking into two sections.



▶ **Example 5** Find the area of the region enclosed by $x = y^2$ and $y = x - 2$, integrating with respect to y.

Solution. As indicated in Figure 6.1.10 the left boundary is $x = y^2$, the right boundary is $y = x - 2$, and the region extends over the interval $-1 \le y \le 2$. However, to apply (4) the equations for the boundaries must be written so that x is expressed explicitly as a function of y. Thus, we rewrite $y = x - 2$ as $x = y + 2$. It now follows from (4) that

$$A = \int_{-1}^{2} [(y + 2) - y^2]\, dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$

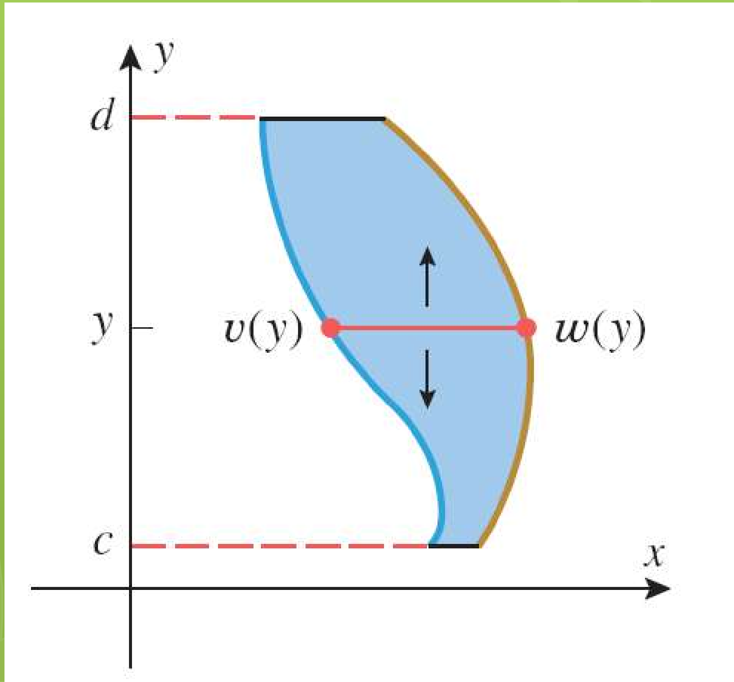


- You get exactly the same answer whether you break the area into two sections or if you reverse x and y.
- This is a much easier and quicker calculation that we had to perform when we reversed x and y.
- We avoided having to do two separate integrals and add our results.

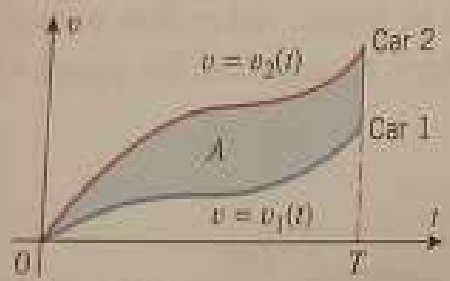
Formula and Picture for Reversing the Roles of x and y

6.1.4 AREA FORMULA If w and v are continuous functions and if $w(y) \geq v(y)$ for all y in $[c, d]$, then the area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below by $y = c$, and above by $y = d$ is

$$A = \int_c^d [w(y) - v(y)]\, dy \tag{4}$$



Application of Area Between Two Curves



▲ Figure 6.1.8

▶ **Example 3** Figure 6.1.8 shows velocity versus time curves for two race cars that move along a straight track, starting from rest at the same time. Give a physical interpretation of the area A between the curves over the interval $0 \le t \le T$.

Solution. From (1)

$$A = \int_0^T [v_2(t) - v_1(t)]\, dt = \int_0^T v_2(t)\, dt - \int_0^T v_1(t)\, dt$$

Since v_1 and v_2 are nonnegative functions on $[0, T]$, it follows from Formula (4) of Section 5.7 that the integral of v_1 over $[0, T]$ is the distance traveled by car 1 during the time interval $0 \le t \le T$, and the integral of v_2 over $[0, T]$ is the distance traveled by car 2 during the same time interval. Since $v_1(t) \le v_2(t)$ on $[0, T]$, car 2 travels farther than car 1 does over the time interval $0 \le t \le T$, and the area A represents the distance by which car 2 is ahead of car 1 at time T. ◀

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