Section 0.4

"Before Calculus": Inverse Functions; Inverse Trigonometric Functions

All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Inverse Functions

- In everyday language the term "inversion" conveys the idea of a reversal.
- Therefore, in mathematics, the term inverse is used to describe functions that reverse (undo) one another .
  - f(x) = the original function
  - $f^{-1}(x)$=the inverse function (NOT THE RECIPROCAL)
- To find the inverse of a function, you must
  - 1. change f(x), if it is in the original, to y and
  - 2. interchange x and y,
  - 3. then solve for y.
  - 4. The result will be the inverse.

Inverse Example

If $f(x) = 3x^2$, follow the steps to find the inverse:

1. change f(x) to y — $y = 3x^2$

2. interchange x and y — $x = 3y^2$

3. solve for y — $\frac{x}{3} = y^2$

$$\mp\sqrt{\left(\frac{x}{3}\right)} = y$$

4. The result will be the inverse. — $f^{-1}(x) = \mp\sqrt{\left(\frac{x}{3}\right)}$

Inverse Functions (con't)

- To prove algebraically that two functions are inverses of each other, you must show that $f^{-1}(f(x))=x$ and that $f(f^{-1}(x))=x$.
- See following slide for example.

Example 4 from pg 41 in book

First, check $f(f^{-1}(x)) = x$?

$$f(f^{-1}(x)) = f(\mp\sqrt{\left(\frac{x}{3}\right)}) = 3\left(\mp\sqrt{\left(\frac{x}{3}\right)}\right)^2 = 3(\frac{x}{3}) = x \ YEAH!!$$

Now, check $f^{-1}\big(f(x)\big) = x$?

$$f^{-1}\big(f(x)\big) = f^{-1}(3x^2) = \mp\sqrt{(\frac{3x^2}{3})} = \mp\sqrt{(x^2)} = \mp x, \ not \ x$$

Therefore, for these to be inverses, we must limit the domain of the original parabola to $x \geq 0$.

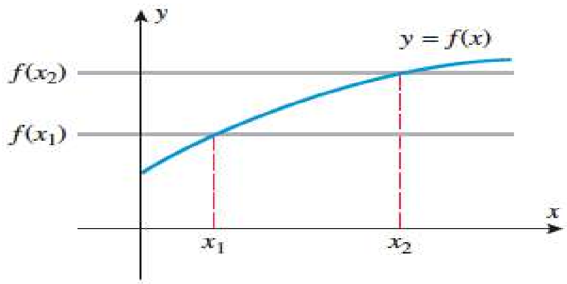
See the next slide for details.

Inverse Functions (con't)

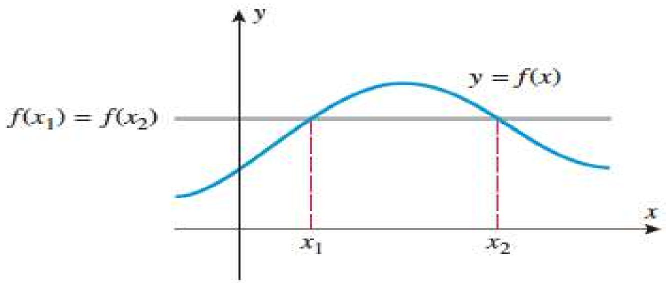
- Note: Sometimes, it is necessary to restrict the domain of an inverse $f^{-1}(x)=x$ or of an original f(x) in order to obtain a function (see examples on page 44).
- A function f(x) has an inverse iff it is one-to-one (invertible), each x has one y and each y has one x (must pass vertical and horizontal line tests). See next page for examples.

Inverse Functions (con't)

0.4.4 THEOREM (*The Horizontal Line Test*) *A function has an inverse function if and only if its graph is cut at most once by any horizontal line.*

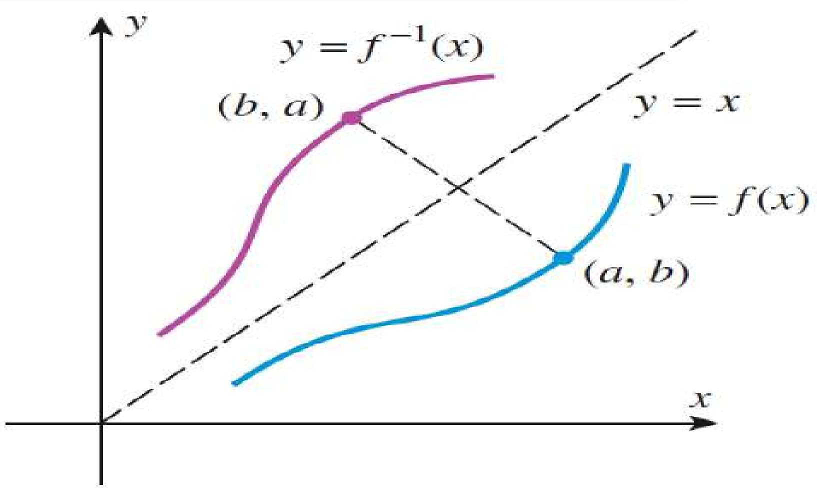


One-to-one, since $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$



Not one-to-one, since $f(x_1) = f(x_2)$ and $x_1 \neq x_2$

0.4.5 THEOREM *If f has an inverse, then the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about the line $y = x$; that is, each graph is the mirror image of the other with respect to that line.*



Domain and Range of Inverse Functions

- The domain of the inverse = the range of f(x)
- The range of the inverse = the domain of f(x)

- This should make sense because all of the points are just reversed between the original and the inverse.

- Example: If f(x) = {(2,1),(3,3),(4,2),(5,7)} then $f^{-1}(x)$ = {(1,2),(3,3),(2,4),(7,5)}

- You will use this idea to solve a problem or two on the section 0.4 assignment.

Inverse Trigonometric Functions

- These are the definitions. Please see graphs on next page.

0.4.6 DEFINITION The *inverse sine function*, denoted by $\sin^{-1}$, is defined to be the inverse of the restricted sine function

$$\sin x, \quad -\pi/2 \le x \le \pi/2$$

0.4.7 DEFINITION The *inverse cosine function*, denoted by $\cos^{-1}$, is defined to be the inverse of the restricted cosine function

$$\cos x, \quad 0 \le x \le \pi$$

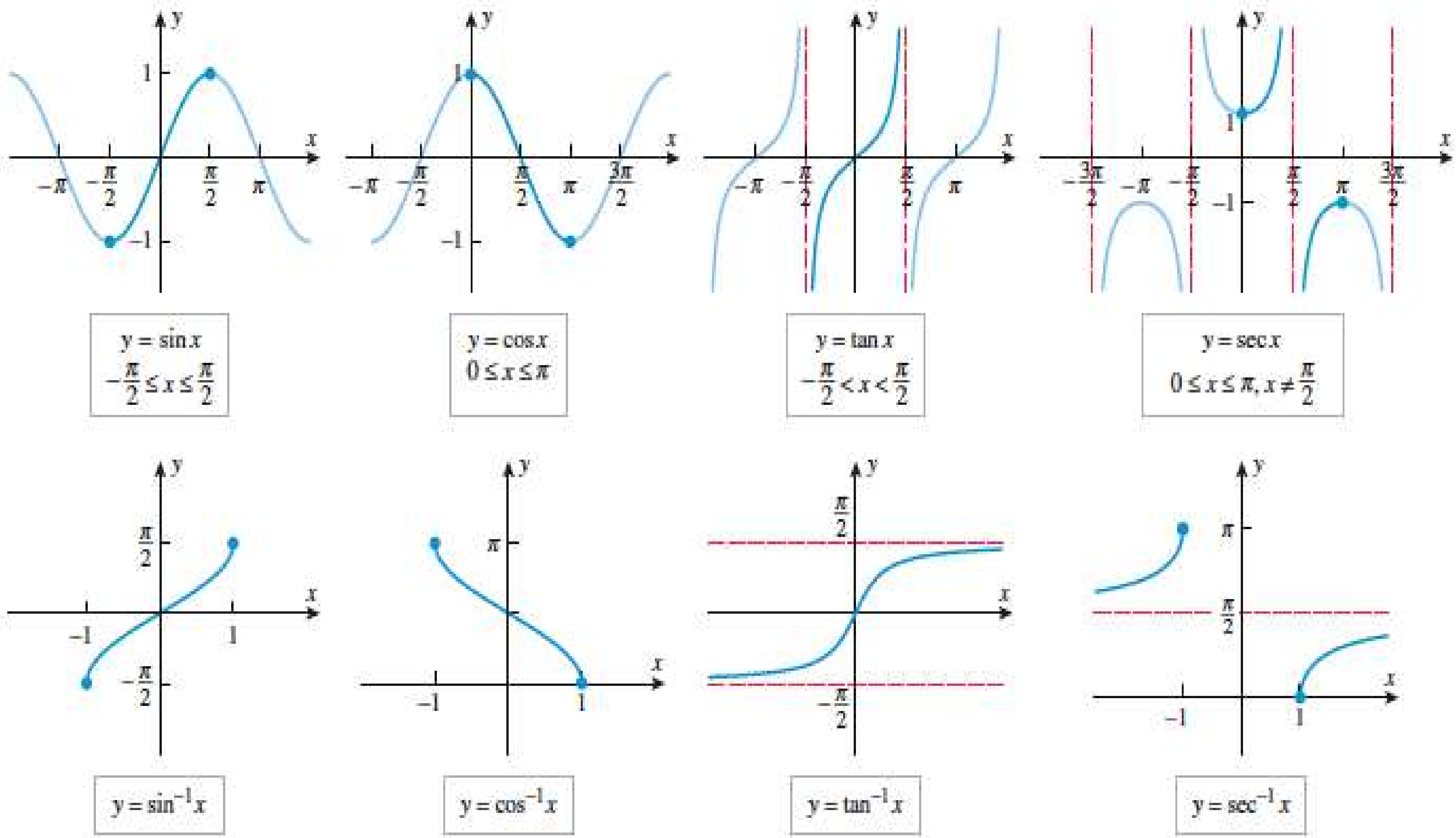
0.4.8 DEFINITION The *inverse tangent function*, denoted by $\tan^{-1}$, is defined to be the inverse of the restricted tangent function

$$\tan x, \quad -\pi/2 < x < \pi/2$$

0.4.9 DEFINITION* The *inverse secant function*, denoted by $\sec^{-1}$, is defined to be the inverse of the restricted secant function

$$\sec x, \quad 0 \le x \le \pi \text{ with } x \ne \pi/2$$

Graphs of Trig. Functions and Their Inverses



Evaluating Inverse Trig. Functions

- Use the information from the unit circle to evaluate inverse trigonometric functions.
- Example:
  - Since $\cos(\Pi/3) = ½$, it is associated with the point (Π/3, ½).
  - For inverse cosine, the point is reversed: (½, Π/3).
  - To evaluate $\cos^{-1}(½)$ you must find the angle whose cosine is ½. It just requires working backward.
  - Answer: $\cos^{-1}(½) = \Pi/3$
  - Make sure your answer is in the domain of inverse cosine (see slide #9) or you may have to pick an angle in a different quadrant. Remember All Students Take Calculus.

Identities for Inverse Trigonometric Functions

- Please read pages 47 and 48 in your book and we will go through this part in class together.