VMHS Math Circle

VI. The Grand Algebra Review (Part I: Equations/Equation Solving)

Now, since we've got more obscure topics aside, we're going to jumping into a review of the main areas of math that will be tested on the AMC: algebra, number theory, combinatorics, and geometry. Topics in pre-calculus are intended for only the AMC 12, so a review of that will be put aside for much later. What the first part of this algebra review will cover are many different types of equations and how you can solve them in the context of a problem you have. There are some topics you should definitely know concerning equation-solving:

- Single-variable equations and solving those
- Knowing how to factor (both numbers and polynomials)
- Multiplication/division/exponentiation of polynomials
- Equation systems
- Quadratic formula/completing the square/discriminant in the quadratic formula

I'm definitely going to revisit these topics here, but I'd cover whatever info is necessary for the AMC.

Equation Systems

First up are equation systems. Concerning that list I just gave earlier, you should be able to solve equation systems. What the AMC will do is try to make you figure out the systems of equations you need to have in order to solve various problems. You're definitely not going to be handed a system of equations right away without a twist that you need to get around. What you need to do is find two different relationships (equations) between two variables, make equations, and solve them from there.

Let's start with some easy examples:

Assume that one pound of apples costs 3 dollars (way too expensive for a pound of apples, by the way), while one pound of bananas costs 4 dollars (I don't need to repeat myself about how badly I'm estimating costs). If we have 5 total pounds of apples and bananas, and the total cost of that amount is \$18, how many pounds of bananas do we have?

Let's call "a" and "b" the amount of pounds we have of apples and bananas, respectively. We know that the total cost of this is \$18, and the total weight is 5 pounds. We are able to make a system.

$a + b = 5 \rightarrow b = 5 - a$

$3a + 4b = 18$

Substituting for b, we have: $3a + 4(5 – a) = 18$. This becomes $3a + 20 – 4a = 18$. Thus, $a = 2$. Solving for b, we get $b = 5 – 2 = 3$ pounds of bananas.

One green ball costs twice as much as a blue ball. If the total cost of both balls is \$6, what is the cost of a green ball?

Let's say a green ball costs "g" dollars while a blue ball costs "b" dollars. Proceed to make the following system:

$2b = g$

$g + b = 6$

Substitute for g, and we get that $2b + b = 6$. This becomes $3b = 6$, so $b = 2$. We need to find the cost of green balls, so substitute 2 for b in our first equation to get g. $2b = g$, so $2(2) =$ \$4. Note that the first equation is not $2g = b$, because this means that one blue ball costs twice as much as a green ball. Remember to be logical when making equations.

How does the AMC make this more difficult? Well, equation systems are pretty basic material, but sometimes, the AMC can place this skill into more difficult contexts. You might even have to craft a system of equations when solving a geometry or number theory problem (I'll get to that when I actually review geometry and number theory). However, one common context is digits. This will become clear with an example:

When a two-digit number is reversed, the new number is 9 more than the original number. If the sum of the digits of both numbers is 7, what is the original number?

Okay, big thing when you have "digit" problems! Express a certain number as $10^n a_1 + 10^{n-1} a_2 + \ldots + 10^0 a_n$, when you're considering digits. If you don't get what I mean, I'm basically saying to express a 1 digit number as a_1, a 2 digit number as $10a_1 + a_2$, a 3 digit number as $100a_1 + 10a_2 + a_3$, a 4 digit number as $1000a_1 + 100a_2 + 10a_3 + a_4$, and so on. Since we are working with 2-digit numbers, let's express our original number as 10a + b, with "a" and "b" being our digit values. Remember that digit values are from 0 to 9 inclusive. We make our system.

$10a + b + 9 = 10b + a \rightarrow 9 = 9b - 9a$

$a + b = 7 \rightarrow a = 7 - b$

Substitute for a, and we get $9 = 9b - 9(7 - b)$. We then have $9 = 9b - 63 + 9b$, which rearranges into $72 = 18b$. $b = 4$, and solving for a, we get $a = 7 - 4 = 3$. Our original number is expressed as 10a + b, which is the number $10(3) + 4 = 34$.

Diophantine Equations

Some equations are less specific. They require you to basically choose what values you need to form solutions. These are Diophantine equations, equations in which you focus on integer solutions since there may exist more than one unknown variable which we can't necessarily get a single value for. Diophantine equations can have one or multiple solutions. Let's get to some examples to show what I'm talking about.

What are all ordered pairs of positive integers (x, y) that satisfy the equation $(9 - x)(3 + y) = 21$?

Since we have two quantities multiplying each other on the left hand side of the equation, it'd help immensely to find all pairs of positive factors that multiply up to 21. Note that x cannot be greater than 9 since if so, then the equation has no solutions for every positive integer y (a negative times a positive is negative, and 21 is clearly negative). Considerations aside, these pairs are: (1, 21) and (7, 3). Huge note: Since the quantities $(9 - x)$ and $(3 + y)$ are different, we definitely want to consider cases when our factors are flipped. Hence, we have four pairs of factors in which order matters: (1, 21), (7, 3), (3, 7), and (21, 1). Now, we try to find ordered pairs of positive integers (x, y) that make our quantities $9 - x$ and $3 + y$ multiply to 21. These are:

(8, 18) (2, 0) (6, 4) and (-12, -2).

Our problem asks for ordered pairs of positive integers. Considering that zero is not a positive integer, our pairs are (8, 18) and (6, 4). If you didn't consider the cases if your factor pairs were flipped, you would've gotten away with it.

When a two-digit number is reversed, the new number is 18 more than the original number. What is the smallest original number for which this is possible (sound familiar? Well, I gave you a very similar problem, but with arguably less information…)?

Apply the same reasoning we used in the earlier problem that was like this and make an equation.

$10a + b + 18 = 10b + a$

Rearranging gets us 18 = 9b – 9a → 18 = 9(b – a) → 2 = (b – a). Since digits "a" and "b" range from 0 to 9, inclusive, we look for the smallest possible values of "a" and "b," each ranging from 0 to 9, that satisfy our equation. Doing so gets us a = 1 and b = 3, which corresponds to 13 when we plug those into 10a + b. Note that we cannot use a = 0 and b = 2 because the problem is looking for a two-digit integer. 2 is only a one-digit integer.

Polynomials and Roots

Finding the roots of a quadratic is pretty big toward the AMC. Other polynomials and the relationships of roots come up as a major AMC topic. First things first, let's examine the **golden mathematical rule of roots and polynomials**.

Vieta's Formulas:

- Given a polynomial (this is when the exponent of the highest power of x is even)
$a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \ldots + a_{n-1}x^1 + a_n$, with roots $r_1, r_2, \ldots, r_n$,
let s_p be the sum of the products of distinct roots of this polynomial, with the roots taken "p" at a time. It follows that

$$s_1 = r_1 + r_2 + \ldots + r_n$$
$$s_2 = r_1r_2 + r_1r_3 + \ldots + r_{n-1}r_n$$
$$s_3 = r_1r_2r_3 + r_1r_2r_4 + \ldots + r_{n-2}r_{n-1}r_n$$
$$\ldots$$
$$s_{n-1} = r_1r_2r_3\ldots r_n$$

and that $s_p = (-1)^p(a_{1+p}/a_1)$ by Vieta's.

- From what we established before, the sequence $a_2/a_1, a_3/a_1, \ldots, a_n/a_1$ should alternate in sign. Note that I'm not referring to the actual numbers, I'm referring to the variable expressions.

- Also, the product of the roots for an even-degree polynomial is equal to a_n/a_1, and the product of the roots of an odd-degree polynomial is equal to $-a_n/a_1$. The sum of the roots for an even-degree polynomial is equal to $-a_2/a_1$, and the sum of the roots of an odd-degree polynomial is equal to $-a_2/a_1$.

To make what I said much clearer, let's consider some polynomials with differing degrees and relate their roots to various coefficients.

For the sake of simplicity, let's start with quadratics.
For the quadratic polynomial $P(x) = 3x^2 + 5x – 4$, let $a_1 = 3$, $a_2 = 5$, and $a_3 = -4$, and let our roots be r_1 and r_2. By Vieta's, we have that $r_1 + r_2 = (-1)^1(a_2/a_1) = -a_2/a_1$ and that $r_1r_2 = (-1)^2(a_3/a_1) = (a_3/a_1)$.

Now, let's apply this to a higher-degree polynomial.
For the fourth-degree polynomial $P(x) = 2x^4 + 5x^3 + 2x + 8$, let $a_1 = 2$, $a_2 = 5$, $a_3 = 0$, $a_4 = 2$, and $a_5 = 8$ and let our roots be r_1, r_2, r_3, and r_4. It follows by Vieta's that $r_1 + r_2 + r_3 + r_4 = -(a_2/a_1)$, and $r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 = (-1)^2(a_3/a_1) = (a_3/a_1)$. Moreover, $r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 = (-1)^3(a_4/a_1) = -(a_4/a_1)$, and $r_1r_2r_3r_4 = (a_5/a_1)$.

We're going to do this one last time, and we're actually going to know what our roots are to check that Vieta's works. Let's consider the following cubic polynomial P(x) in factored form:

$$P(x) = (2x – 3)(x + 6)(x – 2)$$

The roots of this polynomial are 3/2, -6, and 2. Let's multiply this polynomial out. Doing so gets us $2x^3 + 5x^2 – 36x + 36$.

Let $a_1 = 2$, $a_2 = 5$, $a_3 = -36$, and $a_4 = 36$. The product of the roots is (3/2)(-6)(2) = -18 = $(-1)^3(a_4/a_1)$. The sum of the roots is 3/2 + -6 + 2 = -5/2 = $(-1)(a_2/a_1)$. The sum of the products of all combinations of two of our three roots is (3/2)(2) + (3/2)(-6) + (-6)(2) = -18 = $(-1)^2(a_3/a_1)$.

That's basically it for Vieta's formulas, since what the AMC will ask you that pertains to this topic will require the knowledge I gave you. I understand that what I said here can be pretty confusing… If anything, you can talk to me in person at school so I could make my explanations more active and personal. Plus, I'd be able to draw diagrams pretty freely to get things across more clearly. Rest assured, Vieta's gets much more clear in practice, so I'm going to throw in an example problem.

Consider a fourth-degree polynomial $Q(x) = 2x^4 + 6x^3 + 9$. If the sum and product of the roots of this quadratic are "a" and "b," respectively, what is $(a^2 – b^2)/(a – b)$?

You definitely want to factor this expression into (a + b)(a – b)/(a – b). (a – b) cancels out, and we are left with a + b, the sum of the roots plus the product of the roots. To relate this problem to what I said with Vieta's earlier, let $a_1 = 2$, let $a_2 = 6$, and let $a_4 = 9$ (Note that these line up with the powers of x they're supposed to line up with. a_1 should always be in front of the x raised to the highest power, and a_n should be in front of x raised to the zero power, which is 1.). Vieta's states that, since we have an even-degree polynomial, the product of the roots is $a_3/a_1 = 9/2$, while the sum of the roots is $-(a_2/a_1) = -(6/2)$. This gets us a + b = -3 + 9/2 = 3/2.

I call Vieta's the golden rule of polynomials because literally 90% of AMC problems that ask you about polynomials and roots have to deal with Vieta's. Seriously, it's that good.

Now that we've covered that, there is a term concerning polynomials that you should know. This term is "**monic**," meaning that the **leading coefficient of the term with the highest power of x is 1**. The following are monic polynomials:

$$x^3 + x + 3$$
$$x^4 + 2x^3 + x^2 + x + 16$$
$$x^2 – 1$$

There are some other things you should know about when finding roots. I'm going to line them up here.

Rational Roots, Descartes' Rule of Signs, and Polynomial Long Division for Roots:

If you find a root "r" of a polynomial, you can divide that polynomial by x – r evenly. This allows you to find even more roots of that polynomial. **To find possible rational roots, you take all possible factors of the last term (with x to the lowest power) and divide them by all possible factors of the first term (with x to the highest power).** Here's an example.

Find all roots of $x^4 + 2x^3 – 7x^2 – 8x + 12 = 0$.

Using the rational roots test to find all possible roots, we want to find all possible factors of 12 and all possible factors of 1. For 12, this is ±1, ±2, ±3, ±4, ±6, ±12. For 1, this is ±1. Now we divide. Since we have a 1 in the denominator, our possible roots are still going to be ±1, ±2, ±3, ±4, ±6, or ±12. Now we guess. Something that will help us is **Descartes' Rule of Signs**. **This states that the number of positive and negative real roots is equal to the number of sign changes going across a polynomial. For positive roots, we want to find the number of sign changes if we just plug in x into x. For negative roots, we want to find the number of sign changes if we plug in -x into x.**

If we plug in x into x, we notice, reading the polynomial left to right, there are two sign changes; hence, two positive roots.

If we plug in -x into x, we notice, reading the polynomial left to right, there are two sign changes; hence, two negative roots.

Descartes' rule of signs is something to help narrow down your choices of roots. It steers you in the right direction in the process of checking roots.

Let's try 1. We see that 1 works! Thus, we can divide this polynomial by $x – 1$. I'll tell you what we get flat-out:

$$x^3 + 3x^2 – 4x – 12.$$

Now we can try more rational roots and have a simpler time checking them. Let's try -2. That works, too! Now, let's divide $x^3 + 3x^2 – 4x – 12$ by $x + 2$. We get: $x^2 + x – 6$. Luckily, this factors into $(x + 3)(x – 2)$. Our roots are -3, -2, 1, and 2.

Now, we move on to something that is, in most cases, simpler: quadratics. If you don't know what a quadratic is, you'd be better off closing this document and turning off your computer. Kidding. To those who don't know, quadratics are polynomials with degree 2 (in other words, your usual $ax^2 + bx + c$ polynomial). A lot of times, the AMC asks you to find roots or special things concerning roots. There might even be Diophantine equations concerning quadratics, so quadratic equations definitely form a diverse and abundant AMC topic.

With that being said, remember to utilize your quadratic formula! Not a lot of quadratics that you must solve for in the the AMC are going to be that easy to solve. This will become clear when I bring in examples.

One thing that is good to know is the fact that there are various forms to express a quadratic. These are the vertex and standard forms. For a quadratic Q(x), vertex form is expressed as:

$$Q(x) = a(x – h)^2 + k.$$

The standard form of Q(x) would be expressed as:

$$Q(x) = ax^2 + bx + c.$$

Remember that the variables a, b, c, h, and k represent numeric constants (which are real in most cases). What's really good to know is how to factor quadratics, even in a generic way. When I say this, I mean that should be able to factor a quadratic with variable coefficients. For example, given a quadratic $x^2 + bx + c$, you should be able to factor it as $(x – r_1)(x – r_2)$ and know that by Vieta's, $c = r_1r_2$ and $b = -(r_1 + r_2)$.

To better explain this, there is one tactic, although a bit obscure for the AMC, can be very useful in helping to factor various expressions and find solutions: Simon's Favorite Factoring Trick (SFFT for short). SFFT was popularized by this one college professor who attended UC Santa Barbara. This states that an equation $xy + xk + yj +jk$ can be factored into the form $(x + j)(y + k)$, and we can manipulate an equation that isn't in that form by adding a certain number or expression to both sides for us to be able to solve that equation. Usually, SFFT is used for Diophantine equations.

Here are a couple of problem-solving examples I'm going to throw out.

Given the quadratic $Q(x) = nx^2 + 6x + k$, what pairs of positive integers (n, k), with n and k not necessarily distinct, result in only one unique solution for Q(x)?

All right. Since we're looking for single unique solutions to Q(x), this screams the words "double root." You can try to factor this into $(\sqrt{n} + \sqrt{k})^2$, but doing this won't get you anywhere much (at least according to my experiences). One thing you can do is set up a system of equations using Vieta's and the concept that we have a double root "r". Doing so nets us this system:

$$r^2 = k/n$$

$$2r = -6/n \rightarrow r = -3/n.$$

Plugging in $-3/n$ for r in the first equation results in $(9/n^2) = (k/n) \rightarrow 9 = kn$. Thus, we want to find positive integers for n and k that satisfy $nk = 9$. It follows that we have ordered pairs of positive integers (n, k) in (3, 3) and (1, 9), all the pairs of factors that multiply to 9.

On the other hand, you can use the quadratic formula, which gets us:

$$x = (-6 \pm \sqrt{(36 – 4nk)})/2n.$$

In order to get only one solution, we have to "zero out" the discriminant, which basically means set whatever the $b^2 - 4ac$ is to zero and solve. Doing so gets rid of that plus or minus, since both adding and subtracting 0 gets us the same number, obviously. In this case, we have:

$36 - 4nk = 0 \rightarrow -4nk = -36 \rightarrow nk = 9$.

Doesn't this equation look familiar? Proceed as we did with the first solution to get (3, 3) and (1, 9). The technique of "zeroing out" the discriminant is a bit obscure. However, it can help with some pretty tough problems involving finding tangent lines to ellipses or circles without using multivariable calculus or whatever advanced math method there is. I'll get to these types of questions at a later point.

For what value of "k" does the quadratic equation $x^2 + kx + 32 = 0$ have a solution in x = -8 or -4?

It helps immensely to multiply the expressions (x + 8) and (x + 4) to get our desired quadratic since plugging -8 or -4 in makes our desired quadratic equal to 0. This results in $x^2 + 12x + 32 = 0$. Our answer is 12. Pretty simple, huh? Those who would use Vieta's would get across so much more simply by merely adding -8 and -4, flipping the sign at the end because the sum of the roots in this quadratic is -k/1. Hence, we get -(-8 + -4) = -(-12) = 12. Seriously. Vieta's is that good.

Find all ordered pairs of positive integers (x, y) (I swear, this will be the last time I mention ordered pairs in this document) that satisfy the equation xy + 2x + 5y = 20.

Using Simon's Favorite Factoring Trick, we're definitely looking for something to add to both sides such that the left hand side with all those variables can factor. Looking forward, it would seem really nice to have (x + 5)(y + 2) on the left hand side of the equation, since FOIL'ing gets us xy + 2x + 5y, three terms that already fit with the left hand side of our equation, plus a 10. What do we have to add to both sides to get that simple expression? That 10 we were left over with. Hence, we are left with (x + 5)(y + 2) = 30. We find all pairs of factors that multiply to 30. These are (1, 30), (2, 15), (3, 10), (6, 5), (5, 6), (10, 3), (15, 2), and (30, 1). Obviously, we're looking for positive integers, so a pair of factors cannot exist such that any factor is less than or equal to x or y. This definitely rules out factor pairs (1, 30), (2, 15), (3, 10), (5, 6), (15, 2), and (30, 1). Therefore, we are left with (6, 5) and (10, 3). It follows that our ordered pairs (x, y) are (1, 3) and (5, 1).

TIPS:

- Remember to check whatever equations you make in a problem and see whether they're right! For example, if you want to express the fact that it takes twice as long for person A to travel to a certain destination than it takes for person B, express it as a = 2b, not 2a = b, with "a" and "b" being the time it takes for persons a and b to travel, respectively. Why? Because if we multiply person B's time by two, we get person A's time.
- When the problem gives you a certain number and expects integer sums, factors, or whatnot concerning that number, **factor factor factor that number**! This is a very important tip that I cannot stress enough!
- Check your signs! Check your numbers, and don't make stupid mistakes! Remember to take note of specific conditions in the problems you get, and actually understand the question.
- Don't forget to FOIL when you are multiplying two radical expressions with real numbers attached!
- Since I've upheld Vieta's for about a million times, whenever you get a polynomial or quadratic question, think Vieta's! Know how the coefficients of a polynomial are related to its roots!
- Always consider $\mathbf{10^n a_1 + 10^{n-1} a_2 + \ldots + 10^0 a_n}$ when working with digits. This will lead to some Diophantine equations that you can solve.

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Boom. That concludes the first part of our grand algebra review. From now on, I'm going to be using boxes to conclude various parts of the reviews of our main topics (like real mathematicians do with proofs). It's just a little novelty that I like to include. Practice AMC problems are going to come at the very end. Since this is algebra, and this is all one topic, I feel like the best way to prepare you is to add some uncertainty to the problems that will come. That is, I won't place actual AMC questions right after a certain topic that I explain. If I actually did that, you'd actually know what to expect with said questions. Doing so will definitely put you in that testing environment and supply a fresh experience. Jumping into a unique problem without any preexisting knowledge of it will show you where you are, depending on whether or not you can solve it. If you can't solve a certain problem, that's totally fine. Learn to do the things you can't do, for it **will** make you better. Struggle is crucial for improvement.