7.6 The Inverse of a Square Matrix

What You Should Learn

- Verify that two matrices are inverses of each other.
- Use Gauss-Jordan elimination to find inverses of matrices.
- Use a formula to find inverses of 2×2 matrices.
- Use inverse matrices to solve systems of linear equations.

The Inverse of a Matrix

The Inverse of a Matrix

This section further develops the algebra of matrices. To begin, consider the real number equation

$$ax = b.$$

To solve this equation for x, multiply each side of the equation by a^{-1} (provided that $a \neq 0$).

$$ax = b$$

$$(a^{-1}a)x = a^{-1}b$$

$$(1)x = a^{-1}b$$

$$x = a^{-1}b$$

The Inverse of a Matrix

The number a^{-1} is called the *multiplicative inverse* of *a* because

$$a^{-1}\, a = 1.$$

The definition of the multiplicative **inverse of a matrix** is similar.

Definition of the Inverse of a Square Matrix

Let A be an $n \times n$ matrix and let I_n be the $n \times n$ identity matrix. If there exists a matrix A^{-1} such that

$$AA^{-1} = I_n = A^{-1}A$$

then A^{-1} is called the **inverse** of A. The symbol A^{-1} is read "A inverse."

Example 1 – *The Inverse of a Matrix*

Show that *B* is the inverse of *A*, where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

Solution:

To show that *B* is the inverse of *A*, show that *AB* = *I* = *BA*, as follows.

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 + 2 & 2 - 2 \\ -1 + 1 & 2 - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 1 – *Solution*

cont'd

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 + 2 & 2 - 2 \\ -1 + 1 & 2 - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As you can see,

$$AB = I = BA.$$

Finding Inverse Matrices

Finding Inverse Matrices

When a matrix *A* has an inverse, *A* is called **invertible** (or **nonsingular**); otherwise, *A* is called **singular.** A nonsquare matrix cannot have an inverse.

To see this, note that if *A* is of dimension $m \times n$ and *B* is of dimension n × m (where $m \neq n$), then the products *AB* and *BA* are of different dimensions and so cannot be equal to each other.

Not all square matrices have inverses, as you will see later in this section. When a matrix does have an inverse, however, that inverse is unique. Example 2 shows how to use systems of equations to find the inverse of a matrix.

Example 2 – *Finding the Inverse of a Matrix*

Read slides 10-13, do not copy.

Find the inverse of $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}.$

Solution:

To find the inverse of *A* try to solve the matrix equation

for *X*.

$$\overset{A}{\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}} \overset{X}{\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 2 – *Solution*

cont'd

Equating corresponding entries, you obtain the following two systems of linear equations.

$$\begin{cases} x_{11} + 4x_{21} = 1 \\ -x_{11} - 3x_{21} = 0 \end{cases}$$

Linear system with two variables, *x*11 and *x*21

$$\begin{cases} x_{12} + 4x_{22} = 0 \\ -x_{12} - 3x_{22} = 1 \end{cases}$$

Linear system with two variables, *x*12 and *x*22

Solve the first system using elementary row operations to determine that

$x_{11} = –3$ and $x_{21} = 1$.

Example 2 – *Solution*

cont'd

From the second system you can determine that

$x_{12} = -4$ and $x_{22} = 1$.

Therefore, the inverse of A is

$$A^{-1} = X$$

$$= \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}.$$

You can use matrix multiplication to check this result.

Example 2 – *Solution*

cont'd

Check

$$AA^{-1} = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$A^{-1}A = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Finding Inverse Matrices

Finding an Inverse Matrix

Let A be a square matrix of dimension $n \times n$.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain

$$[A \ \vdots \ I].$$

2. If possible, row reduce A to I using elementary row operations on the *entire* matrix

$$[A \ \vdots \ I].$$

The result will be the matrix

$$[I \ \vdots \ A^{-1}].$$

If this is not possible, then A is not invertible.

3. Check your work by multiplying to see that

$$AA^{-1} = I = A^{-1}A.$$

The Inverse of a 2 × 2 Matrix

The Inverse of a 2 × 2 Matrix

Using Gauss-Jordan elimination to find the inverse of a matrix works well (even as a computer technique) for matrices of 3 × 3 dimension or greater.

For 2 × 2 matrices, however, many people prefer to use a formula for the inverse (see next slide) rather than Gauss-Jordan elimination. This simple formula, which works *only* for 2 × 2 matrices, is explained as follows. If *A* is the 2 × 2 matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then *A* is invertible if and only if

$$ad - bc \neq 0.$$

The Inverse of a 2 × 2 Matrix

If *ad* – *bc* ≠ 0, then the inverse is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Formula for inverse of matrix *A*

The denominator

ad – *bc*

is called the *determinant* of the 2 × 2 matrix *A*.

Example 4 – *Finding the Inverse of a 2 × 2 Matrix*

If possible, find the inverse of each matrix.

a. $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$

b. $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

Solution:

a. For the matrix *A*, apply the formula for the inverse of a 2 × 2 matrix to obtain

$$ad - bc = (3)(2) - (-1)(-2)$$

$$= 4.$$

Example 4 – *Solution*

cont'd

Because this quantity is not zero, the inverse is formed by interchanging the entries on the main diagonal, changing the signs of the other two entries, and multiplying by the scalar $\frac{1}{4}$, as follows.

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

Substitute for *a*, *b*, *c*, *d* and the determinant.

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Multiply by the scalar $\frac{1}{4}$

Example 4 – *Solution*

cont'd

b. For the matrix B, you have

$$ad - bc = (3)(2) - (-1)(-6)$$

$$= 0$$

which means that B is not invertible.

Systems of Linear Equations

Systems of Linear Equations

You know that a system of linear equations can have exactly one solution, infinitely many solutions, or no solution.

If the coefficient matrix *A* of a *square* system (a system that has the same number of equations as variables) is invertible, then the system has a unique solution, which is defined as follows.

A System of Equations with a Unique Solution

If A is an invertible matrix, then the system of linear equations represented by $AX = B$ has a unique solution given by

$$X = A^{-1}B.$$

Systems of Linear Equations

The formula $X = A^{-1}B$ is used on most graphing calculators to solve linear systems that have invertible coefficient matrices.

That is, you enter the $n \times n$ coefficient matrix $[A]$ and the $n \times 1$ column matrix $[B]$. The solution X is given by $[A]^{-1}[B]$.

Example 5 – *Solving a System of Equations Using an Inverse*

Use an inverse matrix to solve the system.

$$\begin{cases} 2x + 3y + z = -1 \\ 3x + 3y + z = 1 \\ 2x + 4y + z = -2 \end{cases}$$

Solution:

Begin by writing the system as $AX = B$

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

Example 5 – *Solution*

cont'd

Then, use Gauss-Jordan elimination to find A^{-1}.

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

Finally, multiply B by A^{-1} on the left to obtain the solution.

$$X = A^{-1}B$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

Example 5 – *Solution*

cont'd

So, the solution is

$x = 2$, $y = –1$, and $z = –2$.