Section 5.4

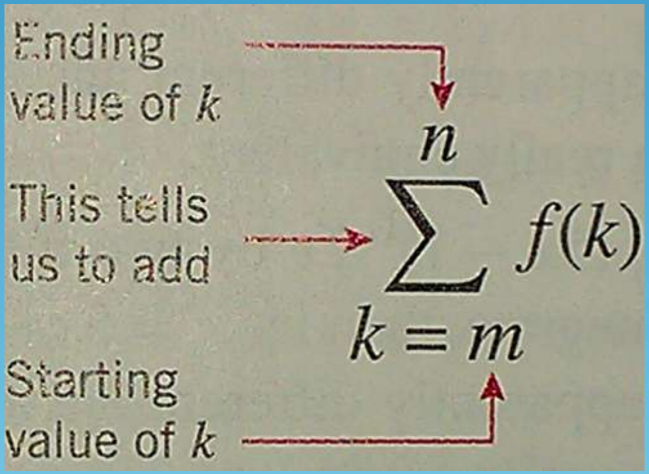
Integration: "the Definition of Area as a Limit; Sigma Notation"

All graphics are attributed to:

- *Calculus,10/E* by Howard Anton, Irl Bivens, and Stephen Davis

Introduction

- Our main goal in this section is to use the rectangle method we discussed in section 5.1 to give a precise mathematical definition of the "area under a curve".
- You probably remember sigma notation from Algebra II or PreCalculus. We use the Greek letter sigma to denote sums:



- m is called the lower limit of summation and n is called the upper limit of summation. k is called the index of summation and some books use different letters.

Changing the Limits of Summation

- A sum can be written in more than one way using sigma notation with different limits of summation and correspondingly different summands to fit the needs of the problem you are trying to solve.
- Each sigma notation below means the same thing:

$$\sum_{i=1}^{5} 2i = 2 + 4 + 6 + 8 + 10 = \sum_{j=0}^{4} (2j + 2) = \sum_{k=3}^{7} (2k - 4)$$

- On occasion, we will want to change the sigma notation for a given sum to a sigma notation with different limits of summation.

Properties of Sums

- We have seen very similar properties for limits, derivatives, and integrals, since they are all related.

5.4.1 THEOREM

(a) $$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k \qquad \text{(if } c \text{ does not depend on } k\text{)}$$

(b) $$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

(c) $$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

Summation Formulas

- We will be using these frequently in this section, but you do not need to memorize them this year. ☺

5.4.2 THEOREM

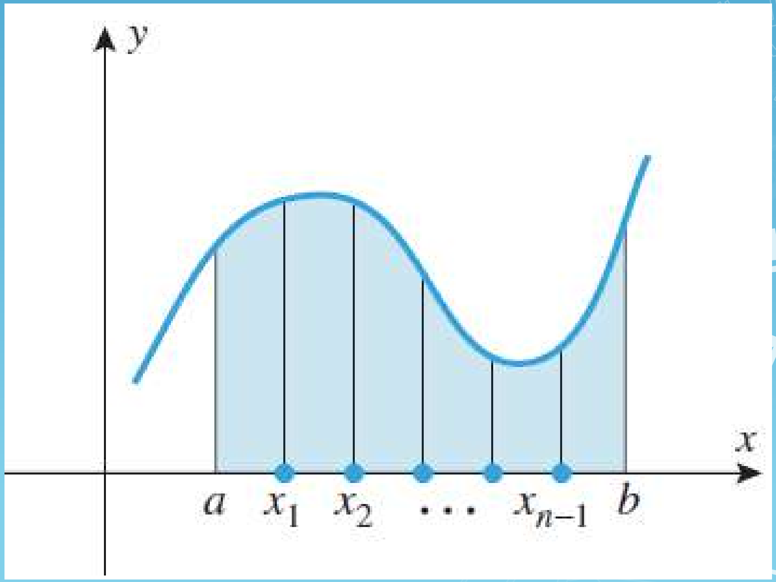
(a) $$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

(b) $$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(c) $$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

A Definition of Area

- We are going to be dealing with functions that are continuous and nonnegative on an interval [a,b] at first.
- We are trying to find the area of the blue shaded region bounded above by the curve y=f(x), bounded below by the x-axis, and bounded on the sided by the vertical lines x=a and x=b.
- We will divide that interval into n equal subintervals as you can see in the picture:



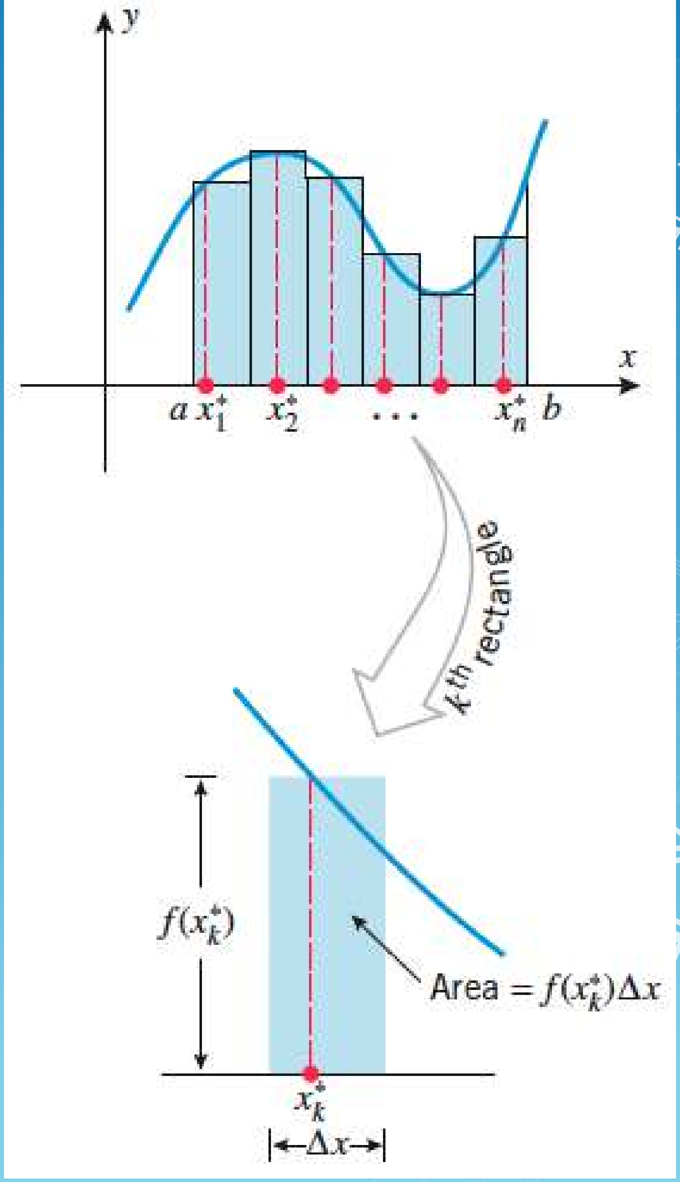
A Definition of Area - continued

- The distance between a and b is b-a.
- Since we divided that distance into n subintervals, each is :

$$\Delta x = \frac{b-a}{n}$$

- In each subinterval, draw a rectangle whose height is the value of the function f(x) at an arbitrarily selected point in the subinterval (a.k.a. x_k*) which gives f(x_k*).
- Since the area of each rectangle is base * height, we get the formula you see on the right for each rectangle:
- Area=

b*h = Δx * f(x_k*) = f(x_k*)Δ x .



A Definition of Area - continued

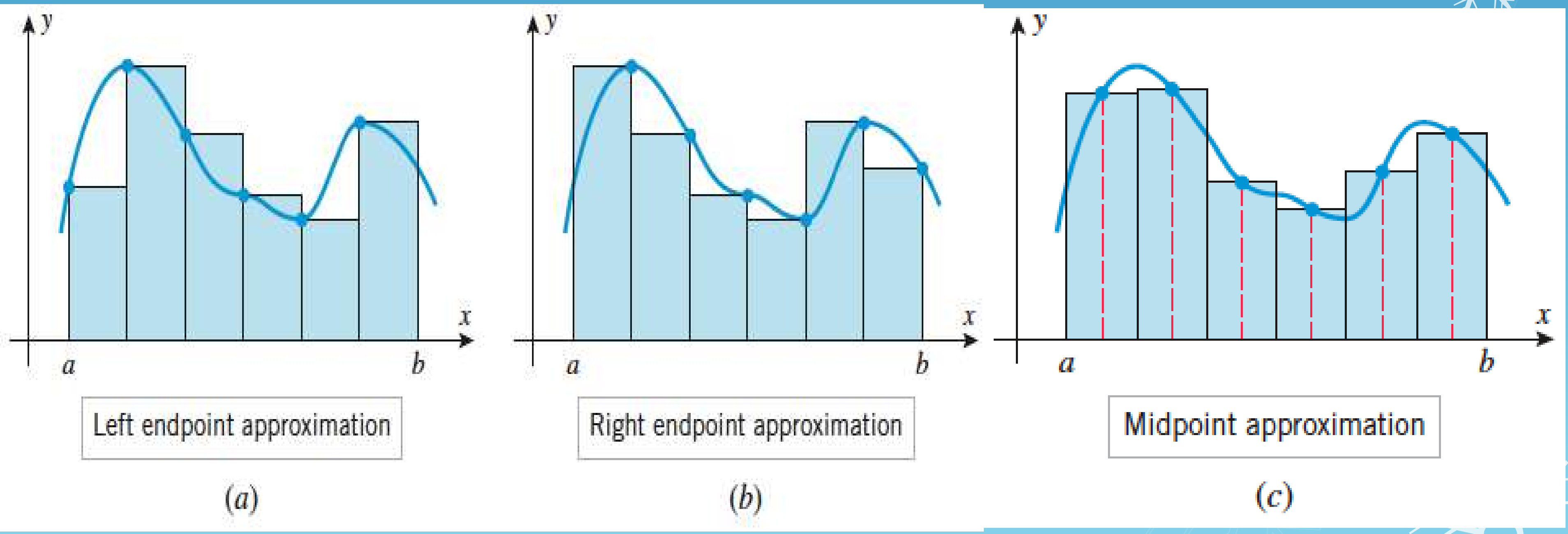
- Remember, that was the area for each rectangle. We need to find the sum of the areas of all of the rectangles between a and b which is why we use sigma notation.
- As we discussed in a previous section, the area estimate is more accurate with the more number of rectangles used. Therefore, we will let n approach infinity.

5.4.3 DEFINITION (*Area Under a Curve*) If the function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$, then the ***area*** A under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \qquad (2)$$

Choices for x_k*

- x_k* can be chosen arbitrarily, but for this year we will either choose each to be the left endpoint of each subinterval, the right endpoint of each subinterval, or the midpoint of each subinterval.
- You can see what that looks like in the following graphs:



How our choice of x_k* affects these problems:

- Keep this list handy so that you know what to substitute in for x_k* depending upon whether you are using the left endpoint, right endpoint, or midpoint (they are on page 345 in your book).

Thus, the left endpoint, right endpoint, and midpoint choices for $x_1^*, x_2^*, \ldots, x_n^*$ are given by

$$x_k^* = x_{k-1} = a + (k-1)\Delta x \quad \text{Left endpoint} \tag{3}$$

$$x_k^* = x_k = a + k\Delta x \quad \text{Right endpoint} \tag{4}$$

$$x_k^* = \tfrac{1}{2}(x_{k-1} + x_k) = a + \left(k - \tfrac{1}{2}\right)\Delta x \quad \text{Midpoint} \tag{5}$$

Example of how to use all of this information:

► **Example 4** Use Definition 5.4.3 with x_k^* as the right endpoint of each subinterval to find the area between the graph of $f(x) = x^2$ and the interval [0, 1].

Solution. The length of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

so it follows from (4) that

$$x_k^* = a + k\Delta x = \frac{k}{n}$$

Thus,

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n} (x_k^*)^2 \Delta x = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} k^2$$

$$= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \qquad \text{Part (b) of Theorem 5.4.2}$$

$$= \frac{1}{6}\left(\frac{n}{n} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n}\right) = \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

from which it follows that

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to+\infty} \left[\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = \frac{1}{3}$$

Another Example

- Step 1: Use the formula $\Delta x = \dfrac{b-a}{n}$
- Step 2: Pick the appropriate x_k^* from slide #13 for the problem you are working on.
- Step 3: Substitute the results from #1&2 into $f(x_k^*)\Delta x$.
- Step 4: Then take the limit of the sum of that product as the number of rectangles (n) approaches infinity and simplify.

▶ **Example 5** Use Definition 5.4.3 with x_k^* as the midpoint of each subinterval to find the area under the parabola $y = f(x) = 9 - x^2$ and over the interval [0, 3].

Solution. Each subinterval has length

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

so it follows from (5) that

$$x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x = \left(k - \frac{1}{2}\right)\left(\frac{3}{n}\right)$$

Thus,

$$\begin{aligned} f(x_k^*)\Delta x = [9 - (x_k^*)^2]\Delta x &= \left[9 - \left(k - \frac{1}{2}\right)^2\left(\frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right) \\ &= \left[9 - \left(k^2 - k + \frac{1}{4}\right)\left(\frac{9}{n^2}\right)\right]\left(\frac{3}{n}\right) \\ &= \frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3} \end{aligned}$$

from which it follows that

$$\begin{aligned} A &= \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \\ &= \lim_{n\to+\infty} \sum_{k=1}^{n} \left(\frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3}\right) \\ &= \lim_{n\to+\infty} 27\left[\frac{1}{n}\sum_{k=1}^{n} 1 - \frac{1}{n^3}\sum_{k=1}^{n} k^2 + \frac{1}{n}\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right) - \frac{1}{4n^2}\left(\frac{1}{n}\sum_{k=1}^{n} 1\right)\right] \\ &= 27\left[1 - \frac{1}{3} + 0 \cdot \frac{1}{2} - 0 \cdot 1\right] = 18 \end{aligned}$$

Theorem 5.4.4 ◀

Other Helpful Limits

- These formulas will often help to make the solution shorter.

5.4.4 THEOREM

(a) $\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} 1 = 1$

(b) $\lim_{n \to +\infty} \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{1}{2}$

(c) $\lim_{n \to +\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{1}{3}$

(d) $\lim_{n \to +\infty} \frac{1}{n^4} \sum_{k=1}^{n} k^3 = \frac{1}{4}$

Net Signed Area

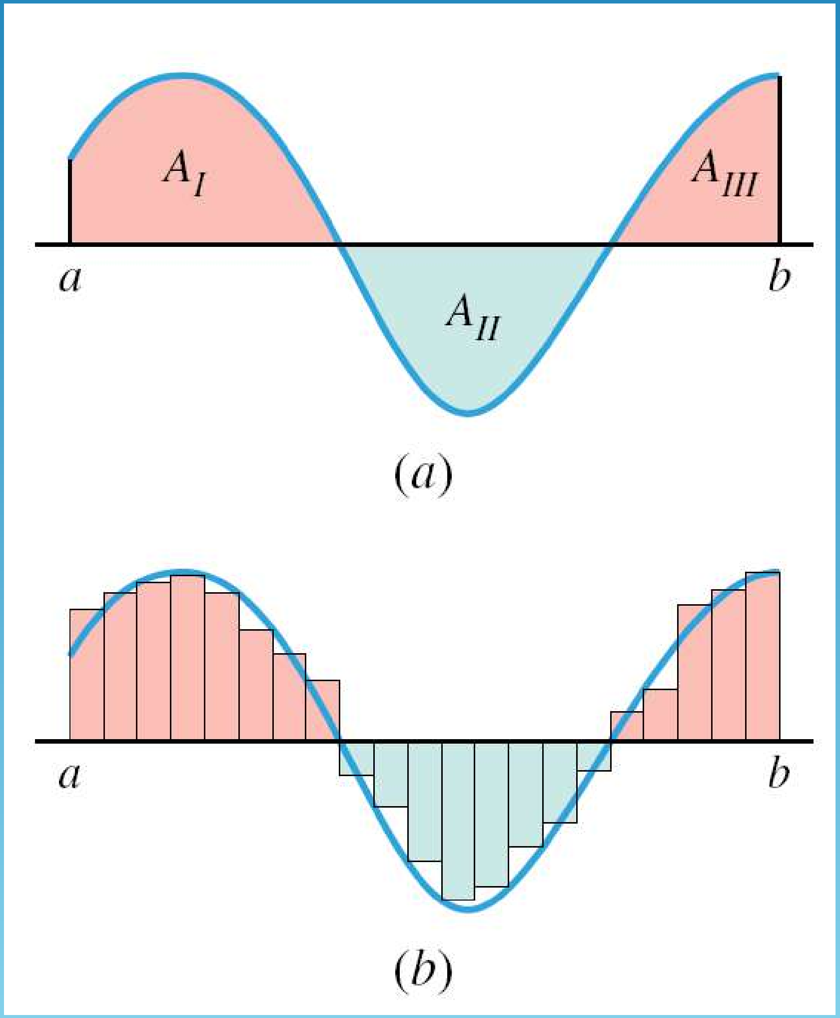
- Net signed area takes into account whether the area(s) is above or below the x-axis.

5.4.5 DEFINITION (*Net Signed Area*) If the function f is continuous on $[a, b]$, then the *net signed area* A between $y = f(x)$ and the interval $[a, b]$ is defined by

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \tag{9}$$

- To find the total area instead, you will have to break the area up into portions that are above the x-axis vs. those that are below the x-axis and find each area separately. After taking their absolute values, you may then add them together (see picture on next slide).

Net Signed Area Example



Rialto Bridge – Venice, Italy