Polynomial and Rational Functions

2.5 The Fundamental Theorem of Algebra

What You Should Learn

- Use the Fundamental Theorem of Algebra to determine the number of zeros of a polynomial function.
- Find all zeros of polynomial functions, including complex zeros.
- Find conjugate pairs of complex zeros.
- Find zeros of polynomials by factoring.

The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra

Only write down the red part: We know that an nth-degree polynomial can have at most n real zeros. In the complex number system, this statement can be improved. That is, in the complex number system, every nth-degree polynomial function has precisely n zeros.

This important result is derived from the **Fundamental Theorem of Algebra**, first proved by the German mathematician Carl Friedrich Gauss (1777–1855).

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

The Fundamental Theorem of Algebra

Read this, but do not write down this slide:

Using the Fundamental Theorem of Algebra and the equivalence of zeros and factors, you obtain the **Linear Factorization Theorem.**

Linear Factorization Theorem

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, . . . , c_n$ are complex numbers.

The Fundamental Theorem of Algebra

Only write down the red part:

Note that neither the Fundamental Theorem of Algebra nor the Linear Factorization Theorem tells you *how* to find the zeros or factors of a polynomial.

Such theorems are called *existence theorems*. To find the zeros of a polynomial function, you still must rely on other techniques such as factoring/zero product property or the rational root theorem (p/q).

Example 1 – *Zeros of Polynomial Functions*

a. The first-degree polynomial $f(x) = x - 2$ has exactly *one* zero: $x = 2$.

b. Counting multiplicity, the second-degree polynomial function

$$f(x) = x^2 - 6x + 9$$

$$= (x - 3)(x - 3)$$

has exactly *two* zeros: $x = 3$ and $x = 3$ (This is called a *repeated zero or double root/multiplicity* 2.)

Example 1 – *Zeros of Polynomial Functions*

cont'd

c. The third-degree polynomial function

$$f(x) = x^3 + 4x$$
$$= x(x^2 + 4)$$
$$= x(x - 2i)(x + 2i)$$

has exactly *three* zeros: $x = 0$, $x = 2i$, and $x = -2i$

d. The fourth-degree polynomial function

$$f(x) = x^4 - 1$$
$$= (x - 1)(x + 1)(x - i)(x + i)$$

has exactly *four* zeros: $x = 1$, $x = -1$, $x = i$, and $x = -i$

Finding Zeros of a Polynomial Function

Finding Zeros of a Polynomial Function

Only write down the red part:

Remember that the *n* zeros of a polynomial function can be real or complex, and they may be repeated. Example 2 illustrates several cases.

Example 2 – *Real and Complex Zeros of a Polynomial Function*

Confirm that the third-degree polynomial function

$f(x) = x^3 + 4x$

has exactly three zeros: $x = 0$, $x = 2i$, and $x = –2i$.

Solution:

Factor the polynomial completely as $x(x – 2i)(x + 2i)$. So, the zeros are

$x(x – 2i)(x + 2i) = 0$

$x = 0$

$x – 2i = 0 \qquad x = 2i$

$x + 2i = 0 \qquad : –2i$

Example 2 – *Solution*

cont'd

In the graph in Figure 2.33, only the real zero $x = 0$ appears as an x-intercept.

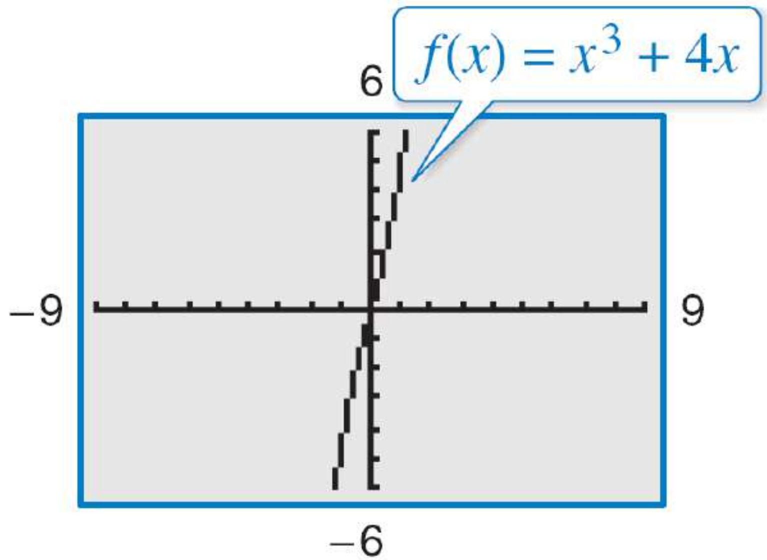

Figure 2.33

Conjugate Pairs

Conjugate Pairs

The two complex zeros are conjugates. That is, they are of the forms $a + bi$ and $a - bi$.

Complex Zeros Occur in Conjugate Pairs

Let $f(x)$ be a polynomial function that has *real coefficients*. If $a + bi$, where $b \neq 0$, is a zero of the function, then the conjugate $a - bi$ is also a zero of the function.

Be sure you see that this result is true only when the polynomial function has *real coefficients*. For instance, the result applies to the function $f(x) = x^2 + 1$, but not to the function $g(x) = x - i$.

Example 4 – *Finding a Polynomial with Given Zeros*

Find a fourth-degree polynomial function with real coefficients that has –1, –1 and 3*i* as zeros.

Solution:

Because 3*i* is a zero and the polynomial is stated to have real coefficients, you know that the conjugate –3*i* must also be a zero.

So, from the Linear Factorization Theorem, $f(x)$ can be written as

$$f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i)$$

Example 4 – *Solution*

cont'd

For simplicity, FOIL and let a = 1 to obtain

$$f(x) = (x^2 + 2x + 1)(x^2 + 9)$$

$$= x^4 + 2x^3 + 10x^2 + 18x + 9.$$

Factoring a Polynomial

Factoring a Polynomial

Only write down the red part:

The Linear Factorization Theorem states that you can write any *n*th-degree polynomial as the product of *n* linear factors.

$$f(x) = a_n(x - c_1)(x - c_2)(x - c_3) \dots (x - c_n)$$

This result, however, includes the possibility that some of the values of c_i are complex.

The following theorem states that even when you do not want to get involved with "complex factors," you can still write $f(x)$ as the product of linear and/or quadratic factors.

Factoring a Polynomial

Factors of a Polynomial

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

Only write down the red part:

A quadratic factor with no real zeros is said to be **prime** or **irreducible over the reals.** Be sure you see that this is not the same as being irreducible over the rationals. For example, the quadratic

$x^2 + 1 = (x - i)(x + i)$

is irreducible over the reals (and therefore over the rationals).

Factoring a Polynomial

On the other hand, the quadratic

$x^2 - 2 =$ $(x - \sqrt{2})(x + \sqrt{2})$

is irreducible over the rationals, but *reducible* over the reals.

Example 6 – *Factoring a Polynomial*

Write the polynomial $f(x) = x^4 - x^2 - 20$

a. as the product of factors that are irreducible over the *rationals*,

b. as the product of linear factors and quadratic factors that are irreducible over the reals, and

c. in completely factored form.

Example 6 – *Solution*

a. Begin by factoring the polynomial into the product of two quadratic polynomials.

$$x^4 - x^2 - 20 = (x^2 - 5)(x^2 + 4)$$

Both of these factors are irreducible over the rationals.

b. By factoring over the reals, you have

$$x^4 - x^2 - 20 = (x^2 + 4)\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)$$

where the quadratic factor is irreducible over the reals.

c. In completely factored form, you have

$$x^4 - x^2 - 20 = (x - 2i)(x \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)$$

Factoring a Polynomial

Read, but do not write down this slide:

The results and theorems have been stated in terms of zeros of polynomial functions. Be sure you see that the same results could have been stated in terms of solutions of polynomial equations. This is true because the zeros of the polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

are precisely the solutions of the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0.$$

Example 7 – *Finding The Zeros of Polynomial Function*

Find all the zeros of

$$f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$$

given that $1 + 3i$ is a zero of f.

Solution:

Because complex zeros occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. This means that both

$$x - (1 + 3i) \text{ and } x - (1 - 3i)$$

are factors of f.

Example 7 – *Solution*

Multiplying these two factors produces

$$[x - (1 + 3i)][x - (1 - 3i) = [(x - 1) - 3i][(x - 1) + 3i]$$

$$= (x - 1)^2 - 9i^2$$

$$= x^2 - 2x + 10.$$

Using long division, you can divide $x^2 - 2x + 10$ into f to obtain the following.

Example 7 – *Solution*

cont'd

Using long division, you can divide $x^2 - 2x + 10$ into f to obtain the following.

$$
\begin{array}{r}
x^2 - x - 6 \\
500x^2 - 2x + 10 \,\big)\overline{\, x^4 - 3x^3 + 6x^2 + 2x - 60} \\
\underline{x^4 - 2x^3 + 10x^2} \phantom{{}+ 2x - 60} \\
-x^3 - 4x^2 + 2x \phantom{{}- 60} \\
\underline{-x^3 + 2x^2 - 10x} \phantom{{}- 60} \\
-6x^2 + 12x - 60 \\
\underline{-6x^2 + 12x - 60} \\
0
\end{array}
$$

Example 7 – *Solution*

cont'd

So, you have

$$f(x) = (x^2 - 2x + 10)(x^2 - x - 6)$$

$$= (x^2 - 2x + 10)(x - 3)(x + 2)$$

and you can conclude that the zeros of *f* are

$$x = 1 + 3i,\ x = 1 - 3i,\ x = 3,\ x = -2.$$