



# Exponential and Logarithmic Functions



**3.4**

## **Solving Exponential and Logarithmic Equations**



# What You Should Learn

- Solve simple exponential and logarithmic equations.
- Solve more complicated exponential equations.
- Solve more complicated logarithmic equations.
- Use exponential and logarithmic equations to model and solve real-life problems.



# Introduction



# Introduction

There are two basic strategies for solving exponential or logarithmic equations. The first is based on the One-to-One Properties and the second is based on the Inverse Properties.

For  $a > 0$  and  $a \neq 1$  the following properties are true for all  $x$  and  $y$  for which

$$\log_a x \text{ and } \log_a y$$

are defined.



# Introduction

## *One-to-One Properties*

$$a^x = a^y \text{ if and only if } x = y.$$

$$\log_a x = \log_a y \text{ if and only if } x = y.$$

## *Inverse Properties*

$$a^{\log_a x} = x$$

$$\log_a a^x = x$$



## Example 1 – Solving Simple Exponential and Logarithmic Exponential

<i>Original Equation</i>	<i>Rewritten Equation</i>	<i>Solution</i>	<i>Property</i>
<b>a.</b> $2^x = 32$	$2^x = 2^5$	$x = 5$	One-to-One
<b>b.</b> $\log_4 x - \log_4 8 = 0$	$\log_4 x = \log_4 8$	$x = 8$	One-to-One
<b>c.</b> $\ln x - \ln 3 = 0$	$\ln x = \ln 3$	$x = 3$	One-to-One
<b>d.</b> $\left(\frac{1}{3}\right)^x = 9$	$3^{-x} = 3^2$	$x = -2$	One-to-One
<b>e.</b> $e^x = 7$	$\ln e^x = \ln 7$	$x = \ln 7$	Inverse
<b>f.</b> $\ln x = -3$	$e^{\ln x} = e^{-3}$	$x = e^{-3}$	Inverse
<b>g.</b> $\log_{10} x = -1$	$10^{\log_{10} x} = 10^{-1}$	$x = 10^{-1} = \frac{1}{10}$	Inverse
<b>h.</b> $\log_3 x = 4$	$3^{\log_3 x} = 3^4$	$x = 81$	Inverse



# Introduction

The strategies used in Example 1 are summarized as follows.

## Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One Properties of exponential or logarithmic functions.
2. Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.
3. Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.





# Solving Logarithmic Equations



# Solving Logarithmic Equations

To solve a logarithmic equation, you can write it in exponential form.

$$\ln x = 3$$

Logarithmic form

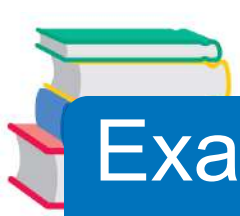
$$e^{\ln x} = e^3$$

Exponentiate each side.

$$x = e^3$$

Exponential form

This procedure is called *exponentiating* each side of an equation. It is applied after the logarithmic expression has been isolated.



## Example 6 – Solving Logarithmic Equations

Solve each logarithmic equation.

a.  $\ln 3x = 2$

b.  $\log_3(5x - 1) = \log_3(x + 7)$

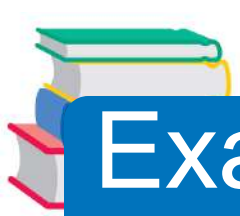
**Solution:**

a.  $\ln 3x = 2$

Write original equation.

$$e^{\ln 3x} = e^2$$

Exponentiate each side.



# Example 6 – Solution

cont'd

$$3x = e^2$$

Inverse Property

$$x = \frac{1}{3}e^2$$

Multiply each side by  $\frac{1}{3}$ .

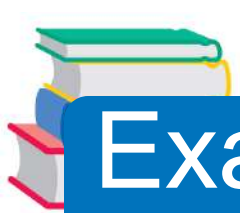
$$x \approx 2.46$$

Use a calculator.

The solution is  $x = \frac{1}{3}e^2$

$$\approx 2.46$$

Check this in the original equation.



# Example 6 – *Solution*

cont'd

**b.**  $\log_3(5x - 1) = \log_3(x + 7)$

Write original equation.

$$5x - 1 = x + 7$$

One-to-One Property

$$x = 2$$

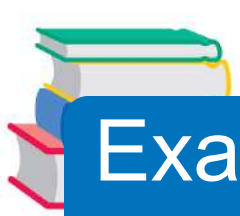
Solve for  $x$

The solution  $x = 2$ . Check this in the original equation.



# Solving Logarithmic Equations

Because the domain of a logarithmic function generally does not include all real numbers, you should be sure to check for extraneous solutions of logarithmic equations.



## Example 10 – *The Change of-Base Formula*

Prove the change-of-base formula:  $\log_a x = \frac{\log_b x}{\log_b a}$  .

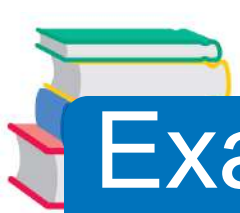
**Solution:**

Begin by letting

$$y = \log_a x$$

and writing the equivalent exponential form

$$a^y = x.$$



# Example 10 – *Solution*

cont'd

Now, taking the logarithms *with base b* of each side produces the following.

$$\log_b a^y = \log_b x$$

$$y \log_b a = \log_b x$$

Power Property

$$y = \frac{\log_b x}{\log_b a}$$

Divide each side by  $\log_b a$ .

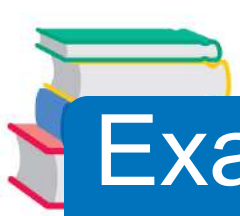
$$\log_a x = \frac{\log_b x}{\log_b a}$$

Replace with  $\log_a x$ .





# Applications



## Example 12 – *Doubling an Investment*

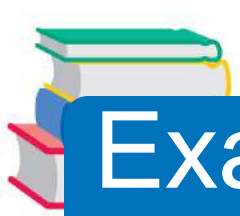
You have deposited \$500 in an account that pays 6.75% interest, compounded continuously. How long will it take your money to double?

### Solution:

Using the formula for continuous compounding, you can find that the balance in the account is

$$\begin{aligned} A &= Pe^{rt} \\ &= 500e^{0.0675t}. \end{aligned}$$

To find the time required for the balance to double, let  $A = 1000$ , and solve the resulting equation for  $t$ .



# Example 12 – Solution

cont'd

$$500e^{0.0675t} = 1000$$

Substitute 1000 for A.

$$e^{0.0675t} = 2$$

Divide each side by 500.

$$\ln e^{0.0675t} = \ln 2$$

Take natural log of each side..

$$0.0675t = \ln 2$$

Inverse Property

$$t = \frac{\ln 2}{0.0675}$$

Divide each side by 0.0675.

$$t \approx 10.27$$

Use a calculator.

The balance in the account will double after approximately 10.27 years.