3

Exponential and Logarithmic Functions

3.4

Solving Exponential and Logarithmic Equations

What You Should Learn

- Solve simple exponential and logarithmic equations.
- Solve more complicated exponential equations.
- Solve more complicated logarithmic equations.
- Use exponential and logarithmic equations to model and solve real-life problems.

Introduction

Introduction

There are two basic strategies for solving exponential or logarithmic equations. The first is based on the One-to-One Properties and the second is based on the Inverse Properties.

For $a > 0$ and $a \neq 1$ the following properties are true for all x and y for which

$$\log_a x \textit{ and } \log_a y$$

are defined.

Introduction

One-to-One Properties

$a^x = a^y$ if and only if $x = y$.

$\log_a x = \log_a y$ if and only if $x = y$.

Inverse Properties

$a^{\log_a x} = x$

$\log_a a^x = x$

Example 1 – *Solving Simple Exponential and Logarithmic Exponential*

Original Equation	*Rewritten Equation*	*Solution*	*Property*
a. $2^x = 32$	$2^x = 2^5$	$x = 5$	One-to-One
b. $\log_4 x - \log_4 8 = 0$	$\log_4 x = \log_4 8$	$x = 8$	One-to-One
c. $\ln x - \ln 3 = 0$	$\ln x = \ln 3$	$x = 3$	One-to-One
d. $\left(\frac{1}{3}\right)^x = 9$	$3^{-x} = 3^2$	$x = -2$	One-to-One
e. $e^x = 7$	$\ln e^x = \ln 7$	$x = \ln 7$	Inverse
f. $\ln x = -3$	$e^{\ln x} = e^{-3}$	$x = e^{-3}$	Inverse
g. $\log_{10} x = -1$	$10^{\log_{10} x} = 10^{-1}$	$x = 10^{-1} = \frac{1}{10}$	Inverse
h. $\log_3 x = 4$	$3^{\log_3 x} = 3^4$	$x = 81$	Inverse

Introduction

The strategies used in Example 1 are summarized as follows.

Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One Properties of exponential or logarithmic functions.
2. Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.
3. Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.

Solving Logarithmic Equations

Solving Logarithmic Equations

To solve a logarithmic equation, you can write it in exponential form.

$\ln x = 3$ — Logarithmic form

$e^{\ln x} = e^3$ — Exponentiate each side.

$x = e^3$ — Exponential form

This procedure is called *exponentiating* each side of an equation. It is applied after the logarithmic expression has been isolated.

Example 6 – *Solving Logarithmic Equations*

Solve each logarithmic equation.

a. $\ln 3x = 2$

b. $\log_3(5x - 1) = \log_3(x + 7)$

Solution:

a. $\ln 3x = 2$ Write original equation.

$e^{\ln 3x} = e^2$ Exponentiate each side.

Example 6 – *Solution*

cont'd

$3x = e^2$ Inverse Property

$$x = \frac{1}{3}e^2$$ Multiply each side by $\frac{1}{3}$.

$x \approx 2.46$ Use a calculator.

The solution is $x = \frac{1}{3}e^2$

≈ 2.46

Check this in the original equation.

Example 6 – *Solution*

cont'd

b. $\log_3(5x - 1) = \log_3(x + 7)$ Write original equation.

$5x - 1 = x + 7$ One-to-One Property

$x = 2$ Solve for *x*

The solution $x = 2$. Check this in the original equation.

Solving Logarithmic Equations

Because the domain of a logarithmic function generally does not include all real numbers, you should be sure to check for extraneous solutions of logarithmic equations.

Example 10 – *The Change of-Base Formula*

Prove the change-of-base formula: $\log_a x = \dfrac{\log_b x}{\log_b a}$.

Solution:

Begin by letting

$$y = \log_a x$$

and writing the equivalent exponential form

$$a^y = x.$$

Example 10 – *Solution*

cont'd

Now, taking the logarithms *with base b* of each side produces the following.

$$\log_b a^y = \log_b x$$

$y\log_b a = \log_b x$ Power Property

$$y = \frac{\log_b x}{\log_b a}$$ Divide each side by $\log_b a$.

$$\log_a x = \frac{\log_b x}{\log_b a}$$ Replace with $\log_a x$.

Applications

Example 12 – *Doubling an Investment*

You have deposited \$500 in an account that pays 6.75% interest, compounded continuously. How long will it take your money to double?

Solution:

Using the formula for continuous compounding, you can find that the balance in the account is

$$A = Pe^{rt}$$

$$= 500e^{0.0675t}.$$

To find the time required for the balance to double, let $A = 1000$, and solve the resulting equation for t.

Example 12 – *Solution*

cont'd

$$500e^{0.0675t} = 1000$$ Substitute 1000 for *A*.

$$e^{0.0675t} = 2$$ Divide each side by 500.

$$\ln e^{0.0675t} = \ln 2$$ Take natural log of each side..

$$0.0675t = \ln 2$$ Inverse Property

$$t = \frac{\ln 2}{0.0675}$$ Divide each side by 0.0675.

$$t \approx 10.27$$ Use a calculator.

The balance in the account will double after approximately 10.27 years.