5-1 Angles and Degree Measure

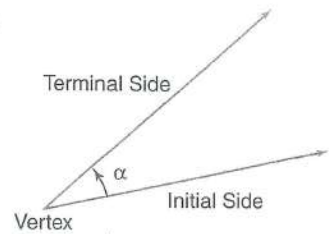
OBJECTIVES

- Convert decimal degree measures to degrees, minutes, and seconds and vice versa.
- Find the number of degrees in a given number of rotations.
- Identify angles that are coterminal with a given angle.

Real World Application

NAVIGATION The sextant is an optical instrument invented around 1730. It is used to measure the angular elevation of stars, so that a navigator can determine the ship's current latitude. Suppose a navigator determines a ship in the Pacific Ocean to be located at north latitude 15.735°. How can this be written as degrees, minutes, and seconds? *This problem will be solved in Example 1.*

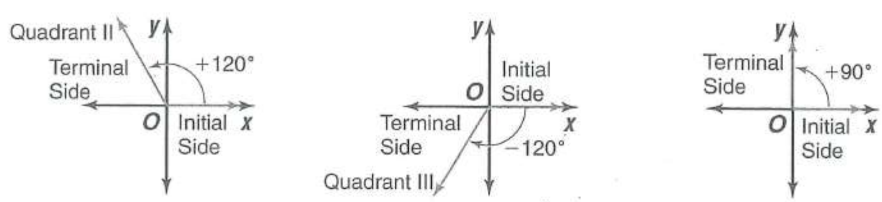
An angle may be generated by rotating one of two rays that share a fixed endpoint known as the **vertex.** One of the rays is fixed to form the **initial side** of the angle, and the second ray rotates to form the **terminal side.**



The measure of an angle provides us with information concerning the direction of the rotation and the amount of the rotation necessary to move from the initial side of the angle to the terminal side.

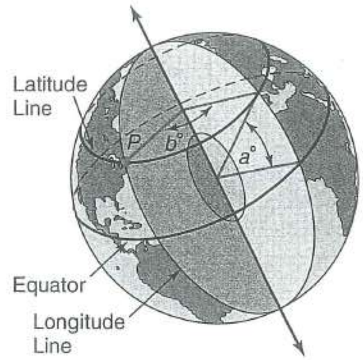
- If the rotation is in a counterclockwise direction, the angle formed is a *positive angle.*
- If the rotation is clockwise, it is a *negative angle.*

An angle with its vertex at the origin and its initial side along the positive x-axis is said to be in **standard position.** In the figures below, all of the angles are in standard position.



The most common unit used to measure angles is the **degree.** The concept of degree measurement is rooted in the ancient Babylonian culture. The Babylonians based their numeration system on 60 rather than 10 as we do today. In an equilateral triangle, they assigned the measure of each angle to be 60. Therefore, one sixtieth $\left(\frac{1}{60}\right)$ of the measure of the angle of an equilateral triangle was equivalent to one unit or degree (1°). The degree is subdivided into 60 equal parts known as **minutes** (1′), and the minute is subdivided into 60 equal parts known as **seconds** (1″).

Angles are used in a variety of real-world situations. For example, in order to locate every point on Earth, cartographers use a grid that contains circles through the poles, called *longitude lines,* and circles parallel to the equator, called *latitude lines.* Point P is located by traveling north from the equator through a central angle of $a°$ to a circle of latitude and then west along that circle through an angle of $b°$. Latitude and longitude can be expressed in degrees as a decimal value or in degrees, minutes, and seconds.



Example 1 **NAVIGATION** **Refer to the application at the beginning of the lesson.**

Real World Application

a. Change north latitude 15.735° to degrees, minutes, and seconds.

$15.735° = 15° + (0.735 \cdot 60)'$ *Multiply the decimal portion of the degree*
$= 15° + 44.1'$ *measure by 60 to find the number of minutes.*
$= 15° + 44' + (0.1 \cdot 60)''$ *Multiply the decimal portion of the minute*
$= 15° + 44' + 6''$ *measure by 60 to find the number of seconds.*

15.735° can be written as 15° 44′ 6″.

b. Write north latitude 39° 5′ 34″ as a decimal rounded to the nearest thousandth.

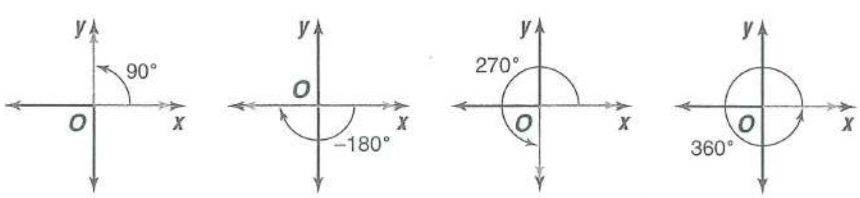
$39°\ 5'\ 34'' = 39° + 5'\left(\frac{1°}{60'}\right) + 34''\left(\frac{1°}{3600''}\right)$ or about 39.093°

39° 5′ 34″ can be written as 39.093°.

Graphing Calculator Tip

▶**DMS** on the **[ANGLE]** menu allows you to convert decimal degree values to degrees, minutes, and seconds.

If the terminal side of an angle that is in standard position coincides with one of the axes, the angle is called a **quadrantal angle.** In the figures below, all of the angles are quadrantal.



A full rotation around a circle is 360°. Measures of more than 360° represent multiple rotations.

Example 2 **Give the angle measure represented by each rotation.**

a. 5.5 rotations clockwise

$5.5 \times -360 = -1980$ *Clockwise rotations have negative measures.*

The angle measure of 5.5 clockwise rotations is −1980°.

b. 3.3 rotations counterclockwise

$3.3 \times 360 = 1188$ *Counterclockwise rotations have positive measures.*

The angle measure of 3.3 counterclockwise rotations is 1188°.

Two angles in standard position are called **coterminal angles** if they have the same terminal side. Since angles differing in degree measure by multiples of 360° are equivalent, every angle has infinitely many coterminal angles.

Coterminal Angles If α is the degree measure of an angle, then all angles measuring $\alpha + 360k°$, where k is an integer, are coterminal with α.

The symbol α is the lowercase Greek letter alpha.

Any angle coterminal with an angle of 75° can be written as $75° + 360k°$, where k is the number of rotations around the circle. The value of k is a positive integer if the rotations are counterclockwise and a negative integer if the rotations are clockwise.

Examples

3 **Identify all angles that are coterminal with each angle. Then find one positive angle and one negative angle that are coterminal with the angle.**

a. 45°

All angles having a measure of $45° + 360k°$, where k is an integer, are coterminal with 45°. A positive angle is $45° + 360°(1)$ or 405°. A negative angle is $45° + 360°(-2)$ or $-675°$.

b. 225°

All angles having a measure of $225° + 360k°$, where k is an integer, are coterminal with 225°. A positive angle is $225° + 360(2)°$ or 945°. A negative angle is $225° + 360(-1)°$ or $-135°$.

4 **If each angle is in standard position, determine a coterminal angle that is between 0° and 360°. State the quadrant in which the terminal side lies.**

a. 775°

In $\alpha + 360k°$, you need to find the value of α. First, determine the number of complete rotations (k) by dividing 775 by 360.

$$\frac{775}{360} \approx 2.152777778$$

Then, determine the number of remaining degrees (α).

Method 1

$$\alpha \approx 0.152777778 \text{ rotations} \cdot 360°$$
$$\approx 55°$$

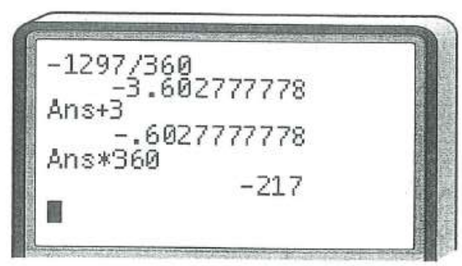
Method 2

$$\alpha + 360(2)° = 775°$$
$$\alpha + 720° = 775°$$
$$\alpha = 55°$$

The coterminal angle (α) is 55°. Its terminal side lies in the first quadrant.

b. −1297°

Use a calculator.

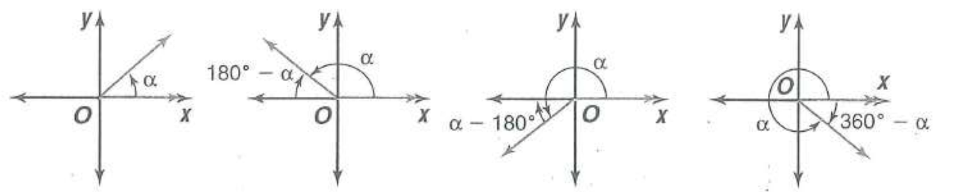


The angle is $-217°$, but the coterminal angle needs to be positive.

$360° - 217° = 143°$

The coterminal angle (α) is 143°.
Its terminal side lies in the second quadrant.

If α is a nonquadrantal angle in standard position, its **reference angle** is defined as the acute angle formed by the terminal side of the given angle and the x-axis. You can use the figures and the rule below to find the reference angle for any angle α where $0° < \alpha < 360°$. If the measure of α is greater than 360° or less than 0°, it can be associated with a coterminal angle of positive measure between 0° and 360°.



Reference Angle Rule

For any angle α, $0° < \alpha < 360°$, its reference angle α' is defined by
a. α, when the terminal side is in Quadrant I,
b. $180° - \alpha$, when the terminal side is in Quadrant II,
c. $\alpha - 180°$, when the terminal side is in Quadrant III, and
d. $360° - \alpha$, when the terminal side is in Quadrant IV.

Example 5 **Find the measure of the reference angle for each angle.**

a. 120°

Since 120° is between 90° and 180°, the terminal side of the angle is in the second quadrant.

$180° - 120° = 60°$

The reference angle is 60°.

b. −135°

A coterminal angle of −135 is 360 − 135 or 225. Since 225 is between 180° and 270°, the terminal side of the angle is in the third quadrant.

$225° - 180° = 45°$

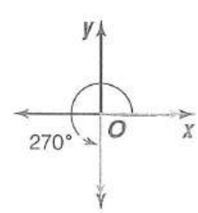
The reference angle is 45°.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Describe** the difference between an angle with a positive measure and an angle with a negative measure.
2. **Explain** how to write 29° 45′ 26″ as a decimal degree measure.
3. **Write** an expression for the measures of all angles that are coterminal with the angle shown.
4. **Sketch** an angle represented by 3.5 counterclockwise rotations. Give the angle measure represented by this rotation.



Guided Practice

Change each measure to degrees, minutes, and seconds.

5. 34.95°
6. −72.775°

Write each measure as a decimal to the nearest thousandth.

7. −128° 30′ 45″
8. 29° 6′ 6″